# Matrix results: useful for EE640 course cyclic matrices, Lyapunov equation solvability, positive (semi-) definiteness

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## 1 Definitions

'P : $\iff$ Q' means that property-P is <u>defined</u> by statement Q (and is thus they both are equivalent by definition). Below are some definitions.

- A polynomial $p(s)$ is called Hurwitz if all its roots are in the <u>open</u> left half complex plane.
- A square matrix $A \in \mathbb{R}^{n\times n}$ is called Hurwitz if all its eigenvalues are in the <u>open</u> left half complex plane.
- An eigenvalue $\lambda$ is called distinct (or non-repeated) if its algebraic multiplicity $n_a(\lambda)$ equals 1. Such eigenvalues are also called 'simple'. If $n_a(\lambda) > 1$, then $\lambda$ is called a repeated-eigenvalue.
- An eigenvalue $\lambda$ of matrix $A$ is called 'semi-simple' if $\lambda$ is a repeated eigenvalue and $n_a(\lambda) = n_g(\lambda)$ (where $n_g(\lambda)$ denotes the geometric multiplicity of $\lambda$).
- An eigenvalue $\lambda$ of matrix $A$ is called 'defective' if $n_g(\lambda) < n_a(\lambda)$.
- $A$ is almost Hurwitz $: \iff$ all eigenvalues in closed LHP <u>and</u> those on $j\mathbb{R}$ (if any) satisfy $n_g(\lambda) = n_a(\lambda)$.
- A symmetric matrix $Q \in \mathbb{R}^{n\times n}$ is called positive definite (and denoted by $Q > 0$) if $v^T Q v > 0$ for every nonzero vector $v \in \mathbb{R}^n$.
- A symmetric matrix $Q \in \mathbb{R}^{n\times n}$ is called positive semi-definite (also called non-negative definite, and denoted by $Q \geqslant 0$) if $v^T Q v \geqslant 0$ for every vector $v \in \mathbb{R}^n$.
- A matrix pair $(Q, A)$ is called observable if the system $\frac{d}{dt}x = Ax$ and $y = Qx$ satisfies observability.

## 2 Facts about Lyapunov equation solvability

**Fact 1: (about Hurwitz)** For $A \in \mathbb{R}^{n\times n}$, following are equivalent (F.A.E):

1. $A$ is Hurwitz
2. $A^T$ is Hurwitz
3. $\lim \|e^{At}x_0\|_2 \to 0$ as $t \to \infty$ for every $x_0 \in \mathbb{R}^n$
4. For every $Q > 0$, there exists $P > 0$ such that $A^T P + PA = -Q$
5. For every $Q \geqslant 0$ with with $(Q, A)$ observable, there exists $P > 0$ such that $A^T P + PA = -Q$

**Fact 2: (about almost-Hurwitz)** For $A \in \mathbb{R}^{n\times n}$, following are equivalent (F.A.E):

1. $A$ is almost Hurwitz
2. $A^T$ is almost Hurwitz
3. $\|e^{At}x_0\|_2$ is a bounded function of time over $[0, \infty)$ for every $x_0 \in \mathbb{R}^n$
4. There exist symmetric matrices $Q \geqslant 0$ and $P > 0$ such that $A^T P + PA = -Q$

('bounded' function $f(t)$ for $t \in [0, \infty)$ in the above fact means there is an $M > 0$ such that $|f(t)| < M$ for all $t \in [0, \infty)$. Of course, $M$ depends on $f$.)

# 3 Positive and non-negative definite matrices

A $k \times k$ minor of a matrix $P$ is the square $k \times k$ matrix obtained by choosing some $k$ rows and some $k$ columns (need not be consecutive, and row and column indices need not be the same). If row/column indices are same, then the minor is called 'principal minor'. If for each $k$, the **first** $k$ positive integers are taken (i.e. 1, 2, 3, $\dots k$), then the principal minor is called 'leading principal minor'.
**Fact 3:** For a matrix $P \in \mathbb{R}^{n\times n}$ satisfying $P = P^T$, following are equivalent.

1. $P$ is positive definite.
2. All eigenvalues are positive.
3. All principal minors (of all sizes) of $P$ have a positive determinant.
4. All leading principal minors (of all sizes) of $P$ have a positive determinant.
5. $P \geqslant 0$ and $P$ is nonsingular.
6. $P \geqslant 0$ and no eigenvalue is zero.
7. There exists a full-column-rank matrix $S$ such that $P = S^T S$.
8. There exists a nonsingular matrix $S$ such that $P = S^T S$.

**Fact 4:** For a matrix $P \in \mathbb{R}^{n\times n}$ satisfying $P = P^T$, following are equivalent.

1. $P$ is positive semi-definite (i.e. $P$ is non-negative definite).
2. All eigenvalues are non-negative.
3. All principal minors (of all sizes) of $P$ have a non-negative determinant.
4. There exists a matrix $S$ such that $P = S^T S$.

$P$ being non-negative **implies** (but not equivalent to) each of the statements:

1. $v^T P v = 0$ implies $Pv = 0$.
2. $v^T P v = 0$ implies each of the $S$ in the last statement (of the above fact) satisfies $Sv = 0$.

Thus kernel of $P$ is same as kernel of each $S$ of the last statement (of the above fact).

# 4 Cyclic matrices

For a given matrix $A \in \mathbb{R}^{n\times n}$, the characteristic polynomial $\chi_A(s)$ is defined as $\det(sI - A)$. The polynomial $\chi_A(s)$ has real coefficients, is of degree exactly $n$ and is monic. The minimal polynomial $m_A(s)$ is defined as the monic polynomial of minimum degree that satisfies $m_A(A) = 0$. Since the characteristic polynomial $\chi_A(s)$ is already satisfied by the matrix $A$, i.e. $\chi_A(A) = 0$ (the Cayley Hamilton theorem), we note that, in general, degree of $m_A(s) \leqslant n$.
**Fact 4: Cyclic matrix equivalent properties**. Let $A \in \mathbb{R}^{n\times n}$ be a matrix. (The first one is the definition). The following are equivalent.

1. $A$ is cyclic ($:\iff$ for **some vector** $v \in \mathbb{R}^n$, the vectors $v$, $Av$, $A^2 v$, $\dots$, $A^{n-1}v$ are independent).
2. $n_g(\lambda) = 1$ for each eigenvalue of $A$.
3. The minimal polynomial $m_A(s)$ of the matrix $A$ is same as the characteristic polynomial $\chi_A(s)$.
4. rank of $(\lambda I - A) \geqslant n - 1$ for every complex number $\lambda$.
5. In some basis, i.e. there exists a nonsingular matrix $S$ such that $S^{-1}AS$ is an unreduced upper Hessenberg matrix. ('upper Hessenberg matrix' is not within EE640 syllabus.)
6. $A$ is similar to each of the four companion forms.

A special case of above fact is that a matrix $A$ with **distinct** eigenvalues is cyclic, and also diagonalizable. In fact, any cyclic and diagonalizable matrix has to have distinct eigenvalues.

# 5 Miscellaneous

If subspace $S_1 \subseteq S_2$ and suppose both have same dimensions, then $S_1 = S_2$.