Behaviors: more kinds of representations, controllability and elimination theorem

Notes for lecture 10 (May 12th, 2014)

In this lecture we will see some more kinds of representations: latent variable representations, image representations, state space systems and we will see the definition of controllability of a behavior.

Recall that

- \( m \): number of inputs,
- \( p \): number of outputs
- \( w \): number of ‘manifest’ variables: typically \( m + p \)
- \( n \): (minimum) number of states (McMillan degree)

1 More on polynomial matrices

In the previous lecture, we studied the Smith canonical form

\[
U(\xi)R(\xi)V(\xi) = \begin{bmatrix} D(\xi) & 0 \\ 0 & 0 \end{bmatrix},
\]

with \( D(\xi) \in \mathbb{R}^{r \times r}[\xi] \) being diagonal with all nonzero and monic polynomials: \( r \) is the (normal) rank of the polynomial matrix of \( R(\xi) \). We motivated \( U \) as elementary row operations that do not change the set of solutions to \( R(\frac{d}{d\xi})w = 0 \). We will not delve on significance of \( V \): we just note here that \( V \) is a ‘coordinate transformation’ that involves not just linear combinations of various components of variable \( w \), but also derivatives of these components.

Unimodularity of \( V \) ensures this transformation is one-to-one and onto, and hence is a coordinate transformation.

Please verify following facts (no need to submit these).

**Fact 1.1** Most of the following can be solved by partitioning \( U \) and \( V \) of equation (1) conforming to that of the RHS. Also consider partitioning \( U^{-1} \) and \( V^{-1} \).

- For any polynomial matrix \( R \), there exists a unimodular \( U \) such that \( U(\xi)R(\xi) = \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \), with \( R_1 \) having full row rank.

- Suppose \( R \) is full row rank. Then \( R \) can be factored into \( R(\xi) = F(\xi)R_c(\xi) \) with \( F \) square and nonsingular and \( R_c(\xi) \) being left-prime\(^1\). Then,

\(^1\)A full row rank polynomial matrix \( R_c(\xi) \) is called left-prime if its Smith form equals \([I \ 0]\), for an identity matrix \( I \) of suitable size.
– $det \, F$ is equal to a constant multiple of the diagonal matrix $D$ that arises in the Smith form of $R$.
– $F$ and $R_c$ are, in general, not unique.
– Any polynomial vector $p \in \mathbb{R}^w[\xi]$ such that $R(\xi)p(\xi) = 0$ satisfies $R_c(\xi)p(\xi) = 0$

• If $R_c(\xi)$ is left-prime, then $R_c(\xi)$ is full row rank.

• Suppose $R_c$ is full row rank. Then, the following are equivalent.
  – $R_c(\xi)$ is left-prime,
  – $R_c(\lambda)$ is full row rank for every complex number $\lambda \in \mathbb{C}$,
  – Whenever\(^2\) $R_c(\xi)$ can be factored into $R_c(\xi) = F(\xi)R_2(\xi)$ with $F$ nonsingular, then $F$ is unimodular,
  – There exists a polynomial right inverse $Q \in \mathbb{R}^{w\times w}[\xi]$, i.e. $R_c(\xi)Q(\xi) = I$.

In view of the first fact above, we might as well assume any kernel representation we begin with $R(\frac{d}{dt})w = 0$ has $R(\xi)$ of full row rank. We call such a representation minimal kernel representation: this is without loss of generality.

The following exercise relates Jordan canonical form of $A$, its algebraic/geometric eigenvalues to the Smith canonical form of $\xi I - A$. Of course, determinant of $D$ in the Smith form is the characteristic polynomial of $A$. Further, the sizes are same and the degrees of the polynomials in $D$ have to add up to size of $A$.

**Exercise 1.2** Suppose $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Let $n_a(\lambda_i)$ and $n_g(\lambda_i)$ denote the algebraic and geometric multiplicity of an eigenvalue $\lambda_i$.

• Suppose every eigenvalue has geometric multiplicity one, then show that $d_n$ in the Smith form of $\xi I - A$ equals the characteristic polynomial of $A$.

• Consider the Smith canonical form of $\xi I - A$. Show that the number of ‘ones’ along the diagonal in the Smith canonical form equals $n - \max_{\lambda_i} n_g(\lambda_i)$.

• Show that the number of polynomials $d_i$ that have $(\xi - \lambda_i)$ as a factor is the geometric multiplicity of $\lambda_i$.

• Find\(^3\) the Jordan canonical form of $A$, where $A$ is such that $\xi I - A$ has the two polynomials $1$ and $(\xi - 2)^2$ along the diagonal.

\(^2\)This statement motivates the use of ‘left-prime’.

\(^3\)More generally, the Smith form of $\xi I - A$ contains all the information about the Jordan canonical form of $A$, and conversely, given the Jordan canonical form of $A$, the Smith form of $\xi I - A$ can be found.
• Use PBH test to show that \((A, B)\) is controllable if and only if \([\xi I - A \ B]\) is left-prime.

• Show that the roots of the polynomials in the Smith form of \([\xi I - A \ B]\) are the uncontrollable eigenvalues of \(A\).

2 Controllability

The set of LTI behaviors described by differential equations in \(w\) number of variables is denoted by \(\mathcal{L}^w\). Equivalently, \(\mathcal{B} \in \mathcal{L}^w\) if \(\mathcal{B}\) is the set of solutions to \(R\left(\frac{d}{dt}\right)w = 0\), for a polynomial matrix \(R \in \mathbb{R}^{\times\times[\xi]}\).

A system \(\mathcal{B} \in \mathcal{L}^w\) is called controllable if for any \(w_1, w_2 \in \mathcal{B}\), there exist \(w_3 \in \mathcal{B}\) and \(T \geq 0\) such that

\[
w_3(t) = \begin{cases} w_1(t) & \text{for } t \leq 0, \\ w_2(t) & \text{for } t \geq T. \end{cases}
\]

Theorem 2.1 Let \(\mathcal{B} \in \mathcal{L}^w\) and suppose \(R\left(\frac{d}{dt}\right)w = 0\) is a minimal kernel representation. Then, the following are equivalent.

1. \(\mathcal{B}\) is controllable,
2. \(R(\xi)\) is left-prime, i.e. \(R(\lambda)\) is full row rank for every complex number \(\lambda \in \mathbb{C}\).
3. \(\mathcal{B}\) has an image representation: there exists \(M(\xi) \in \mathbb{R}^{\times\times[\xi]}\) such that

\[
\mathcal{B} = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^w) \mid \text{there exists } \ell \text{ such that } w = M\left(\frac{d}{dt}\right)\ell\}.
\]

3 Elimination of ‘latent variables’

Consider again:

\[
\mathcal{B} = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^w) \mid \text{there exists } \ell \text{ such that } w = M\left(\frac{d}{dt}\right)\ell\}.
\]

Often, additional variables (like \(\ell\)) than the ones of interest (here: \(w\)). Call all auxiliary variables: latent variables.

Sometimes latent variables inevitable when modeling systems from first principles. In general,

\[
\mathcal{B} = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^w) \mid \text{there exists } \ell \text{ such that } R(\frac{d}{dt})w + M(\frac{d}{dt})\ell = 0\}.
\]
Project \((w, \ell)\) behavior to just \(w\)-variables.

Does \(\mathcal{B}\) have a \textit{kernel representation}?

Always possible for \(C^\infty(R, R^w)\), but not for all function-spaces.

**Theorem 3.1** Consider \(\mathcal{B}_{\text{full}}\) described by \(R\left(\frac{d}{dt}\right)w + M\left(\frac{d}{dt}\right)\ell = 0\) (for polynomial matrices \(R\) and \(M\)).

Then, there exists a kernel representation \((R_2\left(\frac{d}{dt}\right)w = 0)\) for \(\mathcal{B}\) defined by

\[
\mathcal{B} = \{w \in C^\infty(R, R^w) \mid \text{there exists } \ell \text{ such that} \quad R\left(\frac{d}{dt}\right)w + M\left(\frac{d}{dt}\right)\ell = 0\}.
\]

Obtain \(R_2\) as follows. Find a unimodular \(U\) such that \(U(\xi)M(\xi) = \begin{bmatrix} M_1 \\ 0 \end{bmatrix}\). Partition \(U\) conformably into \(\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}\). Define \(R_2(\xi) := U_2(\xi)R(\xi)\). Then, \(\mathcal{B} = \{w \in C^\infty(R, R^w) \mid R_2\left(\frac{d}{dt}\right)w = 0\}\). In general \(\text{ker } R\left(\frac{d}{dt}\right) \supseteq \mathcal{B}\): equality for \(C^\infty\).

Note that the equality is not true for \(\mathcal{D}\): the set of compactly supported \(C^\infty\) functions.

Note: \(U_2\) used above is a so-called Maximal Left Annihilator of \(M\). For a polynomial matrix \(M(\xi) \in R^{w \times m}[\xi]\), define a Maximal Left Annihilator (MLA) \(P(\xi) \in R^{\ast \times w}[\xi]\) if the following is satisfied:

- \(P(\xi)M(\xi) = 0\)
- \(P(\xi)\) is full row rank
- If any polynomial matrix \(\hat{P}(\xi)\) satisfies \(\hat{P}(\xi)M(\xi) = 0\), then there exists a polynomial matrix \(F(\xi)\) such that \(\hat{P}(\xi) = F(\xi)P(\xi)\).

Following facts can be verified easily.

**Fact 3.2** Let \(M \in R^{w \times m}[\xi]\) be full column rank.

- In general, an MLA \(P\) of \(M\) is not unique. (Non-uniqueness can be characterized using unimodular matrices.)
- Any MLA \(P\) is left-prime and \(P\) has \(w - m\) rows.

\(^4\text{This third property motivates the use of the word ‘maximal’}\.)
Conversely, if $P \in \mathbb{R}^{(u-n) \times u}[\xi]$ is left-prime and satisfies $P(\xi)M(\xi) = 0$, then $P$ is an MLA of $M(\xi)$.

- If $M(\xi)$ is left-invertible, then any MLA $P$ can be used to obtain all left-inverses of $M$.

- For any nonsingular polynomial matrix $F(\xi) \in \mathbb{R}^{n \times n}[\xi]$, both $M(\xi)$ and $M(\xi)F(\xi)$ have the same set of MLAs.

4 Dissipative systems

Consider $\Sigma = \Sigma^T \in \mathbb{R}^{v \times v}$. A system $\mathcal{B} \in \mathcal{L}_{\text{cont}}^\varepsilon$ is called dissipative if
\[
\int_{-\infty}^{\infty} w^T \Sigma w \, dt \geq 0 \quad \text{for all} \quad w \in \mathcal{B} \cap \mathcal{D}.
\]

For this course, we restrict ourselves to just controllable behaviors. Work on uncontrollable dissipative systems can be found in the literature, and outside the scope of this course.

Variety of Algebraic Riccati Equations: just different supply rates.

For example:

- LQ control: $A^T P + PA - Q + PB^T R^{-1}BP = 0$
- $\mathcal{H}_\infty$ norm (strictly proper): $A^T PPA + C^T C + PB^T BP = 0$
- Passivity: $A^T P + PA + (KB - C^T)(D + D^T)^{-1}(B^T K - C) = 0$

Dissipativity, storage functions will unify these. Of course, well-known that there is a link with Linear Matrix Inequality (LMI). More precisely,
\[
L(P) := \begin{bmatrix} A^T + PA - Q & PB^T \\ PB^T & R \end{bmatrix} \preceq 0
\]

Then, Schur complement with respect to $R$ is exactly the ARE for the LQ control problem.

References
