Tutorial Sheet: EE 640 (Multivariable Control)

29th Sept 2015. This is a long tutorial sheet. Please try all. (4 pages.)

Q-1: Suppose A is cyclic. And let $b \in \mathbb{R}^n$ s.t. (A, b) controllable.
Clearly $L := [b \;\; Ab \cdots A^{n-1}b]$ is invertible & brings A to the companion form $\begin{bmatrix} 0 & & \\ 1 & & \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$, $\hat{b} = e_1$. In this exercise, prove that last row of L^{-1} (say ~~q~~ q_1^T) helps to bring A to

$$\begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ * & * & \cdots & * \end{bmatrix} \;\&\; \hat{b} = e_n.$$

(a) Last row of L^{-1} satisfies $[0 \; 0 \; 0 \cdots 1] = q^T L$
why?
Check that q satisfies $q^T b = 0$, $q^T A b = 0, \ldots$ $q^T A^{n-2} b = 0$
$q^T A^{n-1} b = 1$.

(b) (Like b is for right side of A), show that q^T is such that

$$\begin{bmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{bmatrix} =: Q$$

is invertible. (Evaluate QL & argue invertibility of product QL, hence Q).

(c) Find $\hat{A}$ that satisfies $\hat{A} Q = Q A$ ($\hat{A}$ - "bottom" companion form. There are top/bottom left/right companion forms, depending on where $\chi_A(s)$ coefficients are.).

$\boxed{\text{Also find } Qb =: \hat{b}}$

- Look up how van der Monde matrix diagonalizes (suitable) companion form.
- check that inverse of a companion form is also companion form (when $\chi_A(0) \neq 0$)

Q-2: Suppose $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.
Suppose $L := [b_1 \; Ab_1 \cdots A^{\mu_1 - 1} b_1 \;\; b_2 \; Ab_2 \cdots A^{\mu_2 - 1} b_2 \;\; b_3 \cdots A^{\mu_m - 1} b_m]$
with μ_i controllability indices for $i = 1, \ldots m$.
Define q_i^T as the σ_i^{th} row of L^{-1}
where $\sigma_1 = \mu_1$, $\sigma_2 = \mu_1 + \mu_2$, $\ldots$ $\sigma_m = \sum_{i=1}^{m} \mu_i = n$.

(a) Find $\det(QL)$ with Q defined as:

(PTO)

$$Q := \begin{bmatrix} q_1^T \\ q_1^T A \\ q_1^T A^{\mu_1 - 1} \\ q_2^T \\ q_2^T A \\ \vdots \\ q_m^T A^{\mu_m - 1} \end{bmatrix}$$

Find

$Q A Q^{-1}$

(invertibility of Q shown due to $\det(QL)$:).

Evaluate $\hat{B} := QB$

<u>Suggest</u> a simple "column operation" to have $\hat{B}$ as just some suitable m columns e_j of $I_{n \times n}$.

<u>Q-3</u>

Solve Q-2 for $B \in \mathbb{R}^{n \times 2}$ ($m = 2$ only)

and find F such that $\hat{A} + \hat{B}F$ is cyclic (assuming (A, B) is controllable). Show that $\exists\, b \in \operatorname{Im} \hat{B}$ such that $(\hat{A} + \hat{B}F, b)$ is controllable.

<u>Q-4</u>: Consider $\begin{bmatrix} sI - A & -B \\ I & 0 \end{bmatrix}$ (i.e. x is output y) u is input.

Suppose the system is controllable.
Show that the system zeros are unchanged by any <u>state</u> feedback.

<u>Q-5</u>: Consider $\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}$ with this matrix full <u>column</u> rank.

- Assume observable.
  Show that at a system zero λ_0, $\exists\, u_0 \neq 0$ such that $y(t) \equiv 0$ (for $u_0 e^{\lambda_0 t}$ input)
- Where did observability play a role?

PTO

Q-6 Suppose $[P(s)\ \ Q(s)]$ is wide, full row rank matrix (as a polynomial matrix)
(think of $[sI-A\ \ -B]$ as an application.)

Suppose $[P\ \ Q] \in \mathbb{R}^{n\times(n+m)}[s]$ with $\det P \neq 0$
(P square & nonsingular).

Suppose $[P\ \ Q]\begin{bmatrix} N \\ D \end{bmatrix} = 0$ with $\begin{bmatrix} N \\ D \end{bmatrix}$ full column rank
& $\begin{bmatrix} N \\ D \end{bmatrix} \in \mathbb{R}^{(n+m)\times m}[s]$.

(a) Show $\det D \neq 0$. (Hint: assume $w \neq 0$ such that $Dw = 0$. Use this to show P cannot be nonsingular (with $Nw \neq 0$ due to $\begin{bmatrix} N \\ D \end{bmatrix}$ f.c.r.)

(b) Prove $-P^{-1}Q = ND^{-1}$
even if N, D are not ~~relat~~ mutually coprime

(c) Assume $[P\ \ Q]$ is left coprime (also called "irreducible" (understood for both tall/wide)
and $\begin{bmatrix} N \\ D \end{bmatrix}$ also irreducible.

Prove $\det P = c \det D$ $(c \neq 0, c \in \mathbb{R})$
why was irreducible required? (Show, without that, not true.)

(d) In fact, though P & D are square but of different sizes, they have same nonunity polynomials in their Smith form. (polynomials in the Smith form are called "invariant polynomials".)

Proof of d we postpone. (not too hard though.)

Q-7: Show that for $A \in \mathbb{R}^{n\times n}$
A is cyclic $\iff$ $(sI-A)$ has invariant polynomials
$\underbrace{1, 1, 1, \ldots, 1}_{n-1 \text{ times}}, \chi_A(s)$

Q-8 Suppose A is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$ Find invariant polynomials (in the Smith form of $sI-A$)

PTO

Q-9: Suppose $[sI-A \;\; -B]$ is irreducible and

$\begin{bmatrix} N \\ D \end{bmatrix} \in \mathbb{R}^{(n+m)\times m}[s]$ is irreducible s.t. $(sI-A)N - BD = 0$.

(a) Show that A cyclic $\Rightarrow$ D loses rank by at most 1 for any pole λ of $(sI-A)^{-1}B$.

(b) Suppose $UDV = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \chi_A(s) \end{bmatrix}$, U, V unimodular (A : cyclic)

and consider $NV =: [f_1 \; f_2 \cdots f_m]$ are column of UNV. $f_i \in \mathbb{R}^n$.

first conclude that

$\begin{bmatrix} f_m(s) \\ \chi_A(s) \end{bmatrix} \in \mathbb{R}^{(n+1)}[s]$ never loses rank for any $\lambda \in \mathbb{C}$.

(c) The transfer function $(sI-A)^{-1}B = (NV)\, S^{-1} U^{-1}$

but LHS is strictly proper while NV is polynomial, S^{-1} is $\begin{bmatrix} 1 & & \\ & 1 & \\ & & \chi_A^{-1} \end{bmatrix}$ and U^{-1} is (unimodular & polynomial).

(d) Using strictly proper / polynomial arguments as above, Conclude that U's last column is constant : say w.

(e) Now finally conclude that (A, Bw) is controllable: in fact, $[sI-A \;\; -Bw] \begin{bmatrix} f_m \\ \hline \chi_A \end{bmatrix} \begin{matrix} \}n \\ 1 \end{matrix} = 0$ and $\begin{bmatrix} f_m \\ \chi_A \end{bmatrix}$ is right prime.

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This proves that

(A, B) controllable & A cyclic $\Rightarrow \exists w$ s.t. (A, Bw) is controllable.

(The original "purely" state space proof is supposedly much more complicated: Kailath, page 504, (1980, Linear Systems.)