Passivity/dissipativity-preserving model reduction Algorithm (Lecture 14)

Madhu N. Belur

Control & Computing group, Electrical Engineering Dept, IIT Bombay

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Outline

- Remaining behavioral-notation and
- (Half-line) dissipativity definition and results
- Dissipativity-preserving model-order reduction
Recall a behavior $\mathcal{B} \in \mathcal{L}^w_{\text{cont}}$ was called $\Sigma$-dissipative if
\[ \int_{\mathbb{R}} w^T \Sigma w dt \geq 0 \text{ for all } w \in \mathcal{B} \cap \mathcal{D}. \]

Call $\mathcal{B}$ dissipative on $\mathbb{R}^-$ if for all $w \in \mathcal{B} \cap \mathcal{D}$ and for all $T$
\[ \int_{-\infty}^{T} w^T \Sigma w dt \geq 0. \]
('bounded from below')
(like physical storage)

and on $\mathbb{R}^+$ if $\int_{T}^{\infty} w^T \Sigma w dt \geq 0$.

- dissipative $\iff \exists$ storage function $Q_{\Psi}(w)$
- dissipative on $\mathbb{R}^-$ $\iff \exists$ storage function $Q_{\Psi}(w) \geq 0$
- dissipative on $\mathbb{R}^+$ $\iff \exists$ storage function $Q_{\Psi}(w) \leq 0$
Half-line dissipativity: $\Sigma = \Sigma^T \in \mathbb{R}^{w \times w}$

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Stability and half-line dissipativity

When supply rate $\Sigma$ equals $\gamma^2 u^T u - y^T y$ and for system with input $u$ and output $y$
(Case of maximal input cardinality: $m(\mathcal{B}) = \sigma_+(\Sigma)$)

- dissipativity on $\mathbb{R}_- \iff$ transfer matrix is stable (no poles in CRHP)

- $\mathcal{B}$ is $\Sigma$-dissipative on $\mathbb{R}_- \iff \mathcal{B}^{\perp\Sigma}$ is $-\Sigma$-dissipative on $\mathbb{R}_+$

Dissipativity on $\mathbb{R}_- \iff$ maximum storage function
$Q_{\Psi_{\max}}(w) \geq 0$ (i.e. $K_{\max} \geq 0$)
($Q_{\Psi_{\max}}(w)$: ‘required supply’)

Dissipativity on $\mathbb{R}_+ \iff$ minimum storage function
$Q_{\Psi_{\min}}(w) \leq 0$ (i.e. $K_{\min} \leq 0$)
($Q_{\Psi_{\min}}(w)$: ‘available storage’)

\[\text{Belur} \quad \text{Lecture 14} \quad 4/13\]
Stability and half-line dissipativity

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- dissipativity on $\mathbb{R}^-$ $\iff$ transfer matrix is stable (no poles in CRHP)
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Dissipativity on $\mathbb{R}^-$ $\iff$ maximum storage function
$Q_{\Psi_{\text{max}}}(w) \geq 0$ (i.e. $K_{\text{max}} \geq 0$)
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Belur Lecture 14 4/13
When supply rate $\Sigma$ equals $\gamma^2 u^T u - y^T y$ and for system with input $u$ and output $y$
(Case of maximal input cardinality: $m(\mathcal{B}) = \sigma_+ (\Sigma)$)

- dissipativity on $\mathbb{R}^-$ $\Leftrightarrow$ transfer matrix is stable
  (no poles in CRHP)

- $\mathcal{B}$ is $\Sigma$-dissipative on $\mathbb{R}^-$ $\Leftrightarrow$ $\mathcal{B}^{\perp \Sigma}$ is $-\Sigma$-dissipative on $\mathbb{R}^+

Dissipativity on $\mathbb{R}^-$ $\Leftrightarrow$ maximum storage function
$Q_{\Psi_{\text{max}}} (w) \geq 0$ (i.e. $K_{\text{max}} \geq 0$)
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Dissipativity on $\mathbb{R}^+$ $\Leftrightarrow$ minimum storage function
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($Q_{\Psi_{\text{min}}} (w)$: ‘available storage’)

Belur Lecture 14 4/13
Assume $\mathcal{B} \in \mathcal{L}^w_{\text{cont}}$ has minimal kernel representation

$$R(\frac{d}{dt})w = 0$$

(Full row rank $R(\xi)$) and image representation $w = M(\frac{d}{dt})\ell$

- WLOG, choose $M(\xi)$ such that $M(\lambda)$ is full column rank for all $\lambda \in \mathbb{C}$.
  
  ($\ell$ is ‘observable’ from $w$, also $M$ is called ‘right-prime’)

- $R(\xi) = [P(\xi) \quad Q(\xi)]$ with $\det(P(\xi)) \neq 0$ and $w = (y, u)$, then transfer matrix from $u$ to $y$ is $G(s) = -P(s)^{-1}Q(s)$.

- Number of rows in $R = \text{number of outputs}$

- Number of columns in $M = \text{number of inputs}$
Again assume controllable $\mathcal{B}$, and $R$ is left-prime and $M$ is right-prime

- Corresponding to $w = (y, u)$, also partition
  \[ M(\xi) = \begin{bmatrix} Y(\xi) \\ U(\xi) \end{bmatrix}, \quad G(s) = -P(s)^{-1}Q(s) = Y(s)U(s)^{-1} \]
  (left/right (polynomial) coprime factorization of $G(s)$)

- Amongst all maximal nonsingular minors $P$ in $R(\xi) = [P(\xi) \quad Q(\xi)]$, find one with maximum determinantal degree: $n(\mathcal{B})$: McMillan degree

- Ensures $G(s)$ is proper: $\det U(s)$ has same degree, and is also maximum

- $n(\mathcal{B})$: least number of ‘states’ (defined using a ‘concatenability’ axiom)
Interested in $w$-behavior: manifest behavior $\mathcal{B} \subseteq C^\infty(\mathbb{R}, \mathbb{R}^w)$: kernel representations but conveniently

- image representations: ‘free’ $\ell$ generates all trajectories
- state representations: state $x$ ‘being equal’ allows concatenation of trajectories
- powerful/efficient/accurate manipulation of constant matrices: $(E, A, B, C, D)$

\[ E \frac{d}{dt} x + Fx + Gw = 0 \] and

\[
\begin{pmatrix}
\frac{d}{dt} x = Ax + Bw_1, \\
w_2 = Cx + Dw_1
\end{pmatrix} \iff \text{transfer matrix } w_1 \rightarrow w_2 \text{ is proper}
\]

(last one: i/s/o representation)
Interested in $w$-behavior: manifest behavior $\mathfrak{B} \subseteq C^\infty(\mathbb{R}, \mathbb{R}^w)$: kernel representations but conveniently

- image representations: ‘free’ $\ell$ generates all trajectories
- state representations: state $x$ ‘being equal’ allows concatenation of trajectories
- powerful/efficient/accurate manipulation of constant matrices: $(E, A, B, C, D)$

$$E \frac{d}{dt} x + Fx + Gw = 0 \text{ and }$$

$$\left( \frac{d}{dt} x = A x + B w_1, \quad w_2 = C x + D w_1 \right) \iff \text{transfer matrix } w_1 \to w_2 \text{ is proper}$$

(last one: i/s/o representation)
Interested in \( w \) only.

Dummy variables: \( x \) and \( d \) (with additional properties)

\[
\dot{x} = Ax + Bd \quad \text{and} \quad w = Cx + Dd \quad \text{(driving variable (d.v.))},
\]

\[
\dot{x} = Ax + Bw \quad \text{and} \quad 0 = Cx + Dw \quad \text{(output nulling (o.n.))},
\]

Just like \( \mathcal{B} = \ker \{ R(\frac{d}{dt}) \} \Leftrightarrow \mathcal{B}^\perp = \text{image} \{ R(-\frac{d}{dt})^T \}, \)

Can jump between i/s/o representations of \( \mathcal{B} \) and \( \mathcal{B}^\perp \)

and d.v. representation of \( \mathcal{B} \) and o.n. of \( \mathcal{B}^\perp \) (and \( \mathcal{B}^\perp \Sigma \))

We seek least number of variables in \( d \), and in \( x \):

‘observability’, ‘trimness’
Interested in $w$ only.

Dummy variables: $x$ and $d$ (with additional properties)

\[
\begin{align*}
\dot{x} &= Ax + Bd \quad \text{and} \quad w = Cx + Dd \quad \text{(driving variable (d.v.))}, \\
\dot{x} &= Ax + Bw \quad \text{and} \quad 0 = Cx + Dw \quad \text{(output nulling (o.n.))},
\end{align*}
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Just like $\mathcal{B} = \ker R\left(\frac{d}{dt}\right) \iff \mathcal{B}^\perp = \text{image } R\left(-\frac{d}{dt}\right)^T$, 

Can jump between i/s/o representations of $\mathcal{B}$ and $\mathcal{B}^\perp$

and d.v. representation of $\mathcal{B}$ and o.n. of $\mathcal{B}^\perp$ (and $\mathcal{B}^\perp \Sigma$)

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Can jump between i/s/o representations of $\mathcal{B}$ and $\mathcal{B}^\perp$

and d.v. representation of $\mathcal{B}$ and o.n. of $\mathcal{B}^\perp$ (and $\mathcal{B}^\perp \Sigma$)

We seek least number of variables in $d$, and in $x$: ‘observability’, ‘trimness’
Model order reduction

Given $\mathcal{B} \in \mathcal{L}^w_{\text{cont}}$ and symmetric nonsingular $\Sigma \in \mathbb{R}^{w \times w}$
Suppose $\mathcal{B}$ is strictly $\Sigma$-dissipative on $\mathbb{R}_-$ and suppose $n$ is the McMillan degree of $\mathcal{B}$
Choose $k < n$. Find $\hat{\mathcal{B}} \in \mathcal{L}^w_{\text{cont}}$ such that

1. $\hat{\mathcal{B}}$ has McMillan degree at most $k$
2. $m(\hat{\mathcal{B}}) = m(\mathcal{B})$
3. $\hat{\mathcal{B}}$ is also strictly $\Sigma$-dissipative on $\mathbb{R}_-$
4. $\hat{\mathcal{B}}$ satisfies $(\hat{\mathcal{B}}^*)_{\text{anti-stab}} \subset \mathcal{B}^*$

(Fourth point: trajectories in $\mathcal{B}$ of minimal dissipation are retained into $\hat{\mathcal{B}}$)
$\mathcal{B}^* = M(\frac{d}{dt}) \ker \partial \Phi'(\frac{d}{dt})$ and
strict dissipativity $\Leftrightarrow$ no $j\mathbb{R}$ roots of $\det \partial \Phi'(*\xi)$
Proposed by Sorensen, SCL 2005, and as interpreted in Minh, Trentelman & Rapisarda (MCSS, 2009)

\[ w^T \Sigma w = u^T y, \ w = (u, y) \]

\[
\frac{d}{dt} x = Ax + Bu, \text{ and } y = Cx + Du \text{ for } \mathfrak{B}, \text{ and hence}
\]

\[ \mathfrak{B} \perp \Sigma \text{ represented by } \frac{d}{dt} z = -A^T z + C^T u, \ y = B^T z - D^T u \]

(Try \( \frac{d}{dt} x^T z = u^T y \))

Interconnecting (& assuming strict passivity \( \Rightarrow D + D^T > 0 \))

\[
\begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} = H 
\begin{bmatrix}
x \\
z
\end{bmatrix} \text{ and } 
\begin{bmatrix}
u \\
y
\end{bmatrix} = L 
\begin{bmatrix}
x \\
y
\end{bmatrix} \text{ with } H \text{ and } L \text{ respectively as}
\]

\[
\begin{bmatrix}
A - B(D + D^T)^{-1} C & B(D + D^T)^{-1} B^T \\
-C^T(D + D^T)^{-1} C & -A^T + C^T(D + D^T)^{-1} B^T
\end{bmatrix}, \begin{bmatrix}
-(D + D^T)^{-1} C & (D + D^T)^{-1} B^T \\
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\end{bmatrix}
\]
Algorithm

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\[ \mathcal{B}^\perp \Sigma \text{ represented by } \frac{d}{dt} z = -A^T z + C^T u, \ y = B^T z - D^T u \]

(Try \( \frac{d}{dt} x^T z \neq u^T y \))

Interconnecting (& assuming strict passivity ⇒ \( D + D^T > 0 \))

\[
\begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} = H \begin{bmatrix} x \\ z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u \\ y \end{bmatrix} = L \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with } H \text{ and } L \text{ respectively as}
\]
\[
\begin{bmatrix}
A - B(D+D^T)^{-1}C \\
-C^T(D+D^T)^{-1}C
\end{bmatrix} \quad \begin{bmatrix}
B(D+D^T)^{-1}B^T \\
A^T + C^T(D+D^T)^{-1}B^T
\end{bmatrix}, \quad \begin{bmatrix}
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\end{bmatrix} \quad \begin{bmatrix}
(D+D^T)^{-1}B^T \\
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Algorithm

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\[ \mathfrak{B}^\perp \Sigma \text{ represented by } \frac{d}{dt} z = -A^T z + C^T u, \ y = B^T z - D^T u \]

(Try \( \frac{d}{dt} x^T z = u^T y \))

Interconnecting (\& assuming strict passivity \( \Rightarrow D + D^T > 0 \))

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with \( H \) and \( L \) respectively as

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$$\mathcal{B}^{-\Sigma} \text{ represented by } \frac{d}{dt} z = -A^T z + C^T u, \ y = B^T z - D^T u$$

(Try $$\frac{d}{dt} x^T z \overset{?}{=} u^T y$$)

Interconnecting (& assuming strict passivity $$\Rightarrow D + D^T > 0$$)

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = H \begin{bmatrix} x \\ z \end{bmatrix} \text{ and } \begin{bmatrix} u \\ y \end{bmatrix} = L \begin{bmatrix} x \\ y \end{bmatrix} \text{ with } H \text{ and } L \text{ respectively as}$$

$$\begin{bmatrix} A-B(D+D^T)^{-1}C & B(D+D^T)^{-1}B^T \\ -C^T(D+D^T)^{-1}C & -A^T+C^T(D+D^T)^{-1}B^T \end{bmatrix} \begin{bmatrix} -(D+D^T)^{-1}C & (D+D^T)^{-1}B^T \\ C-D(D+D^T)^{-1}C & D(D+D^T)^{-1}B^T \end{bmatrix}$$
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-(D+D^T)^{-1}C & (D+D^T)^{-1}B^T \\
C-D(D+D^T)^{-1}C & D(D+D^T)^{-1}B^T
\end{bmatrix}
\]
Choose anti-Hurwitz $R \in \mathbb{R}^{k \times k}$ (from ORHP spectral zeros) and corresponding real $X$ and $Y$ such that

$$H \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} R.$$ 

Strict dissipativities $\Rightarrow$ $X$ and $Y$ are both full column rank. They are ‘part’ of maximal ARE solution (known to be symmetric), same argument helps $X^TY \in \mathbb{R}^{k \times k}$ being symmetric and positive definite.

- Obtain $X^TY = QS^2Q^T$ with $Q^T = Q^{-1}$, and $S$ diagonal.
- Define $V := XQS^{-1}$ and $W := YQS^{-1}$,
- $\hat{A} := W^TAV$, $\hat{B} := W^TB$, $\hat{C} := CV$ and $\hat{D} := D$
- Define reduced order system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$. 
Choose anti-Hurwitz $R \in \mathbb{R}^{k \times k}$ (from ORHP spectral zeros) and corresponding real $X$ and $Y$ such that

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- Obtain $X^T Y = QS^2 Q^T$ with $Q^T = Q^{-1}$, and $S$ diagonal.
- Define $V := XQS^{-1}$ and $W := YQS^{-1}$,
- $\hat{A} := W^T AV$, $\hat{B} := W^T B$, $\hat{C} := CV$ and $\hat{D} := D$
- Define reduced order system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$. 
Some verifications

$W^TV$ is identity matrix and $WV^T$ satisfies $(WV^T)^2 = WV^T$

?? $X^TY$ is the largest ARE solution of the reduced system??

Recall: we sought $\hat{B}$:

1. $\hat{B}$ has McMillan degree at most $k$
2. $m(\hat{B}) = m(B)$
3. $\hat{B}$ is strictly $\Sigma$-dissipative on $\mathbb{R}_-$
4. $\hat{B}$ satisfies $(\hat{B}^*)_{\text{anti-stab}} \subset B^*$

With $\hat{X} := \hat{Y} := SQ^T$ (Sorensen, SCL-'05), Minh, et al gets

$$\hat{H} \begin{bmatrix} \hat{X} \\ \hat{Y} \end{bmatrix} = \begin{bmatrix} \hat{X} \\ \hat{Y} \end{bmatrix} R$$

Further, $\hat{L}\hat{X} = LX$ and $\hat{L}\hat{Y} = LY$ give $(\hat{B}^*)_{\text{anti-stab}} \subset B^*$
Lagrange interpolating polynomials
Rational interpolant with degree constraint → ‘Löwner’ matrices
Link with Nevanlinna Pick interpolation problem
Given $N$ pairs $(x_i, y_i) \in \mathbb{C}^2$, find P.R. interpolant $G(s)$
Pick matrix $\Pi$ with $\Pi_{ij}$ defined as

$$\frac{y_i + y_j^*}{x_i + x_j^*} \quad \text{and} \quad \frac{1 - w_i w_j^*}{x_i + x_j^*} \quad \text{and} \quad \frac{1 - w_i w_j^*}{1 - z_i z_j^*}$$

depending on P.R., B.R. (OLHP), B.R. ($|z| = 1$), with

$$w_i := \frac{1 - y_i}{1 + y_i} \quad \text{and} \quad z_i := \frac{1 - x_i}{1 + x_i}$$

“Model reduction by interpolating at (some) spectral zeros”
“Pick matrix $\equiv$ minimum energy required across trajectories in $\ker A(\frac{d}{dt})$” (QDF, Willems & Trentelman, SIAM 1998)