Tutorial Sheet, EE640 8th Oct 2023.

Q-1: Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $C^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(a) Find controllability index & controllability indices. κ_i

(b) Do κ_i add up to n?

(c) Is (A,B) controllable? (A, b_1)?, (A, b_2)? $(B := [b_1\ b_2])$.

(d) ~~A~~ Using answers amongst a, b, c: Is A cyclic?

(e) Find observability indices.

(f) Is arbitrary steering from $p \in \mathbb{R}^4$ to $q \in \mathbb{R}^4$ possible in 2 steps (i.e. $x(0) = p$, $x(2) = q$ for $x(k+1) = Ax(k) + Bu(k)$)

- using both inputs?
- using only u_1?
- using only u_2?

- in 3 steps?
- in 4 steps?
- in 5 steps?

(g) Same question as (f): but for deducing $x(0)$ from $y(0), y(1), y(2), \ldots y(N)$.

- How many minimum instants are needed for inferring $x(0)$ (arbitrary) using both outputs?
- using y_1?
- using y_2?

Q-2: Suppose $\Lambda_{uc}(A,B) := \{\lambda \in \mathbb{C} \mid \text{rank}[A - \lambda I \;\; B] < n\}$.

(a) Show $$\Lambda_{uc}(A,B) = \bigcap_{F \in \mathbb{R}^{m \times n}} \Lambda(A+BF)$$

~~A (A+I~~ ($\Lambda(P) :=$ set of eigenvalues of a square matrix P.)

(b) Let $G(s) = \dfrac{(s+1)}{(s+1)(s+2)}$ — Obtain a controllable realization: (A_1, B_1, C_1, D_1)

- Obtain an observable realization (A_2, B_2, C_2, D_2).

Q-2 b (contd.)

- Obtain a controllable & observable realization.
- obtain a $\underline{\text{minimal}}$ realization of $G(s)$.

$\underline{\text{Q-2}}$ c) Get a series repr. of $G(s) = g_0 + g_{-1}s^{-1} + g_{-2}s^{-2} + g_{-3}s^{-3} + \dots$

& construct

$$M_n := \begin{bmatrix} g_0 & g_{-1} & g_{-2} & \cdots & g_{-n} \\ g_{-1} & g_{-2} & & & \\ g_{-2} & & \ddots & & \\ \vdots & & & & \\ g_{-n} & & & & g_{-2n} \end{bmatrix} \in \mathbb{R}^{? \times ?}$$

Find rank of $M_0, M_1, M_2, M_3, \dots, M_5$.

~~Check if A~~ Argue why M_n's rank cannot keep increasing by

noting $M_n = \begin{bmatrix} \mathcal{O}_n \\ \vdots \\ \vdots \end{bmatrix} [\cdot\ \mathcal{C}_n\ \vdots]$ where $\mathcal{O}_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^n \end{bmatrix}$

$\underline{\text{Q-2}}$: Verify that $sG(s) = CB + \frac{CAB}{s} + \frac{CA^2B}{s^2} + \dots$

(Assume $G(s)$ is strictly proper) ($\underline{\text{Expand}}$ $(sI-A)^{-1}$ in series in s^{-i}).

$\underline{\text{Q-3}}$ (a) Show $(sI-A)^{-1}$ is strictly proper for any $A \in \mathbb{R}^{n\times n}$.

~~Q-4~~ (b) Show diag term of $(sI-A)^{-1}$ has relative degree $=1$ & off-diagonal terms have rel. degree ≥ 2.

(For $g(s) = \frac{n(s)}{d(s)}$,

relative degree of $g :=$ deg d − deg n).

Hint: Use Adjugate of $(sI-A) (:= $ ~~deg~~ $\det(sI-A) \cdot (sI-A)^{-1}$ can be calculated by co-factor method & get bounds on cofactor det. degree.

$\underline{\text{Q-4}}$: Prove the following equivalence. State what result is used.

(a) $\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}$ is $\underline{\text{not}}$ f.c.r. for some $\lambda \in \mathbb{C}$.

(b) $\mathcal{O}$ is not f.c.r. $\mathcal{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

Q-5: Consider Lyapunov Eqn $A^TQ + QA = -C^TC$.
we intend to infer A is Hurwitz by existence of soln $Q > 0$ of above Lyapunov Eqn.

(a) Show that (A, C) observable is necessary for the conclusion $\exists Q > 0 \Rightarrow A$ is Hurwitz.

(b) Show that (A, C) observable is sufficient for the conclusion $\exists Q > 0 \Rightarrow A$ is Hurwitz.

(c) Finally, give example of A to show that $\exists\ Q > 0$, A - Hurwitz such that $A^TQ + QA \not< 0$

Q-6: For (A, B, C, D) state space realization,
Consider $X_{\bar{o}c}$, $X_{\bar{o}\bar{c}}$, X_{oc}, $X_{o\bar{c}}$ as 4 decompositions of the state space

o - observable
$\bar{o}$ - unobservable
c - controllable
$\bar{c}$ - uncontrollable

by choosing basis appropriately,

Association (A1)
$$\begin{cases} x_1 \leftrightarrow X_{oc} \\ x_2 \leftrightarrow X_{\bar{o}\bar{c}} \\ x_3 \leftrightarrow X_{\bar{o}c} \\ x_4 \leftrightarrow X_{o\bar{c}} \end{cases}$$

obtain $\tilde{A}, \tilde{B}, \tilde{C}$ & check number of nonzero off-diagonal blocks.

(a) draw directed arrows from u to y & maximize the number of arrows
(Rules: unobservable part is A-invariant but not vice-versa)
controllable part is A-invariant but not vice-versa).

(b) Check that $\tilde{C}(sI - \tilde{A})\tilde{B}$ = only controllable & observable part.

Instead of A1-association, decide yourself some other association of x_1, x_2, x_3, x_4 & soln (a) & (b).

Q-7: To be sure that inverse of a block lower triangular matrix P is also block lower triangular (assuming P is invertible)
get inverse of $P = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix}$, with P_{11}, P_{22} square (& diff sizes).

Q-8: Let $A = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

Design an observer that places the observer poles at $-2, -2, -2$. (Bring (A, C) to a form like for control pole placement).

Q-9: Consider $A = \begin{bmatrix} 1 & 1 & \\ & 2 & \\ & & 3 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

and $C = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$. Obtain a minimal realization $(\tilde{A}, \tilde{b}, \tilde{c})$.

Check that (after pole zero cancellation), the transfer functions are the same.