Realization from Markov prameters

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Realization of system from Markov parameters

Suppose the Markov parameters are known for a strictly proper SISO transfer function $G(s) = \frac{b(s)}{a(s)}$, for real and coprime polynomials a and b, with $a(s)$ monic and of degree n.
Construct the Hankel matrix formed from the Markov parameters (h_i)

$$H = \begin{bmatrix} h_1 & h_2 & \cdots & h_n & \cdots \\ h_2 & h_3 & \cdots & h_{n+1} & \cdots \\ & & & & \cdots \\ \iddots & \iddots & \iddots & h_{2n-2} & \cdots \\ h_n & \iddots & \iddots & h_{2n-1} & \cdots \\ h_{n+1} & \iddots & \iddots & h_{2n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

Realization of system from Markov parameters

Find the smallest[1] n such that rank$(H(1:n, 1:n)) = n$, and rank $(H(1:m, 1:m)) = n$; for all[2] $m \geqslant n$.
This gives the order of the system (when SISO): n.

- Solve[3]. for a_i's in the row vector of size $n+1$ such that $[a_0\ a_1\ \cdots a_{n-1}\ 1] \times H(1:(n+1), 1:n) = 0$.
  Note: this row-vector is not the zero vector: this is the meaning of not full row rank.
- Claim: since n is the smallest such number, we note that $H(1:n, 1:n)$ is nonsingular, and hence a_n has to be nonzero[4]: thus normalize the entire row to have $a_n = 1$.

[1] Here $H(1:n, 1:n)$ means matrix formed by taking 1^{st} to n^{th} rows and 1^{st} to n^{th} columns of H.

[2] There is some implicit "model order approximation" involved here: if one is interested in a system transfer function of at most order 100, then find smallest n with m searched till 100. There could be a rank increase for $n = 10,000$ maybe, but we might not be interested in that large order a model anyway.

[3] Antsaklis and Michel, Linear Systems, Birkhauser, 1997 Sec [5.4D]

[4] If a_n is zero, then same row vector would prove that $H(1:n, 1:n)$ is singular

- From above step we get coefficients of denominator of the transfer function, which is given as $s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$.
- Claim: the system constructed in observable-canonical form[5] as below:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \qquad b = \begin{bmatrix} h1 \\ h2 \\ \vdots \\ h_n \end{bmatrix} \tag{1}$$

$$C = [1 \; 0 \; \cdots \; 0] \tag{2}$$

[5] Kailath, Linear Systems, Englewood Cliffs, NJ: Prentice-Hall, 1980 Sec [3.2.1]

Verification and (some) justification

- Verify (by CB, CAB, etc) that indeed the Markov parameters are as sought.
- The row-vector (in blue, on the previous slide) being perpendicular to each of the first few columns of the Hankel matrix (i.e. $H(1:n+1, 1:n)$) is merely rewriting equation that equates coefficients of like-powers in $\frac{b(s)}{a(s)} = \frac{h_1}{s} + \frac{h_2}{s^2} + \frac{h_3}{s^3} + \frac{h_4}{s^4} + \frac{h_5}{s^5} + \cdots$

$$b(s) = b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + b_1 s + b_0 = \tag{3}$$

$$a(s)\left(\frac{h_1}{s} + \frac{h_2}{s^2} + \frac{h_3}{s^3} + \frac{h_4}{s^4} + \frac{h_5}{s^5} + \cdots\right) \tag{4}$$

for coefficient of $\frac{1}{s}, \frac{1}{s^2}, \frac{1}{s^3}, \frac{1}{s^4}, \frac{1}{s^5}$, etc.

Example

Given $h_1 = 1, h_2 = -3, h_3 = -9, h_4 = -15, h_5 = -9, h_6 = 57$ and so on.
It is found that $n = 3$.

$$H(1:4, 1:3) = \begin{bmatrix} 1 & -3 & -9 \\ -3 & -9 & -15 \\ -9 & -15 & -9 \\ -15 & -9 & 57 \end{bmatrix}$$

and $[-6 \;\; 11 \;\; -6 \;\; 1]H(1:4, 1:3) = 0$.
So a realization is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ -3 \\ -9 \end{bmatrix}$$

Also $c = [1 \;\; 0 \;\; 0]$, giving transfer function $T(s) = \frac{(s^2 - 9s + 20)}{(s^3 - 6s^2 + 11s - 6)}$

Some questions and references

- Why is coprimeness of $b(s)$ and $a(s)$ coming from? (Check using an example of $a(s)$ of degree 2 and has a common factor (of say degree 1) with $b(s)$ (which itself has to be of degree 1: since we are dealing with strictly proper transfer function only.
- If $a(s)$ coefficients (after making a monic) get decided by h_i, then where is the freedom of h_i for $b(s)$ to be arbitrary (with n degrees of freedom in the coefficients of the $n-1$ degree polynomial $b(s)$?)

References

- Kailath, Linear Systems, Englewood Cliffs, NJ: Prentice-Hall, 1980
- Antsaklis and Michel, Linear Systems, Birkhauser, 1997
- Slides for Multivariable Control system by Ali Karimpour, Ferdowsi University of Mashhad. Click here (link)