Laplace transform, definition and properties

Function $f, f_1, f_2, g : [0, \infty) \to \mathbb{R}$: piecewise continuous

$$F(s) = \mathfrak{L}(f)(s), \text{ with } F(s) := \int_{0^-}^{\infty} f(t)e^{-st}dt$$

with real$(s) > \sigma_0$ large-enough, and inverse[1] defined using σ_0

- Linearity: $\mathfrak{L}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_2 F_1(s) + \alpha_2 F_2(s)$ for any real/complex constants α_1 and α_2
- Delayed f: $\mathfrak{L}(\sigma_T(f)) = e^{-sT}F(s)$ (with $T \geqslant 0$ and f-'zeroed'). $(\sigma_T(f)(t) := f(t-T))$.
- Derivative of f: $\mathfrak{L}(\frac{d}{dt}f) = sF(s) - f(0^-)$ and
- Integral of f: $\mathfrak{L}(\int_0^t f(\tau)d\tau) = \frac{F(s)}{s}$

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1 $$f(t) = \mathfrak{L}^{-1}(F)(t), \text{ with } f(t) := \frac{1}{2\pi j} \lim_{\omega_0 \to \infty} \int_{\sigma_0 - j\omega_0}^{\sigma_0 + j\omega_0} F(s)e^{st}dt$$

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- **Convolution and product:** $(f * g)(t) := \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$, $\mathfrak{L}(f * g) = F(s)G(s)$
- **Dirac delta:** $\delta * f = f$ **and** $\mathfrak{L}(\delta) = 1$
- **IVT:** $f(0^+) = \lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$ **(provided LHS exists, i.e. no impulses/their derivatives at** $t = 0$**.)**
- **FVT:** $f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$ **(provided LHS exists, i.e.** f **neither diverges, nor oscillates)**
- **Time multiplication** $\mathfrak{L}(tf(t)) = -\frac{d}{ds}F(s)$
- **Complex shift:** $\mathfrak{L}(e^{at}f(t)) = F(s-a)$
- **Time scaling:** $\mathfrak{L}(f(\frac{t}{a})) = aF(as)$

Polynomials/exponentials/sinusoids

- $\mathfrak{L}(1) = \frac{1}{s}$ (note: functions are only on $[0, \infty)$)
- $\mathfrak{L}(t) = \frac{1}{s^2}$
- $\mathfrak{L}(e^{at}) = \frac{1}{s-a}$
- $\mathfrak{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$ and $L(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$
  (Use IVT to be sure of which is of which.)
- $\mathfrak{L}(e^{at}\sin(\omega t)) = \frac{\omega}{(s-a)^2+\omega^2}$