Tutorial 2   EE640   Multivariable   21st Aug 2015

2005 year version of Stoorvogel/Trentelman/Hautus book (Same on moodle)
All numbers w.r.t. above book.

1. An $n\times n$ matrix A is called cyclic if there exists a column vector $(n\times 1)$ b s.t. (A,b) is controllable. Show that the foll. are equivalent:
   a. A is cyclic.
   b. $\exists$ an invertible matrix S s.t. $S^{-1}AS$ has a companion form.
   c. $\forall \lambda \in \mathbb{C}$, rank $(\lambda I - A) \geq n-1$.
   d. χ_A is the monic polynomial $p(z)$ of minimal degree for which $p(A)=0$ (i.e. χ_A is the minimal polynomial of A).
   e. Any $n\times n$ matrix B that commutes with A is a polynomial of A. i.e. if $AB=BA$, $\exists$ a polynomial $p(z)$ s.t. $p(A)=B$. (Ref. Exercise 3.2)

2. Prove that (A,B) controllable $\Leftrightarrow$ $(S^{-1}AS, S^{-1}B)$ controllable, for any invertible S. (Ref. § 3.4)

3. Show that if (A,B) is controllable then $(A+BF, B)$ is controllable for any map $F: \mathcal{X}\to\mathcal{U}$. (Ref. Ex. 3.9)

4. Let $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$. Show that, $\exists\, p\in\mathbb{R}^{m}$ s.t. (A,Bp) is controllable iff.
   a. (A,B) is controllable
   b. A is cyclic. (Ref. Ex 3.3)

5. Theorem (3.11): Let (A,B) be not controllable and B not zero. Then there exists an invertible matrix S s.t. the pair $(\bar{A},\bar{B})$, where $\bar{A} := S^{-1}AS$, $\bar{B} = S^{-1}B$ has the form:

$$\bar{A} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \text{ where } (A_{11}, B_1) \text{ is controllable.}$$

6. Corollary (3.30): If λ is an uncontrollable eigenvalue of (A,B) then λ is an eigenvalue of $(A+BF)$, $\forall F$.

7. Verify using the definitions (series) of e^{At} that if $A = T^{-1}BT$, then $e^{At} = T^{-1}e^{Bt}T$.

(a) Find e^{At} for $A = \begin{bmatrix} 2 & 3 \\ 0 & 8 \end{bmatrix}$ by diagonalizing. (More in Sect. 4.5.4 of Polderman/Willems)

(b) Find $P_{t_1} := \int_0^{t_1} B e^{At} e^{A^T t} B^T dt$ for $A = \begin{bmatrix} 2 & 0 \\ 0 & -8 \end{bmatrix}$ & $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $t_1 = 2s$.

⑧ Def^n (3.12) An eigenvalue λ of A is called (A,B)-controllable if

$\text{rank}\,(A-\lambda I \quad B) = n$.

Consider the following: $A = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. find the controllable and uncontrollable eigenvalues. Use definitions in Reference (Chapter 3) and compare with class definitions (lecture).

~~Some~~ known and useful results about +ve definite and +ve semidefinite (symmetric assumed) matrices.

Result 1: Suppose $P = P^T \in \mathbb{R}^{n\times n}$. Then, the following are equivalent.

(a) $P \geq 0$ (called P is non-negative definite or P is +ve semi definite)

(b) $x^T P x \geq 0 \;\; \forall x \in \mathbb{R}^n$ (definition of (a)).

(c) $P = GG^T$ for some matrix G

(d) $P = GG^T$ for a full ~~r~~ column rank matrix G (then, $G \in \mathbb{R}^{n \times (\text{rank}\,P)}$).

(e) all eigenvalues of P are non-negative.

(f) $P = UDU^T$ for an orthogonal U ($UU^T = I$) & D ~~is~~ is diagonal with $d_{ii} \geq 0$ for all i

(g) (I strongly think) $x^T P x = 0 \Rightarrow Px = 0$ ($\Rightarrow Gx = 0$) and $x^T P x > 0$ for atleast one other x (except when $P = 0$)

Note: when $P \geq 0$, then $x^T P x = 0 \Rightarrow Px = 0 \Leftrightarrow Gx = 0$ (G of c or d).

Result 2: $P = P^T \in \mathbb{R}^{n\times n}$. T.F.A.E.

(a) $P > 0$ (called positive definite).

(b) $x^T P x > 0 \;\; \forall x \neq 0$ (definition of (a)).

(c) $P = GG^T$ for some full row rank G

(d) $P = GG^T$ for some nonsingular G.

(e) $P = UDU^T$ for orthogonal U & D diagonal with $d_{ii} > 0$.

(f) all eigenvalues of $P > 0$ (note symmetry assumed on P already).

(g) P^{-1} ~~is +ve~~ exists and is +ve definite.

(h) P is ≥ 0 and P is nonsingular.

Note. G in the two results is not unique.