Tutorial 1

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Q1.) Prove the following statements are (η) equivalent for an n-dimensional row vector and $T > 0$.

(i) $\eta \perp W_T$ (i.e. $\eta \cdot x = 0 \quad \forall \ x \in W_T$)

(ii) $\eta \cdot e^{At} \cdot B = 0$ for $t \in [0, T]$

(iii) $\eta \cdot A^k \cdot B = 0$ for $k = 1, 2, \dots$

(iv) $\eta \cdot [B \ \ AB \ \ A^2B \ \dots \ A^{n-1}B] = 0$

Q2.) Given a system (A, B) and its transformed system $(P^{-1}AP, P^{-1}B)$, prove that rank of the controllability matrix remains unchanged.

3. Suppose $A_1, A_2, \ldots$ are linearly independent set of vectors, and that $A_1 \subset A_2 \subset \ldots$. Show that the union $A = A_1 \cup A_2 \cup \ldots$ is also linearly independent.

4. Find the rank of the following matrices:-

(i) $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

(iii) $C = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$

5. Suppose B is similar to A, say $B = P^{-1}AP$. Show that $B^n = P^{-1}A^nP$ and so, B^n is similar to A^n.

6. Let $\mathcal{A}$ be an algebra over a field K and P be an invertible element in $\mathcal{A}$. Then the mapping $T_P : \mathcal{A} \to \mathcal{A}$ defined by $T_P(A) = P^{-1}AP$ is an algebra isomorphism. i.e. $\forall\ A, B \in \mathcal{A}$ and ank $k \in K$. Prove the following equations,

(i) $T_P(A+B) = T_P(A) + T_P(B)$ (ii) $T_P(kA) = k\,T_P(A)$

(iii) $T_P(AB) = T_P(A)\,T_P(B)$ (iv) T_P is one-to-one and onto.

7) Find the Eigen Value and Eigen Vectors of given matrix and if you can't find it give the reason?

(a) $\begin{bmatrix} 2 & 5 & 4 \\ 0 & 7 & 1 \\ 0 & 0 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 6 & 3 \\ 2 & 0 & 5 \\ 9 & 8 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 9 & 2 & 4 \\ 5 & 6 & 7 & 8 \\ 7 & 4 & 5 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 4 & 9 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8) Find if System $\dot{x} = Ax + Bu$ is controllable or not. if yes then prove it, if not then give me the reason.

(a) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(e) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

9) If we have a System $\dot{x} = Ax + Bu$ and let (A, B) and $(\bar{A}, \bar{B})$ be isomorphic then prove that $(\bar{A}, \bar{B})$ is controllable if & only if (A, B) is controll.

10) Show that if (A, B) is uncontrollable, then

$$\bigcap_{F \in \mathbb{R}^{m \times n}} \sigma(A+BF) \text{ is not empty.}$$

where, $\sigma(\ ) \equiv$ Set of Eigenvalues,

$A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$

11) Consider $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ and consider following statement;

ⓐ $[B \quad AB \; ---- \; A^{n-1}B]$ is full row rank.

ⓑ $\eta e^{At} B = 0 \quad \forall t \in [0, T]$ where, $T > 0$, $\eta \in \mathbb{R}^{1 \times n} \implies \eta = 0$

ⓒ $[A - \lambda I \quad B]$ is full row rank $\forall \lambda \in \mathbb{C}$.

Choose any two and show they are both Equivalant?

12) If a matrix has rank r, what can you say about the solutions to the equation $Ax = 0$?

OR

Q:) How is the rank related to the invertibility of a matrix?