Tutorial Sheet for 27^{th} Oct 2015. Multivariable Control.
EE 640.

Q-1 Consider $\dot{x} = Ax + Bu$, and $x(0) = x_0 \in \mathbb{R}^n$ and $\min_u \int_0^\infty (x^TQx + u^TRu)\,dt$

Consider the ARE

$$A^TP + PA + Q - PBR^{-1}B^TP = 0$$

(A, Q) observable
(A, B) controllable
$Q > 0$, $R > 0$.

(a) Show that $H := \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$ is such that

$$H\begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} A_F$$ for some A_F iff P satisfies ARE. $F := -R^{-1}B^TP$ gives $A_F = A + BF$.

(b) Find A_F

(c) For $\dot{x} = x + u$, and $\int_0^\infty (x^2 + u^2)\,dt = J(x_0, u)$

and $u = fx$, show that the F defined in (a) is such that cost is minimum. (By $\frac{\partial J}{\partial f} = 0$ & $\frac{\partial^2 J}{\partial f^2} > 0$ for the "stabilizing" P.)

(d) Consider scaling

Q & R both by c (scalar) and $c > 0$.

Show that both F & A_F are unaffected (by the scaling).

(Interprete that optimal feedback law & the optimal closed loop dynamics A_F are unaffected by simultaneous scaling of Q & R).

(e) Consider c but with $\int_0^\infty (x^2 + \frac{u^2}{100})\,dt$ (and same system).

Check that optimal closed loop location is further to the left and $\int_0^\infty u^2\,dt$ is higher (for the optimal u^* of each case.).

Q-2:

Consider the controller Lyapunov eqn

$$AX + XA^T = BB^T \quad \text{for } A \text{ anti-Hurwitz}$$

(a) Use ARE & LQR arguments to interpret $x(0)^T X^{-1} x(0)$ as minimum energy required to take ~~x(0)~~ $x(0)$ to 0 (as $t \to \infty$).

Q-2 (Contd). Of course X need not be invertible in general. ~~Sh~~ First use $X \geq 0$ since A is anti Hurwitz & $BB^T \geq 0$. (also use $X = X^T$).

(b) Show $\ker X$ is A^T-invariant.

(c) Show $\ker X = 0 \iff (A, B)$ is controllable.

(d) ~~S~~ Similarly, Consider the observability Lyapunov equation

$$A^T Q + QA = -C^T C \quad \text{with } A \text{ Hurwitz.}$$

prove $Q := \int_0^\infty e^{A^T t} C^T C e^{At} \, dt$ solves the Lyapunov eqn.

(e) prove (A, C) observable $\iff Q > 0$

Q-3: It is said "Systems are "generically" controllable".
(with inputs)

i.e. Almost all systems (with inputs) are controllable. i.e. If (A, B) is uncontrollable, arbitrarily small perturbation to (A_ϵ, B_ϵ) ensures controllability. (In other words, uncontrollable systems form a "thin set" in $\mathbb{R}^{n^2+nm}$.)

(a) For each of the pairs (A, B) below, perturb only one or both of (A, B). (Only some small perturbation suffices.)

(i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ 1 & & & & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

For each case, try achieving controllability by perturbing only one.

(b) for any one pair above calculate $X(\epsilon)$ and check that $X(\epsilon)$ becomes unbounded along uncontrollable direction as $\epsilon \to 0$.

(c) Is it also true that "cyclicity" of a matrix is generic?