EE613, Nonlinear dynamical systems, Endsemester Exam

19th Nov 2007 Max Marks 90 Weightage 45%.

Question paper has three pages on two sheets.

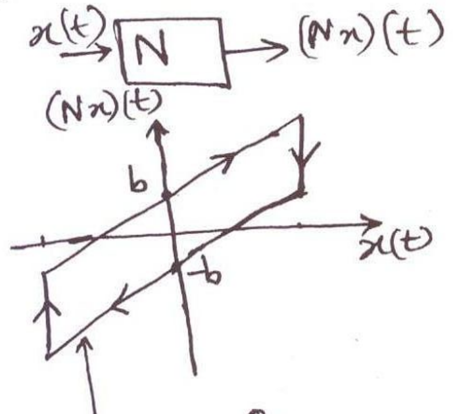
Note: Useful formula given at the end.

For Any clarifications in questions, please feel free to ask me.

Marks distribution given at the end; plan time accordingly.

Prob 1 Consider the "jump hysteresis" nonlinearity which is shown in the figure beside. Plot the output of this nonlinearity for input $r = a \sin \omega t$.

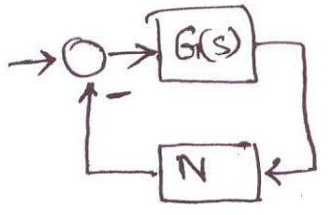
Show that the describing function of N is $\eta(a) = m + \frac{4bj}{\pi a}$.



slope m line

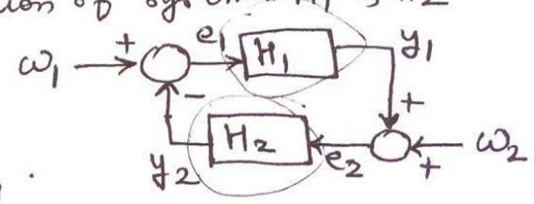
Prob 2: Consider the above nonlinearity connected as below.

$$G(s) = -\left(\frac{s+1}{7s+1}\right).$$



Use the describing function of problem 1 to find existence of periodic solutions and also find frequency & amplitude of oscillations if they exist. $m = 4$, $b = 3$

Prob. 3: Consider feedback interconnection of systems H_1 & H_2 as shown beside.



Suppose H_1 & H_2 are described by state models

$\dot{x}_1 = f_1(x_1, e_1)$, $y_1 = h_1(x_1)$ and

$\dot{x}_2 = f_2(x_2, e_2)$, $y_2 = h_2(x_2)$; respectively. Also assume both H_1 & H_2 are passive. Show that the above feedback interconnection is also passive and show that the autonomous system $\omega_1 = 0$, $\omega_2 = 0$ is stable.

(Assume throughout $f_1(0,0) = 0$, $f_2(0,0) = 0$, $h_1(0) = 0$, $h_2(0) = 0$ and f_i, h_i are locally Lipschitz.)

Prob 4: State La Salle's invariance principle.

Prob 5: State and prove Bellman Gronwall inequality

Prob 6: Define a class KL function.

Prob 7: Consider $G(s) = \frac{s-1}{s+4}$ and the figure of problem 2.

Find the largest sector for nonlinearity N by using

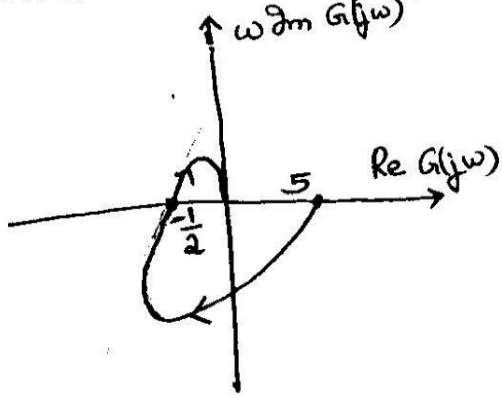
A : Circle criteria

B : Small gain thm

C : Popov criteria

> Note: – First draw
> - Nyquist plot of G(s)
> - Problems 2 & 7 are not related. Only figure is same.

Prob 8: Consider the figure of problem 2. G(s) has Popov plot as shown (almost to scale).



Find K_{max} such that we have absolute stability for nonlinearities in the sector $[0, K_{max}]$.

Assume : 1. Only time invariant nonlinearities

2. G(s) is Hurwitz.

Problems 9 & 10 on next page.

Marks are as follows.

Question No:	1	2	3	4	5	6	7	8	9	10 ← Questions
Marks	10	10	10	5	10	5	15	5	10	10 ← Marks.

Quiz 2, EE 613 Nonlinear dynamical systems

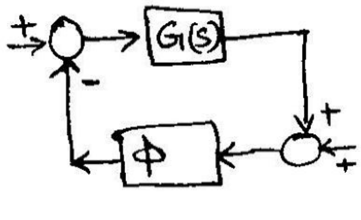
12th Nov, 2007

Marks (Maximum): 60

Weightage: 20%

Problem	1	2	3	4	5	6	7
Marks	5	5	10	10	5	10	15

For all problems below which need a figure, the figure is



Note: 1. First read all <u>seven</u> questions

2. Problems 3 & 4 are closely related.

3. Plan time according to marks.

4. Prob 7: 5 + 10 = 15

State thm — 5; Find k^p_{max} — 10

Prob 1: State KYP lemma for positive real transfer function $G(s)$.

Prob 2: Verify that $G(s) = \frac{s+4}{(s+2)(s+5)}$ is positive real <u>either</u> using KYP lemma <u>or</u> any other way.

Prob 3: Consider $G(s) = \frac{s+4}{(s-1)(s-2)}$. Let $(A, B, C, 0)$ be a state space realization of $G(s)$. Find the range $[k_1, k_2]$ for gain k such that $(A - BkC)$ is a Hurwitz matrix.

Prob 4: Use circle criteria to estimate $[k_1^c, k_2^c]$ such that we have absolute stability for $G(s)$ of problem 3 and any Φ in the sector $[k_1^c, k_2^c]$. (Φ is memoryless and possibly timevarying). Compare this with problem 3's range.

Prob 5: State circle criteria for absolute stability for Φ in the sector $[0, k^c_{max}]$ (Φ is memoryless & possibly time varying.) and $G(s)$ as in figure above.

Prob 6: Find k^c_{max} of problem 5 for $G(s) = \frac{1}{(s+2)(s+5)}$. (only an <u>estimate</u>.)

Prob 7: State Popov criteria <u>and</u> find k^p_{max} for $G(s)$ as in problem 6. (and sector $[0, k^p_{max}]$.)

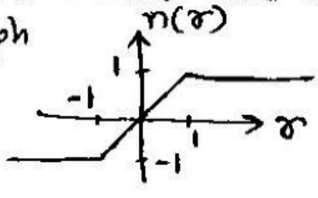
Prob 8: Suppose nonlinearity N in above problem is as below:

$x \rightarrow \boxed{N} \rightarrow (Nx)(t)$

$x \rightarrow \boxed{N_1} \rightarrow \boxed{b} \rightarrow (Nx)(t)$

$(N_1 x)(t) = n(x(t))$ and

this is the standard unit saturation with graph



and b is constant gain b.

Find the minimum value of b to have oscillations.

Prob 9

Consider $b_0 + 0.1$ as the gain, when b_0 is the minimum value above.
For this value of gain, is nonlinearity N within the sector $[0, k_{max})$ of problem 8 ?

Problem 10. Consider the system $\dot{x} = f(x) + G(x)u$, $x(0) = x_0$,
$y = h(x)$ where

f is locally Lipschitz and $G(x)$ & $h(x)$ are continuous. $x(t) \in \mathbb{R}^n$. $y(t) \in \mathbb{R}^q$ $u(t) \in \mathbb{R}^m$.

$f(0) = 0$ and $h(0) = 0$.

State Hamilton Jacobi inequality and its relation to $\mathcal{L}_2$ gain of the above system. $\frac{\partial V}{\partial t} + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} g g^T \frac{\partial V^T}{\partial x} + \frac{1}{2} h^T h < 0$

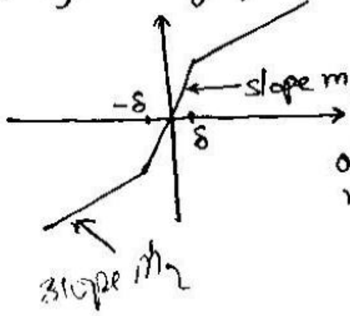
Further, let $f(x) = Ax$, $G(x)u = Bu$, $h(x) = Cx$, for constant matrices A, B & C.

Derive the corresponding Hamilton Jacobi inequality for this case of a linear time invariant system from the general HJ inequality for nonlinear systems.

Paper ends:

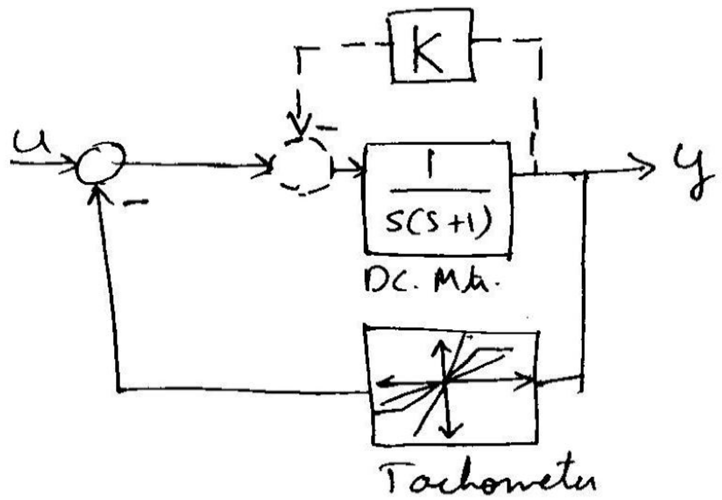
Useful formula. Describing fn for nonlinearity with graph

$$\eta(a) = \begin{cases} m_1 & \text{for } a \leq \delta \\ \frac{2(m_1 - m_2)}{\pi}\left[\sin^{-1}\left(\frac{\delta}{a}\right) + \frac{\delta}{a}\sqrt{1 - \frac{\delta^2}{a^2}}\right] + m_2 \end{cases}$$



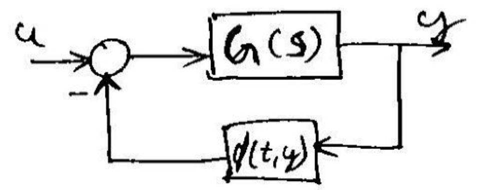
odd, memoryless nonlinearity.

(1)



In a certain DC. mtr. speed control system, the ~~#~~ tachometer (feedback element) due to some damage gives a response in the sector $(-2, 2)$ (the slope of the IP/OP char.). Design a suitable controller (gain K) for the motor so that the interconnected system is stable.

(2)



Given $G(s) = \frac{1}{(s+1)(s+2)}$, find a sector (α, β) such that $\phi \in (\alpha, \beta)$ ensures stability of the interconnected system. Any improvements possible?

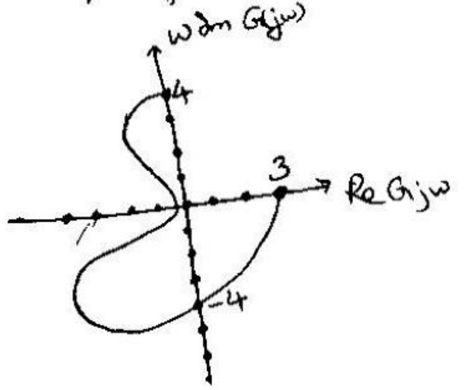
(3) Can you improve the sector if the nonlinear element in Q.2 is known to be time-invariant?

Quiz (4^{th} Nov, 2008) EE613 Nonlinear dynamical systems.

Max marks 35 Weightage: 15 %.
All reasons are to be <u>very brief</u>. Plan time carefully.
Marks distribution at the end. Problems 5, 6 & 7 are related: <u>calculate carefully</u>

1. The Popov plot of a transfer function $G(s)$ is given.

Graph is drawn <u>to scale</u>.



A: Find largest sector of the form $(0, k^1_{max})$ for time varying nonlinearities ϕ in the feedback path.

B. Find largest sector of the form $(0, k^2_{max})$ for time-invariant nonlinearities ϕ in the feedback path.

(Find <u>estimates</u> and justify them.)

C. Which of the above calculations would require stability assumption on G. Give reasons.

D. ~~Which of the above~~ The Popov plot shown above is for ω varying from 0 to +∞. Draw the direction along the curve as $\omega : 0 \longrightarrow +\infty$. Give reasons.

E. What can be said about G:
- strictly proper
- proper but not strictly proper.
- diff in degrees = 1
- diff in degrees > 1

(numerator & denominator degrees)

Give reason (briefly)

2. Prove that any transfer fn $G(s) = \dfrac{a}{s+b}$ with $a, b > 0$ is passive.

Also show that if G_1 & G_2 are passive then $a_1 G_1 + a_2 G_2$ is passive for any $a_1, a_2 > 0$.

(Use definition of passivity.)

3. Consider $G(s) = \dfrac{5}{s-25}$. Find the minimum γ such that the Hamilton Jacobi inequality has a solution $V(x) \geq 0$. (We want $\gamma > 0$.)

4. Define class K function, class KL function and class K_∞ function. Give an example each.

5. Use <u>small gain thm</u> to find the largest sector of memoryless, possibly time varying nonlinearities ϕ that will give absolute stability ~~for~~ when interconnected with $G(s) = \dfrac{s-1}{2s+3}$

6. Use <u>circle criterion for</u> problem 5 to find largest sector..

7. Find the range of <u>linearities</u> such that closed loop is stable for G(s) of problem 5.

Q. No. →	1	2	3	4	5	6	7.
Marks →	10	5	5	6	3	3	3

For clarifications about questions, <u>ask</u>.

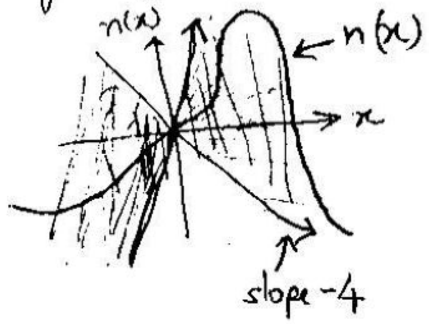
Endsemester Exam EE613, Dec 2007

Max marks : 100 (distribution at the end). Duration : three hours only.

Attempt all questions, spend time according to marks. Read all questions first.

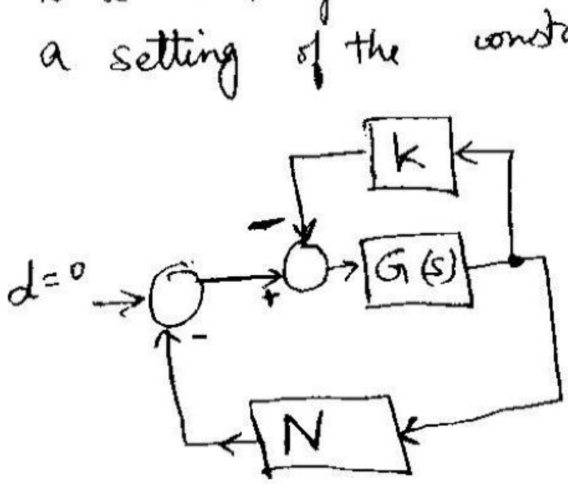
1. Consider nonlinearity N. State conditions on N under which one can define its describing function, and define describing function. Also explain in what sense the describing function of N is an optimal approximation of N.

2. A certain system has a nonlinearity N which is time invariant and memoryless and satisfies $(Nx)(t) = n(x(t))$ and n is in the sector $(-4, 4)$



$x(t) \rightarrow \boxed{N} \rightarrow (Nx)(t)$

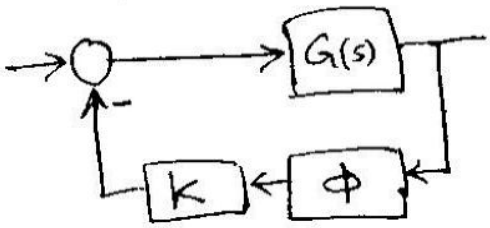
This nonlinearity is connected to a linear system which allows a setting of the constant gain k as follows:



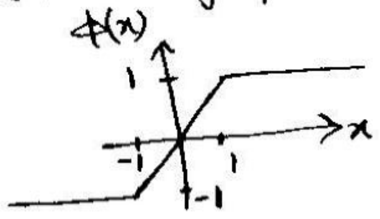
$$G(s) = \frac{1}{s(s+1)}$$

Find the range of k so that any gain k ~~with~~ in this range will result in closed loop stability for any N of the above specified sector and memoryless/time invariant conditions.

6. A certain system $G(s) = \frac{1}{(s+1)(s+2)(s+3)}$ is connected to a saturation nonlinearity as shown:



$\Phi(x)$ has graph



where k is a constant gain that requires to be tuned. Choose the correct statement and solve for the answers.

A: By varying k, we can change both amplitude and frequency of steady state oscillations. If so, find k to get 1 rad/s and amplitude 1 at input of nonlinearity.

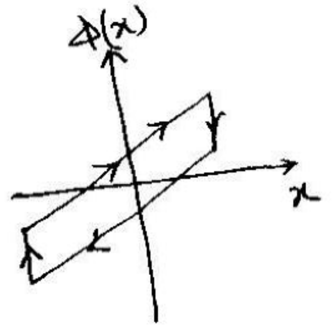
B: By varying k, we can change only frequency ω of steady state oscillations, but not amplitude. If so, find the amplitude and find k to get frequency 1 rad/s.

C: By varying k, we can change only amplitude a, but not frequency ω of steady state oscillations. If so, find frequency ω, and calculate k to get amplitude 1 at input of Φ.

D: By varying k, we can change neither ω nor a. If so, find values of ω & a.

7. The jump hysteresis (graph shown) has describing fn $= 6 + \frac{4j}{\pi a}$.

This is connected to $G(s) = -\frac{(s+1)}{11s+1}$ as in figure of problem 3.



Find frequency & amplitude of oscillations.

(Amplitude a at the input of Φ.

8. When is a function $f : \mathbb{R}^n \to \mathbb{R}^n$ called Lipschitz at a point $x_0 \in \mathbb{R}^n$?

9. Define Lyapunov stability of an equilibrium point $x_0 \in \mathbb{R}^n$ for the system $\dot{x} = f(x)$ with f Lipshitz at x_0.

10. State Lyapunov's theorem of stability of equilibrium point 0 of the system $\dot{x} = f(x)$, f Lipshitz and $f : \mathbb{R}^n \to \mathbb{R}^n$.

11. Prove this theorem (Lyapunov's theorem of stability).

Marks:	10	10	10	10	10	10	10	5	10	10
Question:	1	2	3	4	5	6	7	8	9	10

Question 11: 5 marks. Total = 100.

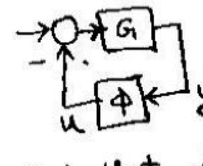
Endsemester exam EE613 (Nonlinear dynamical systems).

Max marks: 100 Weightage 40% Duration 3 hours.

Marks distribution given at the end.

Give brief reasons, and ask for clarifications early. Assume suitably, if required and state your assumptions.

Q-1.. Consider $G(s) = \dfrac{24}{(s+3)(s+2)(s+4)}$ connected to a nonlinearity Φ

as shown 

.

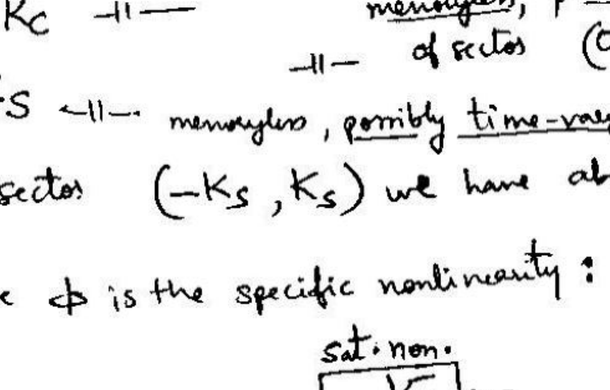
A. Find K_P such that for all memoryless & time invariant nonlinearities Φ in the sector $(0, K_P)$, we have absolute stability

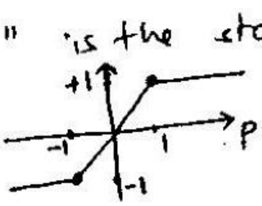
B. Find K_L such that for all memoryless, time invariant linearities Φ in the sector $(0, K_L)$, we have absolute stability

C. Find K_C —"— memoryless, possibly time-varying nonlinearities Φ. —"— of sector $(0, K_C)$ —"—.

D. Find K_S —"— memoryless, possibly time-varying nonlinearities Φ in the sector $(-K_S, K_S)$ we have absolute stability.

E. Suppose Φ is the specific nonlinearity:



where "sat. non." is the standard saturation nonlinearity with graph 

. k is a parameter to set.

Graph of describing function of just sat. non. is given to scale 

Find value of k to obtain closed loop sustained oscillations of amplitude 2 (at y).

Find frequency of these oscillations.

(Value of k is to be approximate and use the graph of η versus a for this.)

(For A, B, C, D above: Nyquist & Popov plots are to be sketched: please draw.)

Q-2 A. Consider $\dot{x} = f(x) + G(x)u$, $y = h(x)$
where $x(t) \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $G(x) \in \mathbb{R}^{n \times m}$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$,
$h : \mathbb{R}^n \to \mathbb{R}^p$. (All functions are continuously differentiable.)
Further $h(0) = 0$, $f(0) = 0$. State the Hamilton-Jacobi inequality and relate existence of $V(x)$ & γ to finite gain $\mathcal{L}_2$ stability of the system. <u>Prove</u> this theorem.

B. Write the same inequality for the system $\dot{x} = -\frac{x}{5} + 2u$, $y = x$.
Find the minimum γ such that the HJI has a solution.

Q-3. For each of the following pairs of statements, choose <u>exactly one</u> of the four relations $\Leftrightarrow$, $\overset{\Rightarrow}{\not\Leftarrow}$, $\overset{\not\Rightarrow}{\Leftarrow}$, $\overset{\not\Rightarrow}{\not\Leftarrow}$ (in place of question mark).

A, B, C below pertain to system $\dot{x} = f(x,u)$, $y = h(x)$, $f(0,0) = 0$ & $h(0) = 0$.

A. (The system is finite gain $\mathcal{L}_2$ stable and unforced system is zero state observable) ? (Origin is asymptotically stable equilibrium pt of unforced system).

B. (Unforced system has origin as exponentially stable eq. pt.) ? (Unforced system has origin as asymptotically stable eq. pt.)

C. (For the unforced system, $\exists\, V(x)$, continuously differentiable and satisfies $V(x) > 0$, but does <u>not</u> satisfy $\dot{V}(x) \leq 0$.) ? (Origin is unstable eq. pt.)

For D, E & F, $K(s)$ is a transfer function $= \frac{n(s)}{d(s)}$, n & d are polynomials with no common factors. D, E & F below are <u>separate</u> questions.

D. K is +ve real ? difference in degrees of d & n ≤ 1

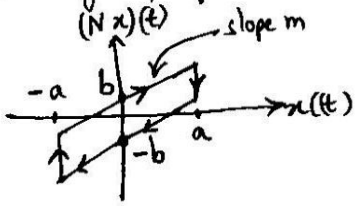
E. K is +ve real ? n has all roots in $\mathbb{C}^-$

F. K is +ve real ? d has all roots in $\mathbb{C}^-$

G. A nonlinearity $\mathcal{N}$ is memoryless & time invariant ? Its describing function η is independent of ω.

Q-4. Let $G(s)$ be a proper transfer function with minimal realization $G(s) = D + C(sI-A)^{-1}B$. State the Kalman Yakubovich Popov Lemma for the strictly +ve real $G(s)$ & relation to A, B, C, D.

Q-5. Consider the jump hysteresis nonlinearity whose input/output relation is shown

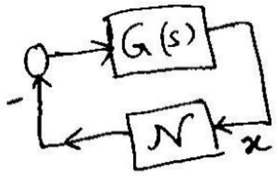


$$(Nx)(t) = m x(t) + b \quad \text{if } \dot{x} > 0$$
$$= m x(t) - b \quad \text{if } \dot{x} < 0.$$

A. Is this nonlinearity memoryless ?

B. Derive its describing function from first principles, i.e., using the definition of describing function of a nonlinearity.

C. Consider this connected to $G(s) = \dfrac{-(s+1)}{11s+1}$ as shown.



Find the amplitude of oscillations and its frequency at input to N for $m = 6$ and $b = \pi$.

D. Find the range of m for which oscillations are expected (as far as describing function method approximation yields) and compare this with the range of linearities that destabilize the system. (Use Nyquist criterion for the latter;) draw Nyquist plot. Comment on frequency of oscillation as m is varied.

4. Verify either using KYP lemma or using the definition that $\dfrac{3s+2}{(s+1)(s+2)}$ is a positive real transfer fn.

5. State assumptions on the state model and the Hamilton Jacobi inequality and its relation to the $\mathcal{L}_2$ gain γ of the system. Specialize the inequality to the case of linear system
$$\dot{x} = Ax + Bu$$
$$y = Cx$$