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An Elementary Derivation of the Routh–Hurwitz Criterion

Ming-Tzu Ho, Aniruddha Datta, and S. P. Bhattacharyya

***Abstract*— In most undergraduate texts on control systems, the Routh–Hurwitz criterion is usually introduced as a mechanical algorithm for determining the Hurwitz stability of a real polynomial. Unlike many other stability criteria such as the Nyquist criterion, root locus, etc., no attempt whatsoever is made to even allude to a proof of the Routh–Hurwitz criterion. Recent results using the Hermite–Biehler theorem have, however, succeeded in providing a simple derivation of Routh's algorithm for *determining the Hurwitz stability or otherwise* of a given real polynomial. However, this derivation fails to capture the fact that Routh's algorithm can also be used to count the number of open right half-plane roots of a given polynomial. This paper shows that by using appropriately generalized versions of the Hermite–Biehler theorem, it is possible to provide a simple derivation of the Routh–Hurwitz criterion which also captures its unstable root counting capability.**

***Index Terms*—Generalized Hermite–Biehler, Routh–Hurwitz, stability.**

I. Introduction

The problem of determining conditions under which all of the roots of a given real polynomial lie in the open left-half complex plane is one of fundamental importance in the study of stability of a dynamic system [1]. This problem has intrigued researchers for more than 100 years now, and one of the earliest solutions, and the most widely known one, is the criterion of Routh–Hurwitz. Indeed, today most undergraduate students are exposed to the Routh–Hurwitz criterion in their first introductory controls course. This exposure, however, is at the purely algorithmic level in the sense that no attempt is made whatsoever to explain why or how such an algorithm works. This is in stark contrast to the treatment given to other stability criteria such as Nyquist or root locus which are rationalized in considerable detail. The principal reason for this is that the classical proof of the Routh–Hurwitz criterion, e.g., [1], relies on the notion of Cauchy indexes and Sturm's theorem, both of which are beyond the scope of undergraduate students. Unfortunately, this material is not covered even in most graduate courses so that the Routh–Hurwitz criterion has become one of the few results in control theory that most control engineers are compelled to accept on faith.

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Very recent results in the area of Parametric Robust Control have started to change this scenario. First, the emergence of Kharitonov's celebrated theorem [2] has focused renewed attention on the Hermite–Biehler theorem, mainly because the original proof of the former relied heavily on the latter. Second, the Hermite–Biehler theorem was used in [3] to provide an elementary derivation of Routh's algorithm for determining the Hurwitz stability of a given real polynomial. However, the derivation of the Routh–Hurwitz criterion given there is incomplete in the sense that it fails to capture the fact that Routh's algorithm, can also be used to count the number of open right half-plane zeros of a real polynomial. Moreover, the approach adopted in that reference does not suggest any obvious fix for removing this discrepancy.

A closer examination of the result in [3] shows that this state of affairs is only to be expected. Indeed, the Hermite–Biehler theorem is applicable to only Hurwitz polynomials and it is, therefore, not surprising that the result in [3] does not permit us to keep a count of the number of open right half-plane zeros. To obtain a simple and complete derivation of the Routh–Hurwitz criterion, it seems logical to first obtain appropriately generalized versions of the Hermite–Biehler theorem applicable to not necessarily Hurwitz polynomials and then exploit these results along the lines of [3]. The main objective of this paper is to do precisely that. In other words, this paper extends the result of [3] and derives the Routh–Hurwitz criterion in its entirety.

The paper is organized as follows. In Section II, we state the relationship between the net phase change of the frequency response of a real polynomial as the frequency ω varies from zero to ∞ and the numbers of its roots in the open left and open right half-planes. In Section III, we use the relationship from Section II to derive two results, each of which in effect is a generalization of the Hermite–Biehler theorem to the case of not necessarily Hurwitz real polynomials. In Section IV, we use the results of Section III to provide a simple derivation of the Routh–Hurwitz criterion. In particular, the ability of Routh's algorithm to count the number of open right half-plane zeros is proven. The singular cases are discussed in Section V. Section VI contains some concluding remarks.

II. Signature and Net Accumulated Phase

In this section, we state a fundamental relationship between the net accumulated phase of the frequency response of a real polynomial and the difference between the numbers of roots of the polynomial in the open left and open right half-planes. To this end, let $\mathcal{C}$ denote the complex plane, $\mathcal{C}^-$ the open left half-plane, and $\mathcal{C}^+$ the open right half-plane. From the very beginning, we focus on polynomials without zeros on the imaginary axis. We consider a real polynomial

$\delta(s)$ of degree n

$$\begin{aligned}\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_n s^n, \delta_i \in \mathcal{R},\\ i = 0, 1, \cdots, n, \delta_n \neq 0\\ \text{such that } \delta(j\omega) \neq 0, \quad \forall \omega \in (-\infty, \infty).\end{aligned}$$

Definition 2.1: Let l and r denote the numbers of roots of $\delta(s)$ in $\mathcal{C}^-$ and $\mathcal{C}^+$, respectively. Then the signature of $\delta(s)$, denoted by $\sigma(\delta)$, is defined as

$$\sigma(\delta) \stackrel{\Delta}{=} l - r.$$

Since

$$n = l + r$$

it follows that $\sigma(\delta)$ and n uniquely determine l and r, and hence the root distribution of $\delta(s)$. Now for every frequency $\omega \in \mathcal{R}$, $\delta(j\omega)$ is a point in the complex plane. Let $p(\omega)$ and $q(\omega)$ be two functions defined pointwise by $p(\omega) = \mathrm{Re}[\delta(j\omega)]$, $q(\omega) = \mathrm{Im}[\delta(j\omega)]$.

With this definition, we have

$$\delta(j\omega) = p(\omega) + jq(\omega) \quad \forall \omega.$$

Furthermore $\theta(\omega) \stackrel{\Delta}{=} \angle\delta(j\omega) = \arctan[q(\omega)/p(\omega)]$. Let $\Delta_0^\infty \theta$ denote the net change in argument $\theta(\omega)$ as ω increases from zero to ∞. Then we can state the following lemma [4, p. 174].

Lemma 2.1: Let $\delta(s)$ be a real polynomial with no imaginary axis roots. Then

$$\Delta_0^\infty \theta = \frac{\pi}{2}\sigma(\delta).$$

III. Generalizations of the Hermite–Biehler Theorem

In this section, we derive two generalizations of the Hermite–Biehler theorem by first developing a procedure for systematically determining the net accumulated phase change of the "frequency response" of a polynomial. We first recall that at any given frequency ω, the phase angle of $\delta(j\omega)$ is given by

$$\theta(\omega) = \tan^{-1}\frac{q(\omega)}{p(\omega)}.$$

Hence the rate of change of phase with respect to frequency at any given frequency ω is given by

$$\begin{aligned}\frac{d\theta(\omega)}{d\omega} &= \frac{1}{1 + \dfrac{q^2(\omega)}{p^2(\omega)}}\,\frac{\dot{q}(\omega)p(\omega) - \dot{p}(\omega)q(\omega)}{p^2(\omega)}\\ &= \frac{\dot{q}(\omega)p(\omega) - \dot{p}(\omega)q(\omega)}{p^2(\omega) + q^2(\omega)}. \qquad (1)\end{aligned}$$

If $p(\omega)$ and $q(\omega)$ are known for all ω, we can integrate (1) to obtain the net phase accumulation. However, to calculate the net accumulation of phase over all frequencies it is *not necessary* to know the precise rate of change of phase at each and every frequency. This is because we know that every time the polar plot $\delta(j\omega)$ makes a transition from the real axis to the imaginary axis or vice versa, there can be at most a net phase change of $\pm(\pi/2)$ radians. The precise sign of the phase change can be determined by examining (1) at the real or imaginary axis crossing of the $\delta(j\omega)$ plot. Since at a real or imaginary axis crossing one of the two terms in the numerator of (1) vanishes and the denominator is always positive, the actual determination of sign of the phase change becomes even simpler.

Now, given any polynomial $\delta(s)$ of degree greater than or equal to one, either the real part or the imaginary part or both of $\delta(j\omega)$ become infinitely large as $\omega \to \pm\infty$. However, if we wish to count the total phase accumulation in integral multiples of real to imaginary axis crossings or imaginary to real axis crossings, it is imperative that the frequency response plot used approach either the real or imaginary axis as $\omega \to \pm\infty$. To accomplish this, one can normalize the plot of $\delta(j\omega)$ by scaling it with $1/f(\omega)$, where $f(\omega) = (1 + \omega^2)^{n/2}$. Since $f(\omega)$ does not have any real roots, this scaling will ensure that the normalized frequency response plot $\delta_f(j\omega) = p_f(\omega) + jq_f(\omega)$ actually intersects either the real axis or the imaginary axis at $\pm\infty$, while at the same time keeping unchanged the finite frequencies at which the $\delta(j\omega)$ plot intersects the real and imaginary axes. The subsequent development in this paper makes use of the normalized frequency response plot for determining the net accumulated phase change as we move from $\omega = 0$ to $\omega = +\infty$.

Using such a normalized frequency response plot, in [5] we obtained an analytical expression for $\sigma(\delta)$. This expression was given in terms of the frequencies at which $\delta_f(j\omega)$ crosses the real and imaginary axes. However, for the purpose of this paper, it is more convenient to derive expressions for $\sigma(\delta)$ which involve either the real or the imaginary axis crossings of $\delta_f(j\omega)$, but not both. Accordingly, in this section, we proceed to carry out such a derivation. In what follows, we will give a detailed derivation only for the expression involving real axis crossings; the expression involving the imaginary axis crossings will be merely stated without proof since its derivation follows along very similar lines.

The following development will make extensive use of the standard signum function $\mathrm{sgn}\colon \mathcal{R} \to \{-1, 0, 1\}$ defined by

$$\mathrm{sgn}[x] = \begin{cases} -1, & \text{if } x < 0\\ 0, & \text{if } x = 0\\ 1, & \text{if } x > 0.\end{cases}$$

Now consider a polynomial $\delta(s)$ of degree n

$$\begin{aligned}\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_n s^n, \delta_i \in \mathcal{R},\\ i = 0, 1, \cdots, n, \delta_n \neq 0\\ \text{such that } \delta(j\omega) \neq 0, \quad \forall \omega \in (-\infty, \infty).\end{aligned}$$

Let $p(\omega)$, $q(\omega)$, $p_f(\omega)$, $q_f(\omega)$ be as already defined, and let

$$0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{m-1}$$

be the real, nonnegative distinct finite zeros of $q_f(\omega)$ with odd multiplicities.[1] Also define $\omega_m = +\infty$.

Then we can make the following simple observations.

1) If ω_i, ω_{i+1} are both zeros of $q_f(\omega)$, then

$$\Delta_{\omega_i}^{\omega_{i+1}}\theta = \frac{\pi}{2}\left[\mathrm{sgn}[p_f(\omega_i)] - \mathrm{sgn}[p_f(\omega_{i+1})]\right]\cdot \mathrm{sgn}[q_f(\omega_i^+)]. \qquad (2)$$

2) If ω_i is a zero of $q_f(\omega)$ while ω_{i+1} is not a zero of $q_f(\omega)$, a situation possible only when $\omega_{i+1} = \infty$ is a zero of $p_f(\omega)$ (n odd), then

$$\Delta_{\omega_i}^{\omega_{i+1}}\theta = \frac{\pi}{2}\,\mathrm{sgn}[p_f(\omega_i)]\cdot \mathrm{sgn}[q_f(\omega_i^+)]. \qquad (3)$$

3)

$$\mathrm{sgn}[q_f(\omega_{i+1}^+)] = -\mathrm{sgn}[q_f(\omega_i^+)], \quad i = 0, 1, 2, \cdots, m-2. \qquad (4)$$

[1] The function $q_f(\omega)$ does not change sign while passing through a real zero of even multiplicity; hence such zeros can be skipped while counting the net phase accumulation.

Equation (2) above is obvious, while (4) simply states that $q_f(\omega)$ changes sign when it passes through a zero of odd multiplicity. Equation (3), on the other hand, can be directly traced to (1).

Using (4) repeatedly, we obtain

$$\operatorname{sgn}[q_f(\omega_i^+)] = (-1)^{m-1-i} \cdot \operatorname{sgn}[q_f(\omega_{m-1}^+)], \quad i = 0, 1, \cdots, m-1. \tag{5}$$

Substituting (5) into (2), we see that if ω_i, ω_{i+1} are both zeros of $q_f(\omega)$, then

$$\Delta_{\omega_i}^{\omega_{i+1}}\theta = \frac{\pi}{2}[\operatorname{sgn}[p_f(\omega_i)] - \operatorname{sgn}[p_f(\omega_{i+1})]] \cdot (-1)^{m-1-i} \cdot \operatorname{sgn}[q_f(\omega_{m-1}^+)]. \tag{6}$$

The above observations enable us to state and prove the following theorem concerning $\sigma(\delta)$.

Theorem 3.1: Let $\delta(s)$ be a given real polynomial of degree n with no roots on the $j\omega$ axis, i.e., the normalized plot $\delta_f(j\omega)$ does not pass through the origin. Let $0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{m-1}$ be the real nonnegative distinct finite zeros of $q_f(\omega)$ with odd multiplicities. Also define $\omega_m = \infty$. Then

$$\sigma(\delta) = \begin{cases} \{\operatorname{sgn}[p_f(\omega_0)] - 2\operatorname{sgn}[p_f(\omega_1)] + 2\operatorname{sgn}[p_f(\omega_2)] \\ \quad + \cdots + (-1)^{m-1} 2\operatorname{sgn}[p_f(\omega_{m-1})] \\ \quad + (-1)^m \operatorname{sgn}[p_f(\omega_m)]\} \cdot (-1)^{m-1}\operatorname{sgn}[q(\infty)], \\ \qquad \text{if } n \text{ is even} \\ \{\operatorname{sgn}[p_f(\omega_0)] - 2\operatorname{sgn}[p_f(\omega_1)] + 2\operatorname{sgn}[p_f(\omega_2)] \\ \quad + \cdots + (-1)^{m-1} 2\operatorname{sgn}[p_f(\omega_{m-1})]\} \\ \quad \cdot (-1)^{m-1}\operatorname{sgn}[q(\infty)], \\ \qquad \text{if } n \text{ is odd.} \end{cases} \tag{7}$$

Proof: First, let us suppose that n is even. Then $\omega_m = \infty$ is a zero of $q_f(\omega)$. The desired expression, i.e., the first one in (7), now follows by repeatedly using (6) to determine $\Delta_0^\infty\theta$, applying Lemma 2.1, and then using the fact that $\operatorname{sgn}[q_f(\omega_{m-1}^+)] = \operatorname{sgn}[q(\infty)]$.

Now let us consider the case that n is odd. Then, $\omega_m = \infty$ is not a zero of $q_f(\omega)$. Hence

$$\begin{aligned} \Delta_0^\infty\theta &= \sum_{i=0}^{m-2} \Delta_{\omega_i}^{\omega_{i+1}}\theta + \Delta_{m-1}^{\infty}\theta \\ &= \sum_{i=0}^{m-2} \frac{\pi}{2}\,[\operatorname{sgn}[p_f(\omega_i)] - \operatorname{sgn}[p_f(\omega_{i+1})]] \\ &\quad \cdot\ (-1)^{m-1-i}\operatorname{sgn}[q_f(\omega_{m-1}^+)] \\ &\quad + \frac{\pi}{2}\operatorname{sgn}[p_f(\omega_{m-1})] \cdot \operatorname{sgn}[q_f(\omega_{m-1}^+)] \\ &\quad [\text{using (6) and (3)}]. \end{aligned} \tag{8}$$

Applying Lemma 2.1, and then using the fact that $\operatorname{sgn}[q_f(\omega_{m-1}^+)] = \operatorname{sgn}[q(\infty)]$, the desired expression follows. □

We now state the result analogous to Theorem 3.1 where the signature $\sigma(\delta)$ of a real polynomial $\delta(s)$ is to be determined using the values of the frequencies, where $\delta_f(j\omega)$ crosses the imaginary axis. The proof is omitted since it follows along essentially the same lines as that of Theorem 3.1.

Theorem 3.2: Let $\delta(s)$ be a given real polynomial of degree n with no roots on the $j\omega$ axis, i.e., the normalized plot $\delta_f(j\omega)$ does not pass through the origin. Let $0 < \omega_1 < \omega_2 < \cdots < \omega_{m-1}$ be the real nonnegative distinct finite zeros of $p_f(\omega)$ with odd multiplicities. Also define $\omega_m = \infty$. Then

$$\sigma(\delta) = \begin{cases} -\{2\operatorname{sgn}[q_f(\omega_1)] - 2\operatorname{sgn}[q_f(\omega_2)] \\ \quad + \cdots + (-1)^{m-2} 2\operatorname{sgn}[q_f(\omega_{m-1})]\} \\ \quad \cdot (-1)^m \operatorname{sgn}[p(\infty)], \\ \qquad \text{if } n \text{ is even} \\ -\{2\operatorname{sgn}[q_f(\omega_1)] - 2\operatorname{sgn}[q_f(\omega_2)] \\ \quad + \cdots + (-1)^{m-2} 2\operatorname{sgn}[q_f(\omega_{m-1})] \\ \quad + (-1)^{m-1}\operatorname{sgn}[q_f(\omega_m)]\} \cdot (-1)^m \operatorname{sgn}[p(\infty)], \\ \qquad \text{if } n \text{ is odd.} \end{cases} \tag{9}$$

IV. Derivation of the Routh–Hurwitz Criterion

In this section, we use Theorems 3.1 and 3.2 to obtain a simple proof of the Routh–Hurwitz criterion. First we consider a real polynomial $\delta(s)$ of degree n

$$\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_n s^n, \delta_n \neq 0$$

and denote

$$\delta(s) = \delta^{\text{even}}(s) + \delta^{\text{odd}}(s)$$

where $\delta^{\text{even}}(s)$, $\delta^{\text{odd}}(s)$ are the polynomials made up of the terms in $\delta(s)$ containing the even and odd powers of s respectively. To avoid singularities of the "first type" and the "second type" [1] in Routh's algorithm, we make the following assumptions.

1) $\delta_{n-1} \neq 0$.
2) $\delta^{\text{even}}(s)$ and $\delta^{\text{odd}}(s)$ are coprime.

To derive Routh's algorithm, we start with the polynomial $\delta(s)$ and construct a polynomial $\delta_1(s)$ of order $n-1$ as follows.

If n is even, then

$$\delta_1(s) = \left[\delta^{\text{even}}(s) - \frac{\delta_n}{\delta_{n-1}} \cdot s \cdot \delta^{\text{odd}}(s)\right] + \delta^{\text{odd}}(s). \tag{10}$$

If on the other hand, n is odd, then

$$\delta_1(s) = \left[\delta^{\text{odd}}(s) - \frac{\delta_n}{\delta_{n-1}} \cdot s \cdot \delta^{\text{even}}(s)\right] + \delta^{\text{even}}(s). \tag{11}$$

The following theorem relates the signature of $\delta(s)$ to that of the reduced-order polynomial $\delta_1(s)$.

Theorem 4.1: Let $\delta(s)$, $\delta_1(s)$ be as already defined. Then

$$\sigma(\delta) - \sigma(\delta_1) = \begin{cases} 1, & \text{if } \delta_n\delta_{n-1} > 0 \\ -1, & \text{if } \delta_n\delta_{n-1} < 0. \end{cases}$$

Proof: Suppose

$$\delta(j\omega) = p(\omega) + jq(\omega). \tag{12}$$

First let us consider the case when n is even. Then from (10)

$$\delta_1(j\omega) = \left[p(\omega) + \frac{\delta_n}{\delta_{n-1}}\omega q(\omega)\right] + jq(\omega). \tag{13}$$

From (12) and (13) it follows that the finite zeros of $q_f(\omega)$ are the same for both $\delta(j\omega)$ and $\delta_1(j\omega)$. Moreover, at these frequencies both $\delta(j\omega)$ and $\delta_1(j\omega)$ have the same real part so that $\operatorname{sgn}[p_f(\omega)]$ is also identical for both these polynomials at these frequencies. Thus, subtracting the second expression on the right-hand side of (7) from the first one, we obtain

$$\sigma(\delta) - \sigma(\delta_1) = -\operatorname{sgn}[p_f(\infty)] \cdot \operatorname{sgn}[q(\infty)].$$

Now for large positive ω

$$p(\omega) \simeq (-1)^{n/2}\delta_n\omega^n$$
$$\text{while } q(\omega) \simeq (-1)^{n-2/2}\delta_{n-1}\omega^{n-1}$$

so that

$$\mathrm{sgn}[p_f(\infty)]\cdot\mathrm{sgn}[q(\infty)] = -\mathrm{sgn}[\delta_n\delta_{n-1}].$$

Thus

$$\sigma(\delta)-\sigma(\delta_1)=\begin{cases}1, & \text{if } \delta_n\delta_{n-1}>0\\ -1, & \text{if } \delta_n\delta_{n-1}<0.\end{cases} \tag{14}$$

We now consider the case that n is odd. Then from (11)

$$\delta_1(j\omega)=p(\omega)+j\left[q(\omega)-\frac{\delta_n}{\delta_{n-1}}\omega p(\omega)\right]. \tag{15}$$

From (12) and (15) it follows that the finite zeros of $p_f(\omega)$ are the same for both $\delta(j\omega)$ and $\delta_1(j\omega)$. Moreover, at these frequencies both $\delta(j\omega)$ and $\delta_1(j\omega)$ have the same imaginary part so that $\mathrm{sgn}[q_f(\omega)]$ is also identical for both these polynomials at these frequencies. Thus, from (9) we obtain

$$\begin{aligned}\sigma(\delta)-\sigma(\delta_1)&=-(-1)^{m-1}(-1)^m\,\mathrm{sgn}[q_f(\infty)]\cdot\mathrm{sgn}[p(\infty)]\\ &=\mathrm{sgn}[p(\infty)]\cdot\mathrm{sgn}[q_f(\infty)].\end{aligned}$$

Now for large positive ω

$$\begin{aligned}p(\omega)&\simeq(-1)^{(n-1)/2}\delta_{n-1}\omega^{n-1}\\ q(\omega)&\simeq(-1)^{(n-1)/2}\delta_n\omega^n\end{aligned}$$

so that

$$\mathrm{sgn}[p(\infty)]\cdot\mathrm{sgn}[q_f(\infty)]=\mathrm{sgn}[\delta_n\delta_{n-1}].$$

Thus

$$\sigma(\delta)-\sigma(\delta_1)=\begin{cases}1, & \text{if } \delta_n\delta_{n-1}>0\\ -1, & \text{if } \delta_n\delta_{n-1}<0\end{cases} \tag{16}$$

and this completes the proof. □

Using Theorem 4.1, we obtain the following corollary.

Corollary 4.1: Let $\delta(s)$ be a given real polynomial and let $\delta_1(s)$ be defined by (10) or (11) as appropriate. Let l, l_1 denote the number of open left half-plane roots of $\delta(s),\ \delta_1(s)$ while $r,\ r_1$ denote the number of open right half-plane roots of $\delta(s),\ \delta_1(s)$. Then

$$\left.\begin{array}{ll} l_1=l-1, r_1=r & \text{if } \delta_n\delta_{n-1}>0\\ l_1=l, r_1=r-1 & \text{if } \delta_n\delta_{n-1}<0\end{array}\right\}. \tag{17}$$

Proof: Now $\sigma(\delta)=l-r$ and $\sigma(\delta_1)=l_1-r_1$. Thus Theorem 4.1 implies that

$$l-r-l_1+r_1=\begin{cases}1, & \text{if } \delta_n\delta_{n-1}>0\\ -1, & \text{if } \delta_n\delta_{n-1}<0.\end{cases} \tag{18}$$

But

$$(l+r)-(l_1+r_1)=1. \tag{19}$$

Adding (18) and (19) we obtain

$$l-l_1=\begin{cases}1, & \text{if } \delta_n\delta_{n-1}>0\\ 0, & \text{if } \delta_n\delta_{n-1}<0.\end{cases} \tag{20}$$

Again, subtracting (18) from (19), we obtain

$$r-r_1=\begin{cases}0, & \text{if } \delta_n\delta_{n-1}>0\\ 1, & \text{if } \delta_n\delta_{n-1}<0.\end{cases} \tag{21}$$

The desired result now follows from (20) and (21). □

Now, given a real polynomial $\delta(s)$, Routh's algorithm is equivalent to reducing the degree of $\delta(s)$ by one at a time using (10) and (11) alternately. This is clearly articulated in [3], [6], and [7], while the Sturm sequence calculation in [1] is equivalent to the alternate application of (10) and (11). Thus Corollary 4.1 leads us to the immediate conclusion that $\delta(s)$ will be Hurwitz iff the leading coefficients of all the polynomials that result from alternately applying (10) and (11) to $\delta(s)$ are of the same sign. Furthermore, it is also clear that the number of open right half-plane roots of $\delta(s)$ is equal to the number of sign changes in the leading coefficients of the successive polynomials. This is exactly the Routh–Hurwitz criterion.

V. Singular Cases

The derivation of the Routh–Hurwitz criterion in the last section dealt with only the so-called "regular" case, i.e., the case in which the degree of $\delta(s)$ can be successively reduced one at a time by the alternate application of (10) and (11) until we finally have a zeroth-order polynomial. This process would, however, terminate prematurely if, while trying to apply (10) or (11), we encounter $\delta_{n-1}=0$. Then we have what are called "singular" cases, and this section is devoted to their treatment.

Starting with a given real polynomial $\delta_0(s)$ of degree n

$$\delta_0(s)=\delta_0^0+\delta_1^0 s+\delta_2^0 s^2+\cdots+\delta_n^0 s^n,$$

suppose using (10) and (11) alternately we obtain a sequence of polynomials $\{\delta_0(s),\delta_1(s),\ \delta_2(s),\cdots,\delta_m(s)\}$, where the leading coefficient of each $\delta_i(s), i=0,1,2,\cdots,m$ is nonzero. Let

$$\delta_m(s)=\delta_0^m+\delta_1^m s+\delta_2^m s^2+\cdots+\delta_{n-m-1}^m s^{n-m-1}+\delta_{n-m}^m s^{n-m}$$

where $\delta_{n-m}^m\neq 0$. Now, if $\delta_{n-m-1}^m=0$, then it is clear that Routh's algorithm stops because to proceed with Routh's algorithm using (10) or (11), we would need to divide by δ_{n-m-1}^m which is now equal to zero. To handle such singularities we consider the three distinct possibilities that can occur.

Case (I): $\delta_{n-m-1}^m=0$ but there exists at least one $k,\ k=3,5,7,9,\cdots$ such that $\delta_{n-m-k}^m\neq 0$, i.e., the first element in any one row of the Routh table vanishes, but there is at least one nonzero element in that row.

If we know beforehand that $\delta_0(s)$ has no imaginary axis roots, then we can proceed as follows. Replace $\delta_{n-m-1}^m=0$ with a "small" nonzero number ϵ of arbitrary sign and then continue to proceed with Routh's algorithm. If a similar singularity is encountered later, introduce another parameter to replace the offending zero element, and so on.

By replacing $\delta_{n-m-1}^m=0$ with ϵ, we in fact modify the original polynomial $\delta_0(s)$. From (10) and (11), for $\delta_{n-m-1}^m=\epsilon$, we can work backward to obtain a modified polynomial $\delta_0(s,\epsilon)$, where the coefficients of $\delta_0(s,\epsilon)$ are rational functions of ϵ. Since $\delta_0(s)$ has no roots on the imaginary axis, it follows by continuity that for ϵ small enough, $\sigma(\delta_0(s))=\sigma(\delta_0(s,\epsilon))$. This is the reason why this modification can be used to handle a singularity of this type and still provide a count of the number of open right half-plane roots.

Case (II): $\delta_{n-m-k}^m=0$ for $k=1,3,5,7,\cdots$, i.e., all the elements in one row of the Routh table vanish.

For this case, since $\delta_{n-m-k}^m=0$ for $k=1,3,5,7,\cdots$, it follows that $\delta_0(s)$ must have one or more pairs of complex conjugate roots symmetrically distributed about the origin of the complex plane. This includes the case of purely imaginary roots as well as the case of purely real roots having opposite signs.

To take care of this kind of singularity, one can simply replace $\delta_0(s)$ with $\delta_0(s-\epsilon)$, where ϵ is a sufficiently "small" positive number and then proceed with Routh's algorithm. The net result is that the number of *closed* right half-plane roots of $\delta_0(s)$ equals the number of sign changes in the leading coefficients of the successive polynomials.

Case (III): Cases (I) and (II) occur at different stages in the same problem when proceeding with Routh's algorithm.

Once again, we can replace $\delta_0(s)$ with $\delta_0(s-\epsilon)$, where ϵ is a sufficiently "small" positive number and then proceed with Routh's algorithm. Alternatively, we can factor out the imaginary axis roots as in [1] and then apply Routh's algorithm to the new polynomial.

Remark 5.1: The derivation of the Routh–Hurwitz criterion in [1] is carried out using the Cauchy Index which disregards the imaginary axis roots. Consequently, in [1], even in singular cases it is possible to obtain a count of the number of open right half-plane roots by appropriately modifying Routh's algorithm. The modifications proposed here, however, allow us to count the number of closed right half-plane roots when the original polynomial has roots on the imaginary axis.

VI. Concluding Remarks

In this paper, we have provided an elementary derivation of the well-known Routh–Hurwitz criterion. A key point in this derivation was the development of appropriately generalized versions of the Hermite–Biehler theorem. The latter enable us to not only derive the Routh test for Hurwitz stability as in [3] but also recover the unstable zero counting capability of Routh's algorithm. The immediate consequence of the result presented here is to make the proof of the Routh–Hurwitz criterion accessible to most people with elementary knowledge of complex numbers. On the other hand, the generalizations of the Hermite–Biehler theorem presented here are likely to have very far reaching implications on the long standing open problem of stabilization using a fixed order compensator. Such problems are currently under investigation and will be addressed in a future paper.

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On the Complexity of Purely Complex μ Computation and Related Problems in Multidimensional Systems

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***Abstract*— In this paper, the following robust control problems are shown to be $\mathcal{NP}$-hard: given a purely complex uncertainty structure $\boldsymbol{\Delta}$, determine if: 1) $\mu_{\boldsymbol{\Delta}}(M) < 1$, for a given rational matrix M; 2) $\|M(\cdot)\|_\mu < 1$, for a given rational transfer matrix $M(s)$; and 3) $\inf_{Q\in\mathcal{H}^\infty} \|\mathcal{F}(T,Q)\|_\mu < 1$, for a given linear fractional transformation $\mathcal{F}(T,Q)$ with rational coefficients. In other words, purely complex μ computation, analysis, and synthesis problems are $\mathcal{NP}$-hard. It is also shown that checking 4) stability and 5) computing the $\mathcal{H}^\infty$ norm of a multidimensional system, are $\mathcal{NP}$-hard problems. Therefore, it is rather unlikely to find nonconservative polynomial time algorithms for solving problems 1)–5) in complete generality.**

***Index Terms*—Complex structured singular value, computational complexity, μ analysis/synthesis, multidimensional systems, $\mathcal{NP}$-hardness.**

Notation

Z	Set of integers.
Q	Set of rational numbers.
$\mathbb{R}$	Set of real numbers.
C	Set of complex numbers.
$\mathbb{D}$	$\{z \in C : \|z\| < 1\}$.
$\bar{\mathbb{D}}$	$\{z \in C : \|z\| \le 1\}$.
T	$\{z \in C : \|z\| = 1\}$.
$\mathcal{H}^\infty$	Space of bounded analytic functions.
j	Square root of -1.
S^n	Set of all n-dimensional column vectors with entries in S.
$S^{m\times n}$	Set of all m-by-n matrices with entries in S.
$F[x_1,\ldots,x_n]$	Set of all polynomials in $x_1,\cdots,x_n$ with coefficients in F.
I_n	n-by-n identity matrix.
$0_{m\times n}$	m-by-n zero matrix.
$u\odot v$	Coordinatewise product of the vectors u and v, $(u\odot v)_k = u_k v_k$.
$\mathbf{B}(\mathbb{R}^n)$	Unit ball of $\mathbb{R}^n$.
$\rho(M)$	Spectral radius of M.
$\bar{\sigma}(A)$	Maximum singular value of A.
$\\|v\\|$	Euclidean norm of the vector v.
$\mathbf{B}(\boldsymbol{\Delta})$	$\{\Delta \in \boldsymbol{\Delta} : \bar{\sigma}(\Delta) \le 1\}$.
$\mu_{\boldsymbol{\Delta}}(M)$	$(\min\{\bar{\sigma}(\Delta) : \Delta \in \boldsymbol{\Delta}, \det(I - M\Delta) = 0\})^{-1}$.
$\\|M(\cdot)\\|_\infty$	$\mathcal{H}^\infty$ norm of $M(s)$, $\sup_{\mathrm{Re}(s)\ge 0}\bar{\sigma}(M(s))$.
$\\|M(\cdot)\\|_\mu$	μ norm of $M(s)$, $\sup_{\mathrm{Re}(s)\ge 0}\mu(M(s))$.

I. Introduction

In control theory, several problems are considered to be difficult, in the sense that they cannot be solved in complete generality by using known polynomial time algorithms. By analyzing their

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