Hamiltonian matrix, orthogonal complements
Notes for lecture 12 (May 19th, 2014)

This lecture contains link between Hamiltonian matrix, ARE and (so-called) stationary trajectories. We will also define the ‘orthogonal complement’ of a behavior.

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1 Lossless systems, orthogonal complement

In the case of dissipativity, the storage function is not unique, in general. (Note that non-uniqueness of the storage function is \textit{not} due to our choice of expressing the storage function in terms of different variables.) There is a special case\(^1\) of dissipativity when the storage function is unique: lossless.

Consider \( \Sigma = \Sigma^T \in \mathbb{R}^{w \times w} \). A system \( \mathcal{B} \in \mathcal{L}_{\text{cont}}^w \) is called \( \Sigma \)-lossless if

\[
\int_{-\infty}^{\infty} w^T \Sigma w \, dt = 0 \quad \text{for all} \quad w \in \mathcal{B} \cap \mathcal{D}.
\]

\textbf{Theorem 1.1} Let \( \mathcal{B} \in \mathcal{L}_{\text{cont}}^w \) and suppose \( \Sigma = \Sigma^T \in \mathbb{R}^{w \times w} \). Suppose \( w = M(\frac{d}{dt})\ell \) is an image representation for \( \mathcal{B} \). Then, the following are equivalent.

\begin{itemize}
  \item \( \mathcal{B} \) is \( \Sigma \)-lossless.
  \item \( M(-\xi)^T \Sigma M(\xi) \) satisfies \( M(-\xi)^T \Sigma M(\xi) = 0 \).
  \item There exists a storage function \( Q_\Psi(w) \) such that
    \[
    \frac{d}{dt} Q_\Psi(w) = w^T \Sigma w \quad \text{for all} \quad w \in \mathcal{B}.
    \] \( (1) \)
\end{itemize}

\(^1\)Note that there are dissipative and non-lossless systems which can have a unique storage function: uniqueness of storage function only ensures non-strictness of the dissipativity, it does not ensure losslessness.
\[ \int_{t_1}^{t_2} w^T \Sigma w \, dt \] is ‘path-independent’: i.e. the value of the integral depends only on values of \( w \) (and its derivatives) at \( t_1 \) and \( t_2 \), and does not depend on which trajectory in \( \mathcal{B} \) \( w \) assumes between \( t_1 \) and \( t_2 \).

Closely related to lossless is the notion of an orthogonal complement of a controllable behavior. Given a controllable behavior \( \mathcal{B} \in \mathcal{L}_{\text{cont}}^w \) and a symmetric, nonsingular matrix \( \Sigma \in \mathbb{R}^{w \times w} \), the \( \Sigma \)-orthogonal complement of \( \mathcal{B} \), (denoted by \( \mathcal{B}^\perp_\Sigma \)), is the set of all the trajectories \( v \in C^\infty(\mathbb{R}, \mathbb{R}^w) \) such that \( \int_{-\infty}^{\infty} v^T \Sigma w \, dt = 0 \) for all \( w \in \mathcal{B} \cap \mathcal{D} \).

When \( \Sigma = I \), the \( \Sigma \)-orthogonal complement \( \mathcal{B}^\perp_\Sigma \) is written as just \( \mathcal{B}^\perp \).

## 2 Euler Lagrange equation

We briefly link the differential equations \( \partial \Phi'(\frac{d}{dt}) \ell = 0 \) with the EL equation (for the simplified case). Consider minimizing or maximizing a performance functional \( \int V(\ell, \dot{\ell}) \, dt \), with \( y \) unconstrained, and \( C^\infty \). Then the optimum trajectories \( y^* \) satisfy

\[ \frac{\partial V}{\partial \ell} - \frac{d}{dt} \frac{\partial V}{\partial \dot{\ell}} = 0. \]

Of course, we are dealing with a situation simplified in many ways.

## 3 Hamiltonian matrix

A matrix \( H \in \mathbb{R}^{n \times n} \) is called a Hamiltonian matrix if \( H \) is similar to \( -H^T \). Hamiltonian matrices arise in different contexts; in our context, they are closely related to kernel of \( \partial \Phi'(\frac{d}{dt}) \). For example, roots of \( \partial \Phi'(\xi) \) and eigenvalues of \( H \) are the same (counted with multiplicity).

Recall that for lossless systems, \( \partial \Phi'(\xi) \) was identically zero. (Vaguely) motivated by this, think of kernel of \( \partial \Phi'(\frac{d}{dt}) \) as ‘lossless trajectories’.

For finite dimensional vector spaces, if \( \mathcal{V} \subseteq \mathbb{R}^n \) is \( \Sigma \)-non-negative, then \( \mathcal{V} \cap \mathcal{V}^\perp_\Sigma \) is a subspace of \( \mathbb{R}^n \) which is \( \Sigma \)-neutral. (A subspace \( \mathcal{V} \) is called \( \Sigma \)-neutral if \( v^T \Sigma v = 0 \) for all \( v \in \mathcal{V} \); neutral is same as lossless. See [GLR05] for an elaborate treatment on indefinite linear algebra.)

We will see that \( \mathcal{B} \cap \mathcal{B}^\perp_\Sigma \), when autonomous, has a state representation that is a Hamiltonian matrix (as the state transition matrix). The link between ARE and ‘the corresponding’ Hamiltonian matrix is due to the following fact (that is best verified oneself).

**Fact 3.1** ([TSH02, Section 13.4]) Let \( F, S, T \in \mathbb{R}^{n \times n} \) with \( S \) and \( T \) are symmetric. Due to

\[
\begin{bmatrix}
X & -I
\end{bmatrix}
\begin{bmatrix}
F & T \\
-S & -F^T
\end{bmatrix}
\begin{bmatrix}
I \\
X
\end{bmatrix}
= F^T X + XF + XTX + S \quad \text{and} \quad
\begin{bmatrix}
X & -I
\end{bmatrix}
\begin{bmatrix}
I \\
X
\end{bmatrix}
= 0,
\]

we note that solutions to the ARE \( F^T X + XF + XTX + S = 0 \) are linked to \( n \)-dimensional

\[We used that \( X \) is symmetric. Also, an \( n \)-dimensional invariant subspace (say image of \( \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \)) of \( H \) requires to have its top \( n \times n \) block \( X_1 \) invertible, for this subspace to yield an ARE solution. In most ARE studies, this invertibility is a key step.
invariant subspaces of the $2n \times 2n$ matrix (say $H$) in the above equation. $H$ is defined as the Hamiltonian matrix corresponding to this ARE.

Further, verify that

$$H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} (F + TX).$$

Thus, if $H$ has no eigenvalues on the imaginary axis, choosing ‘the’ $n$-dimensional invariant subspace corresponding to all OLHP eigenvalues gives an $X$ (assuming invertibility of $X_1$ as mentioned in Footnote 2) that is stabilizing: this is due to $F + TX$ being Hurwitz.

4 Minimal dissipation trajectories

One of the passivity preserving model order reduction methods proposed in [Ant05, Sor05] turns out to ‘retain’ (a lower dimensional subspace of) the set of trajectories of minimal dissipation ([MTR09]).

Consider a nonsingular $\Sigma = \Sigma^T \in \mathbb{R}^{w \times w}$ and suppose $B \in \mathcal{L}_{\text{cont}}^w$ is $\Sigma$-dissipative. As proposed in [MTR09], for a $w \in B$, consider the change $J_w(\delta)$ in dissipation\(^3\) (about $w$) if $w$ is changed to $w + \delta$, for $\delta \in B \cap \mathcal{D}$:

$$J_w(\delta) := \int_{-\infty}^{\infty} (Q_\Delta(w + \delta) - Q_\Delta(w)) \, dt.$$

A trajectory $w \in B$ is said to be a trajectory of minimal dissipation if $J_w(\delta) \geq 0$ for all $\delta \in B \cap \mathcal{D}$. Any small change in $w$ causes increase of net dissipated energy: in that sense, these are local minima (see [MTR09, page 177].)

The link between the set of trajectories (in a $\Sigma$-dissipative behavior $B$) of minimal dissipation (denoted by $B^*$) and $B^\perp_\Sigma$ is [MTR09, Theorem 3.4], which states $B^* = B \cap B^\perp_\Sigma$. Notice that $B \cap B^\perp_\Sigma$ is just the set of those trajectories $w = M(\frac{d}{dt})\ell$, where $\ell$, is no longer free/generic, but in fact, satisfies $\partial \Phi'(\frac{d}{dt})\ell = 0$.

5 Strict dissipativity

Quite unfortunately, the lossless case is not handled by the ARE/ARI. The ARE and the ARI are best suited for ‘strict dissipativity’, which is a kind of opposite to losslessness. Consider $\Sigma = \Sigma^T \in \mathbb{R}^{w \times w}$. A system $B \in \mathcal{L}_{\text{cont}}^w$ is called strictly $\Sigma$-dissipative if there exists an $\epsilon > 0$ such that

$$\int_{-\infty}^{\infty} w^T \Sigma w \, dt \geq \epsilon \int_{-\infty}^{\infty} w^T w \, dt$$

for all $w \in B \cap \mathcal{D}$.

---

\(^3\)A dissipation function $Q_\Delta(w)$ (a function of time, that depends on the trajectory $w$) is defined as the amount of supplied power that didn’t go into storing energy, i.e. $Q_\Delta(w) := w^T \Sigma w - \frac{d}{dt} Q_\Psi(w)$. Since storage functions are not unique, we speak of a dissipation function $Q_\Delta$ corresponding to a storage function $Q_\Psi$. Inspite of this dependence on $Q_\Psi$, along compactly supported trajectories, the ‘net power’ dissipated depends only on $w$: for more details, see [WT98].
General concerns: lossless part in $\mathcal{B}$: moreover, lossless part is non-autonomous. In the LMI (corresponding to dissipativity), one needs to take the Schur complement with respect to the ‘lower right’ block: which ought to be sign-definite: only then the LMI yields the ARI/ARE. This in turn allows defining the Hamiltonian matrix.

Strict dissipativity lays to rest any concerns about singularity of the above lower-right block. (Of course, strict dissipativity at just ‘the $\infty$ frequency’ is good enough: [KBAR14].)

**Theorem 5.1** Assume $\mathcal{B} \in \mathcal{L}_{\text{cont}}^w$ and $\Sigma \in \mathbb{R}^{w\times w}$ is symmetric and nonsingular. Suppose $\mathcal{B}$ is strictly $\Sigma$-dissipative. Then,

1. $\mathcal{B} \cap \mathcal{B}^{\perp \Sigma}$ is autonomous. $n(\mathcal{B} \cap \mathcal{B}^{\perp \Sigma}) = 2n(\mathcal{B})$.

2. The ARE exists.

3. The Hamiltonian matrix exists.

Dissipation at the $\infty$ frequency is denoted by the matrix $P$ in the matrices in the last section. In order to obtain the Hamiltonian matrix as indicated there, some more development relating $\mathcal{B}$ and $\mathcal{B}^{\perp \Sigma}$ is required. Key property is that $\mathcal{B} \cap \mathcal{B}^{\perp \Sigma}$ has a state transition matrix exactly the Hamiltonian matrix $H$: this will be elaborated in this pdf-file by 20th May, 2pm.

The two statements within 1 above can be viewed as regular interconnection and regular feedback interconnection (see [Wil97, JPKB13]) between the ‘plant’ $\mathcal{B}$ and the ‘controller’ $\mathcal{B}^{\perp \Sigma}$ (defined next). Study of these interconnections seems inessential to pursue model-order reduction, and hence we do not pursue further here.

(In Lecture 11, we began with an exercise, which we continue with now.)

**Exercise 5.2** Consider the system $\frac{d}{dt}x = 3x + 2u$ and the performance cost $\int_0^\infty (4x^2 + u^2) \, dt$. Let the initial condition be $x(0) = 4$.

- Find all stationary trajectories $\mathcal{B}^*$ in $\mathcal{B}$ using the Euler-Lagrange equation.

- Find $\mathcal{B}^*$ as the set of trajectories of ‘minimal dissipation’ (as $\mathcal{B} \cap \mathcal{B}^{\perp \Sigma}$).

- Check if a first order representation of $\mathcal{B}^*$ results in a Hamiltonian matrix $H$.

- Compare $H$ with the one linked to the corresponding ARE, and use $H$ to obtain the stabilizing ARE solution ($K_{\text{min}}$, in our case).

**Exercise 5.3** Consider the following circuit.

Let the capacitance $C$ be 1 F and the resistances $R_2$ and $R_C$ be equal to 3 $\Omega$ and 1 $\Omega$ respectively. Find the minimum energy required at the port to charge the capacitor to 4 V (from initially discharged state). Also find the maximum energy one can extract out from the port if the capacitor is initially charged to 4 V. Why is it reasonable that the actual energy stored is ‘exactly in between’ the maximum and the minimum storage functions?
6 Autonomous systems

This section contains briefly about autonomous systems to the extent we need for model order reduction. A behavior $\mathcal{B} \in \mathcal{L}^w$ is called autonomous if

whenever $w_1, w_2 \in \mathcal{B}$ satisfy $w_1(t) = w_2(t) \Rightarrow w_1 = w_2$.

**Theorem 6.1** Let $\mathcal{B} \in \mathcal{L}^w$ have minimal kernel representation $R\left(\frac{d}{dt}\right)w = 0$. Then, the following are equivalent.

1. $\mathcal{B}$ is autonomous.
2. $R(\xi)$ is square and nonsingular.
3. $\mathcal{B}$ is finite dimensional as a vector space over $\mathbb{R}$.
4. $\mathcal{B}$ is a finite linear combination of only\(^4\) exponentials, i.e., assuming for simplicity\(^5\) det $(R)$ has only real distinct roots $\lambda_1, \ldots, \lambda_N$, with $N = \text{deg det } R$:

$$w \in \mathcal{B} \iff w = \sum_{i=1}^{N} a_i v_i e^{\lambda_i t} \text{ for } a_i \in \mathbb{R} \text{ and } v_i \in \mathbb{R}^w \setminus 0.$$ 

Excepting the (trivial) case when $\mathcal{B} = \{0\}$, which is both controllable and autonomous, in general, autonomous means uncontrollable. In fact, autonomous means, not just uncontrollable, but in fact, controllable-part equal to zero. For this course, we stick to just controllable behaviors $\mathcal{B}$ for the purpose of model-order reduction, but $\mathcal{B} \cap \mathcal{B}^\perp \Sigma$ will be autonomous (under strict dissipativity assumptions, etc.: as mentioned in Theorem 5.1).

7 LMI, ARE and Hamiltonian matrix $H$

For three key supply rates $x^T Q x + u^T R u$ (the LQ problem), $u^T y$ (passivity) and $\gamma^2 u^T u - y^T y$ ($\mathcal{L}_\infty$ norm at most $\gamma$), we list the LMI, ARE and Hamiltonian matrix. In each case, assume $\frac{d}{dt} x = Ax + Bu$ and $y = Cx + Du$ is a minimal state space realization (i.e. controllable and observable realization). The LMI can be obtained by $x^T K x$ as a storage function. The ARE is obtained by taking Schur complement w.r.t. the lower right block (say, $P$, the one corresponding to $u^T P u$) and the Hamiltonian matrix is constructed from the ARE as elaborated in [TSH02, Section 13.4].

\(^4\)In ‘exponential functions’, we allow sinusoids and cosinusoids (due to complex exponents) and also polynomial combination of exponentials (when repeated roots).

\(^5\)Real roots ensures no sinusoids/cosinusoids, and distinct ensures no polynomials required: see [PW98] for the general autonomous case.
Supply rate LMI  \( P \) (the dissipation at \( \infty \) frequency)  \( H \)
\[
\begin{bmatrix}
Q & 0 \\
0 & R \\
\end{bmatrix}
\begin{bmatrix}
A^T K + K A^T - Q K B \\
B^T K - R \\
\end{bmatrix}
\begin{bmatrix}
R \\
B^T R^{-1} B - A^T \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & I \\
I & 0 \\
\end{bmatrix}
\begin{bmatrix}
A^T K + K A^T & KB - C^T \\
B^T K - C & -(D + D^T) \\
\end{bmatrix}
(D + D^T)
\begin{bmatrix}
A - BP^{-1} C & BP^{-1} B^T \\
-C^T P^{-1} C & -(A - BP^{-1} C)^T \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
\gamma^2 I & 0 \\
0 & -I \\
\end{bmatrix}
\begin{bmatrix}
A^T K + K A^T + C^T C & KB + C^T D \\
D^T C + B^T K & D^T D - \gamma^2 I \\
\end{bmatrix}
(\gamma^2 I - D^T D)
\begin{bmatrix}
A + BP^{-1} D^T C & BP^{-1} B^T \\
-C^T P^{-1} C & -(A + BP^{-1} D^T C)^T \\
\end{bmatrix}
\]

References


