Tutorial Num 3, EE640 Multivariable Control, 26th Aug 2024

Q-1: Define $P_T := \int_0^T e^{A\tau} BB^T e^{A^T\tau} d\tau$. Show that for $T_1 \geq T_2 (\geq 0)$

$P_{T_1} - P_{T_2}$ is +ve semi-definite.

Q-2: Show that (assuming A is Hurwitz,) P_T, $T=\infty$ satisfies

$$A P_\infty + P_\infty A^T = -BB^T$$

(obtain $\int \frac{d}{d\tau}\left[e^{A\tau} BB^T e^{A^T\tau}\right] d\tau$ in two ways).

Q-3: For $T>0$, show $P_T > 0 \Leftrightarrow (A, B)$ is controllable.

Q-4: Use a counterexample to show that

A Hurwitz $\not\Rightarrow$ for every $P>0$, ~~there is~~ $A^TP + PA < 0$.

(find an A, P to show $A^TP+PA \not< 0$

Q-5: For $A = \begin{bmatrix} -1 & 20 \\ 0 & -1 \end{bmatrix}$, & $Q = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, solve for P such that $A^TP + PA = -Q$.

Q-6: For $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, solve for P satisfying

$A^TP + PA = -BB^T$. Is $P>0$ or <0 or none of these?

Q-7: (A) For $\dot{x} = 5x + 2u$, solve for minimum control-energy input u^* that results in $x(\infty) = 0$ and $x(0) = 3$.

(use controllability gramian P and check that minimum energy (control): $\int_0^\infty u(\tau)^2 d\tau = \frac{-x(0)^2}{P}$

(P satisfies $AP + PA^T = -BB^T$).

(B) use $u = fx$ to get closed loop as $\dot{x} = (A+BF)x$ and thus $u(t) = e^{(A+BF)t} x(0)$ and get $\int_0^\infty u(\tau)^2 d\tau = C(f)$ (assuming $A+BF$ is Hurwitz)

$C(f)$ denotes cost as a function of feedback f in $u = f \cdot x$

$\frac{d}{df} C(f) = 0$ gives optimal f^*: check if answer is same as Q-7A.

Q-7 (contd).

In general, ~~$u^* = F^* B$~~ $u^* = F^* x$ and optimal feedback
$F^* = \cancel{-B^T P^{-1}}\ \cancel{P^{-1} B^T}\ -B^T P^{-1}$

Q-8 For $\dot{x} = -5x + 4$, what is the minimum amount of $\int_0^\infty u^2(\tau)\,d\tau$ to have $x(0) = 3$ and $x(\infty) = 0$?
(Argue that this minimum value $= 0$).

Q-9: Why is Im $[B \ AB \cdots A^{n-1}B]$ an A-invariant subspace?

Q-10: Why is ker $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \rightarrow \mathcal{O} \in \mathbb{R}^{pn \times n}$ an A-invariant subspace?

Q-11: Check that if $v \neq 0$ is in ker $\mathcal{O}$, $v \in \mathbb{R}^n$
then $y(t)$ corresponding to initial condition $x(0) = v$ satisfies $y(t) \equiv 0$. Thus, have you proved ker $\mathcal{O} \neq 0 \Rightarrow$ unobservable?

Q-12: Suppose $u^*_{T_1}(\tau)$ is minimum control energy input to reach state $x(T_1) = q \in \mathbb{R}^n$ from $x(0) = 0$.
Show/argue $T_1 \leq T_2 \Rightarrow \int_0^{T_1} (u_1^*(\tau))^2 d\tau \geq \int_0^{T_2} (u_2^*(\tau))^2 d\tau$.

Q-13: Suppose $\eta \in \mathbb{R}^n$ satisfies $\eta^T \mathcal{C} = 0$ ($\mathcal{C}$ is controllability matrix).
then show that $P\eta = 0$, when P is solution to controllability Lyapunov Eqn.

Q-14: Suppose $\eta \in \mathbb{R}^n$ satisfies $\mathcal{O}\eta = 0$ ($\mathcal{O} \equiv$ observability matrix).
then show $Q\eta = 0$ (Q is solution to observability Lyapunov Eqn $A^T Q + QA = -C^T C$.