Lossless trajectories of singularly passive systems

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Abstract: Lossless trajectories of a passive system are the trajectories that satisfy the dissipation inequality with equality. In other words, for a suitable input-state-output representation, these are trajectories for which the rate of change of stored energy is equal to the power supplied to the system. In this paper, we present a method to design feedback control strategies that restrict the trajectories of a passive system to its lossless trajectories. In particular, we deal with passive systems that do not admit an Algebraic Riccati Equation (ARE) arising from the dissipation inequality; we call such systems singularly passive systems. We show that suitably designed PD feedback controllers help us restrict the trajectories of the system to its lossless trajectories. The design method of the controller is linked to the LMI arising from the KYP Lemma corresponding to a passive system.

Keywords: Passive systems, Lossless trajectories, PD controller, Hamiltonian systems.

1. INTRODUCTION

About five decades ago, J.C, Willems mathematically formalized the theory of dissipativity which, loosely speaking, states that a system is dissipative if the rate of change of its stored energy is less than or equal to the power supplied to it. More precisely, a system with minimal input-state-output (i/s/o) representation

\[
d \frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du,
\]

with \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}, D \in \mathbb{R}^{m \times m}\) is called passive if there exists a symmetric, positive semidefinite \(K \in \mathbb{R}^{n \times n}\) such that all system trajectories satisfy

\[
d \frac{dx}{dt} (x(t)^T K x(t)) \leq 2u(t)^T y(t) \quad \text{for all} \quad t \in \mathbb{R}.
\]

The term \(d \frac{dx}{dt} (x(t)^T K x(t))\) represents the system’s rate of change of stored energy\(^1\) and \(2u(t)^T y(t)\) is the total power supplied to the system [Willems (1972), Trentelman and Willems (1997)]. This definition of passivity neatly fits in with classical passivity theory based on the Kalman-Yakubovich-Popov (KYP) lemma [Yakubovich (1962), Kalman (1963), Popov (1964)]. From the definition of passivity using inequality (2), it is natural that system trajectories that satisfy inequality (2) with equality are termed lossless, since the rate of change of stored energy is equal to the supplied power for such trajectories, thus indicating no energy dissipation. Such trajectories are also referred to as trajectories of minimal dissipation in related works such as Rapisarda and Willems (2005). Thus, for an RLC circuit with a state-space realization as in equation (1) and with states having the physical interpretation of capacitor voltage and inductor current, the lossless trajectories, in theory, correspond to the trajectories at which the circuit should dissipate no energy.

1 The word “energy” in this paper might not refer to the physical and familiar notion of energy, but is more notional. For instance, in an RLC circuit, if we consider the states to be capacitor voltages and inductor currents, and the input/output to be current/voltage in the circuit, then the terms \(x^T K x\) and \(u^T y\) represent, respectively, the physical stored energy and power supplied to the circuit. On the other hand, if the states chosen do not have a physical interpretation, then the terms \(x^T K x\) and \(u^T y\) might not have a physical interpretation.

natural question at this point is: are there ways to restrict the trajectories of a passive system to its lossless trajectories? We investigate this for passive single-input single-output (SISO) systems in this paper.

Lossless trajectories of dissipative systems have been studied in the literature: see Bhat and Bernstein (2002), Rapisarda and Willems (2005), Trentelman et.al. (2009). In Bhat and Bernstein (2002) the authors use classical state-space techniques to study the equipartition of energy of oscillators coupled in a lossless way. This was pursued further for linear oscillatory systems in Rapisarda and Willems (2005), where the authors study quadratic functionals of the system variables and their derivatives. In Trentelman et.al. (2009) the authors introduced the idea of restriction of a system to its minimal dissipation trajectories. The authors showed that suitable eigenspaces of the corresponding Hamiltonian matrix can be used for such restrictions. Using these ideas it is easy to note that for a system with a nonsingular feedthrough term \(D\) in equation (1), lossless trajectories (trajectories of minimal dissipation) can be computed using the solutions of the corresponding algebraic Riccati equation (ARE). These solutions lead to a proportional state feedback controller design that restricts system trajectories of certain passive systems to their lossless trajectories. However, for circuits like the one in Figure 1, the feedthrough term \(D\) is zero, and therefore such passive circuits do not admit an ARE of the form \(A^T K + KA + (KB - C^T)(D + D^T)^{-1}(B^T K - C) = 0\).

![Fig. 1. A network with impedance function \(G(s) = \frac{3s^3 + 12s + 11}{s^2 + 6s^2 + 11s + 6}\).](image)

This leads to the following question: how does one characterize the lossless trajectories of a passive system that does not admit an ARE? Is it possible to design feedback controllers that restrict the trajectories of such systems to their lossless trajectories? We investigate these questions for passive SISO
2.1 Notation

- \( \mathbb{R} \) and \( \mathbb{C} \): Sets of real and complex numbers, respectively.
- \( \mathbb{R}[s] \) and \( \mathbb{R}(s) \): Ring of polynomials with real coefficients and the field of rational functions, respectively.
- \( \mathbb{R}^{n \times p} \): The set of \( n \times p \) matrices with elements from \( \mathbb{R} \).
- \( I_n \): \( n \times n \) identity matrix.
- \( 0_{n \times n} \): \( n \times n \) matrix with all entries zero.
- \( \text{col}(B_1, B_2, \ldots, B_n) \): A matrix of the form \( [B_1^T, B_2^T, \ldots, B_n^T]^T \).
- \( \text{roots}(g(s)) \): Multiset \(^2\) of the roots of polynomial \( g(s) \).
- \( \text{rootsym}(G(s)) \): Multiset of the roots of the numerator of a rational function \( G(s) \).
- \( \sigma(\mathcal{E}(\mathcal{A})) \): Multiset of the roots of \( \det(\mathcal{E}(\mathcal{A})) \). These are also called eigenvalues of \( (\mathcal{E}(\mathcal{A})^T \mathcal{A}) \).
- \( \sigma(\mathcal{H}) \): Multiset of the roots of \( \det(\mathcal{H}) \), i.e., multiset of the eigenvalues of \( \mathcal{H} \).
- \( |A| \): Cardinality of the multiset \( S \).
- \( \text{im}(A) \): Image of matrix \( A \).
- \( \det(A) \): Determinant of a square matrix \( A \).
- \( \delta : \mathbb{R} \rightarrow \mathbb{R} \): Dirac delta impulse function.
- \( \delta[i] \): \( i \)-th distributional derivative of \( \delta \).
- \( \mathcal{E} : \mathbb{R}^p \rightarrow \mathbb{R}^p \): The space of impulsive smooth distributions from \( \mathbb{R} \) to \( \mathbb{R}^p \) (see Hautus and Silverman (1983) for definition).

2.2 Definitions

A few of the definitions that would be essential for development of the results in this paper are collated below.

**Definition 2.1.** A SISO system with transfer function \( G(s) \in \mathbb{R}[s] \) and a minimal \( i/s/o \) representation given by equation (1) is said to be **singly passive** if it satisfies the following conditions:

1. The system is bounded-input bounded-output (BIBO) stable (Anderson and Vongpanitlerd, 2006, Section 3.7).
2. There exists a solution \( K = K^T \geq 0 \) to the LMI (3).
3. \( \text{rootsym}(G(s) + G(-s)) \cap \mathbb{R} = \emptyset \).
4. The feedthrough term \( D \) in equation (1) is zero.

In contrast, a **regularly passive system** \(^3\) is a system that satisfies properties (1), (2) and (3) above, but, instead of property (4), the feedthrough term \( D \) is assumed nonsingular.

Since rank-minimizing solutions of the KYP LMI are important for this paper, we define this below for easy later reference.

**Definition 2.2.** \( K_{\text{opt}} \) is called a rank-minimizing solution of an LMI \( \mathcal{L}(K) \leq 0 \) if the following conditions are satisfied:

1. \( \mathcal{L}(K_{\text{opt}}) \leq 0 \).
2. \( K_{\text{opt}} \) minimizes the rank of \( \mathcal{L}(K) \) over the set of solutions of \( \mathcal{L}(K) \leq 0 \).

In order to find a rank-minimizing solution of the KYP LMI we need the notion of Lambda-set: we define this next.

**Definition 2.3.** Consider a polynomial \( q(s) \in \mathbb{R}[s] \). A Lambda-set \( \Lambda \subset \text{rootsym}(q(s)) \), if it exists, is the set \( \text{rootsym}(p(s)) \) such that \( q(s) = p(s)p(-s) \), where \( p(s) \in \mathbb{R}[s] \) and \( p(s), p(-s) \) are coprime.

Since we are dealing with lossless trajectories in this paper, a formal definition of lossless trajectories is in order. We define this next.

**Definition 2.4.** Consider a passive SISO system \( \Sigma \) with a minimal \( i/s/o \) representation as in equation (1). Let \( \text{col}(x, u, y) \in \mathbb{R}^{n+1+1} \) be a trajectory of \( \Sigma \), i.e., it satisfies equation (1). Then, \( \text{col}(x, u, y) \) is called a lossless trajectory if there exists a solution \( K = K^T \in \mathbb{R}^{n \times n} \) of the corresponding KYP LMI (3) such that \(^5\)

\[
\frac{d}{dt} (x(t)^T K x(t)) = 2u(t)y(t) \quad \forall t \in \mathbb{R}.
\]

3. RANK-MINIMIZING SOLUTIONS OF THE KYP LMI

In this section, we present the first main result of this paper that formulates a method to compute a rank-minimizing solution of the KYP LMI (3) (Theorem 3.1). It is known that for

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2 A system is said to be minimal if it is both controllable and observable.
3 A set that allows for multiple instances of an element is called a multiset.
5 In general, there exist passive systems that allow \( \det(s \delta - \mathcal{H}) \) to have roots on \( \mathbb{R} \). We call such systems **marginally singular/regular** passive systems based on the existence of ARE. Such systems are not included in the class of regularly/singularity passive systems. In this paper we do not deal with marginally singular/regular systems (Condition 3 of Definition 2.1 is violated due to the fact that \( \sigma(\mathcal{E}, \mathcal{H}) = \text{rootsym}(G(s) + G(-s)) \)). A special class of marginally singular passive systems are lossless systems for which \( \det(s \delta - \mathcal{H}) = 0 \). Hence an inherent assumption throughout the paper is that \( \det(s \delta - \mathcal{H}) \neq 0 \) see Bhawal et al. (2016) for different methods to compute KYP LMI solutions for lossless systems.

Since the trajectories of \( \Sigma \) are from the space of impulsive-smooth distributions, Definition 2.4 does not preclude the possibility of multiplication of \( \delta \) and its derivatives with themselves. Hence we treat equation (4) formally here, i.e., equation (4) is said to hold if the expression \( \frac{d}{dt} (x(t)^T K x(t)) - 2u(t)y(t) \) is zero as a function for \( t \in (0, \infty) \) and each of the coefficients of \( \delta, \delta \cdot, \delta \cdot \cdot \cdot, \delta \cdot \cdot \cdot \cdot \) and their powers is zero in the impulsive part of the expression.
regularly passive systems, the corresponding ARE solutions are computed using the following matrix pencil [Dooren (1981)]:

\[
\begin{bmatrix}
0 & A & B \\
C & D & 0
\end{bmatrix}
\]

The matrix pencil \((\mathcal{E} - \mathcal{H})\) is called the Extended Hamiltonian matrix pencil (EHP) and the matrix pair \((\mathcal{E}, \mathcal{H})\) the Hamiltonian matrix pair; see [Ionescu et.al. (1999)] for more on this. The EHP in equation (5) is the Rosenbrock system matrix pencil corresponding to the following output-nulling system [Rosenbrock (1967)]:

\[
\begin{bmatrix}
\frac{d}{dt} x \\
\frac{d}{dt} y
\end{bmatrix} = \begin{bmatrix}
A & 0 \\
\mathcal{B} & \mathcal{C}^T
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
\mathcal{A} & \mathcal{B} \\
\mathcal{C} & \mathcal{D}
\end{bmatrix} \begin{bmatrix}
u \\
u
\end{bmatrix}
\]

The variable \(z\) is called the co-state of the system. We use the symbol \(\Sigma_{\text{Ham}}\) to denote the system in equation (6). The roots of \(\det(\mathcal{E} - \mathcal{H})\) are also known as the spectral zeros of the system in equation (1). Note that in the behavioral theory framework, if \(\Sigma\) is the set of trajectories of the system in equation (1) and \(\mathcal{B}^+\) is the set of trajectories of the orthogonal complement of \(\mathcal{B}\) with respect to the passivity support, then equation (6) is the system with trajectories from the set \(\Sigma \cap \mathcal{B}^+\) [Trentelman et.al. (2009)]. In Trentelman et.al. (2009) the trajectories in \(\mathcal{B} \cap \mathcal{B}^+\) are considered to be continuous smooth trajectories with bounded support. However, in this work we allow the trajectories in \(\Sigma_{\text{Ham}}\) to be in \(\mathcal{E}_{\text{R}}\). This inclusion helps in characterization of the lossless trajectories from regularly passive systems and further leads to the design of a feedback controller.

Due to the structure of the matrix pair \((\mathcal{E}, \mathcal{H})\), it follows that the polynomial \(\det(\mathcal{E} - \mathcal{H})\) admits roots that are symmetric about the real and imaginary axis of the complex plane. Further, regularly/passive systems do not admit EHP with \(j\mathbb{R}\) axis roots; see Footnote 4. Therefore, it is evident that \(\det(\mathcal{E} - \mathcal{H})\) admits Lambda-sets. We use this fact in the next theorem to find rank-minimizing solutions of the KYP LMI.

**Theorem 3.1.** Consider a regularly/passive system \(\Sigma\) with a minimal i/s/o representation as given in equation (1). Let the corresponding EHP be as in equation (5). Assume \(\Lambda\) to be a Lambda-set of \(\det(\mathcal{E} - \mathcal{H})\) with cardinality \(n_{\mathcal{E}}\). Define \(n_{\mathcal{E}} := n - n_{\mathcal{H}}\) and \(\mathcal{B} \equiv \begin{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\
\mathcal{C} & \mathcal{D}
\end{bmatrix}
\end{bmatrix} \). Let \(V_{\mathcal{A}}, V_{\mathcal{B}} \in \mathbb{R}^{n \times n_{\mathcal{E}}}\) and \(V_{\mathcal{A}}, V_{\mathcal{B}} \in \mathbb{R}^{n \times n_{\mathcal{E}}}\) be such that

\[
\begin{bmatrix}
A & 0 \\
C & B
\end{bmatrix} = \begin{bmatrix}
V_{\mathcal{A}} & 0 \\
0 & V_{\mathcal{B}}
\end{bmatrix} \Gamma,
\]

where \(\mathcal{F} \in \mathbb{R}^{n_{\mathcal{E}} \times n_{\mathcal{E}}}\) and \(\mathcal{F}(\mathcal{G}) = \Lambda\). Define \(V_{\mathcal{A}} := \begin{bmatrix} V_{\mathcal{A}} \\
V_{\mathcal{B}}
\end{bmatrix} \in \mathbb{R}^{2n \times n_{\mathcal{E}}}\) and \(W := \begin{bmatrix}
\mathcal{B} & \mathcal{A} \mathcal{R}^{-1} \\
\mathcal{C} & \mathcal{D} \mathcal{R}^{-1}
\end{bmatrix} \in \mathbb{R}^{2n \times n_{\mathcal{E}}}\). Partition \(V_{\mathcal{A}}, W \in \mathbb{R}^{2n \times n_{\mathcal{E}}}\) as

\[
V_{\mathcal{A}}, W := \begin{bmatrix} X_{\mathcal{A}} & X_{\mathcal{B}} \end{bmatrix}
\]

where \(X_{\mathcal{A}}, X_{\mathcal{B}} \in \mathbb{R}^{n \times n_{\mathcal{E}}}\). Then, the following statements hold.

(1) \(X_{\mathcal{A}}\) is invertible.

(2) \(K := X_{\mathcal{B}} X_{\mathcal{A}}^{-1}\) is symmetric.

(3) \(K\) is a rank-minimizing solution of the LMI (3).

(4) \(K\) is positive semi-definite.

Refer to Bhawal et.al. (2022) for a proof of Theorem 3.1.

Interestingly, based on the cardinality of the Lambda-sets of \((\mathcal{E} - \mathcal{H})\), one can classify passive systems, without imaginary-axis spectral poles (see Footnote 4), into three types.
Next we present Lemma 4.2 and Lemma 4.3 that characterize certain special trajectories of the system $\Sigma$ when the initial conditions of the system are from the subspace $\mathbb{im}(V_{\Lambda})$ and $\mathbb{im}(W_{1})$, respectively. In the sequel, we show that these are lossless trajectories of the system. For reasons that would be clear after the lemmas, we call $\mathbb{im}(V_{\Lambda})$ and $\mathbb{im}(W_{1})$ the space of regular and irregular initial conditions of $\Sigma$, respectively.

**Lemma 4.2.** Consider a regularly/singularly passive SISO system $\Sigma$ as in Theorem 3.1 and let $\Sigma_{\text{Ham}}$ be the corresponding output-nulling system as in equation (6). Let $A, \Gamma, V_{\Lambda}, V_{\Sigma}, V_{\Lambda}$ and $n_{a}$ be as defined in Theorem 3.1. Let the initial condition of the system $\Sigma$ be $x_{0} := V_{\Lambda}z_{0}$, where $\bar{z}_{0} := V_{\Sigma}x_{0}$, $\bar{x}_{0} := V_{\Lambda}^{T}\bar{z}_{0}$, $\bar{z}_{0} := V_{\Lambda}^{T}\bar{x}_{0}$, $\bar{u}_{0} := F^{T}x_{0}$ and $\bar{y}_{0} := Cx_{0}$, where $F : \mathbb{R}^{k} \to \mathbb{R}$ such that $V_{\Lambda} = FV_{1\Lambda}$. Then, the following statements are true:

1. For initial condition $col(x_{0}, z_{0})$, $col(\bar{x}_{0}, \bar{z}_{0}, \bar{u}_{0})$ is a system trajectory of $\Sigma_{\text{Ham}}$.
2. For initial condition $x_{0}$, $col(\bar{x}_{0}, \bar{u}_{0}, \bar{y}_{0})$ is a system trajectory of $\Sigma$.

In particular, $col(\bar{x}_{0}, \bar{u}_{0}, \bar{y}_{0})$ is a lossless trajectory of $\Sigma$ in the sense of Definition 2.4.

**Proof.** On using $A, \bar{B}$ and $\bar{C}$ as defined in equation (6) and the fact that $y = Cx$, the proof of statement (1) and (2) of the lemma is similar to that of (Bhawal and Pal, 2019, Lemma 2). We prove next that $col(\bar{x}_{0}, \bar{u}_{0}, \bar{y}_{0})$ is a lossless trajectory of $\Sigma$.

We prove the lemma for singularly passive case and hence $D = 0$ in equation (1). Corresponding to the Lambda set $\Lambda$, we construct $K = X_{\Lambda}^{\prime}X_{\Lambda}$ as described in Theorem 3.1. Since $K$ is a solution of the LMI (3) and $D = 0$, we must have $KB = CT = 0$. Using this in the evaluation of the left hand side of dissipation inequality (2) we get

$$
\frac{d}{dt}(x^{T}Kx) + x^{T}Kx = x^{T}[A^{T}K + KA]x + 2uy + 2uy.
$$

Evaluating equation (10) at trajectories $(\bar{x}_{0}, \bar{u}_{0}, \bar{y}_{0})$, we get

$$
\bar{x}_{0}^{T}(A^{T}K + KA)\bar{x}_{0} + 2u_{0}\bar{y}_{0} = 2\bar{u}_{0}\bar{y}_{0}.
$$

Using Statement 1 of Lemma 4.1 in equation (11) gives

$$
\bar{x}_{0}^{T}(A^{T}K + KA)\bar{x}_{0} + 2u_{0}\bar{y}_{0} = 2u_{0}\bar{y}_{0}.
$$

Therefore, equation (10) and equation (12) gives

$$
\frac{d}{dt}(x^{T}Kx) |_{col(\bar{x}_{0}, \bar{u}_{0}, \bar{y}_{0})} = 2u_{0}\bar{y}_{0}.
$$

This proves that $col(\bar{x}_{0}, \bar{u}_{0}, \bar{y}_{0})$ is a lossless trajectory.

The proof for the regularly passive case follows directly from the fact that $V_{\Lambda}^{T}V_{\Lambda}^{-1}$ is a solution of the corresponding ARE. We skip the detailed proof here.

Note that the lossless trajectories obtained in Lemma 4.2 are exponential in nature with exponents given by the elements of the Lambda-set $\Lambda$. Therefore, we call such trajectories the *slow lossless* trajectories of $\Sigma$.

**Lemma 4.3.** Consider a singularly passive SISO system $\Sigma$ as in Theorem 3.1 and let $\Sigma_{\text{Ham}}$ be the corresponding output-nulling system as in equation (6). Let the initial condition of the system be $x_{0} := \sum_{k=0}^{n_{a}}(-1)^{k}a_{k}(A^{T})^{k}$, where $a_{1}, a_{2}, \ldots, a_{n_{a}} \in \mathbb{R}$. Define $z_{0} := \sum_{k=0}^{n_{a}}(-1)^{k}a_{k}(A^{T})^{k}$, $\bar{x}_{0} := \sum_{k=0}^{n_{a}}(-1)^{k-1}a_{k}(A^{T})^{k-1}B\delta(0)$, $\bar{z}_{0} := \sum_{k=0}^{n_{a}}(-1)^{k-1}(A^{k-1}C^{T})\delta(0)$, $\bar{u}_{0} := \sum_{k=0}^{n_{a}}(-1)^{k}\delta(k)$, and $\bar{y}_{0} = C\bar{z}_{0}$.

Then the following statements are true:

1. For initial condition $col(x_{0}, z_{0})$, $col(\bar{x}_{0}, \bar{z}_{0}, \bar{u}_{0})$ is a system trajectory of $\Sigma_{\text{Ham}}$.
2. For initial condition $x_{0}$, $col(\bar{x}_{0}, \bar{u}_{0}, \bar{y}_{0})$ is a system trajectory of $\Sigma$.

In particular, $col(\bar{x}_{0}, \bar{u}_{0}, \bar{y}_{0})$ is a lossless trajectory of $\Sigma$ in the sense of Definition 2.4.

**Proof.** On using $\bar{A}, B$ and $\bar{C}$ as defined in equation (6) and the fact that $y = Cx$, the proof of statements (1) and (2) are along similar lines to the proof of (Bhawal and Pal, 2019, Lemma 1). We prove next that $col(\bar{x}_{0}, \bar{u}_{0}, \bar{y}_{0})$ is a lossless trajectory of $\Sigma$.

Since $\Sigma_{\text{Ham}}$ corresponds to initial condition $(x_{0}, z_{0}) = \sum_{k=0}^{n_{a}}(\bar{A}^{k}B)\delta(0)$, can be characterized as

$$
\bar{x}_{0} = \begin{bmatrix}
\delta(0) \\
\delta(1) \\
\vdots \\
\delta(n_{a} - 2) \\
\delta(n_{a} - 1)
\end{bmatrix},
$$

$$
\bar{u}_{0} = \begin{bmatrix}
\alpha_{0} \\
\alpha_{1} \\
\vdots \\
\alpha_{n_{a} - 1}
\end{bmatrix},
$$

Therefore, the trajectory $col(\bar{x}_{0}, \bar{u}_{0}, \bar{y}_{0})$ is lossless in the sense of Definition 2.4.

From Lemma 4.3 it is clear that the lossless trajectories of the system corresponding to initial conditions from $\mathbb{im}(W_{1})$ are impulsive in nature. Therefore, we call such trajectories the *fast lossless* trajectories of $\Sigma$. Unlike the slow lossless trajectories, the fast ones do not depend on the roots of $\det(s\delta - \bar{H})$.

Since slow lossless trajectories are obtained when initial conditions are in $\mathbb{im}(V_{\Lambda})$ and fast lossless trajectories are obtained when initial conditions are in $\mathbb{im}(W_{1})$, we call the initial conditions from $\mathbb{im}(V_{\Lambda})$ and those from $\mathbb{im}(W_{1})$, regular and irregular initial conditions, respectively.

The next theorem formulates a complete characterization of the lossless trajectories of a regularly/singularly passive SISO system for an arbitrary initial condition.

**Theorem 4.4.** Consider a regularly/singularly passive SISO system $\Sigma$ with any arbitrary initial condition $x_{0}$ and cardinality...
of corresponding Lambda-sets $n_s$. Let $x_0 =: x_{0s} + x_{0f}$, where $x_{0s} \in \text{im}(V_1\Lambda)$ and $x_{0f} \in \text{im}(W_1)$ with $V_1\Lambda$ and $W_1$ as defined in Theorem 3.1. Define $\bar{x} := \bar{x}_s + \bar{u}$, $\bar{y} := \bar{y}_s + \bar{y}_r$, where $\text{col}(\bar{x}_s, \bar{u}, \bar{y}_s)$ and $\text{col}(\bar{x}_r, \bar{y}_r)$ are trajectories as defined in Lemma 4.2 and Lemma 4.3 corresponding to initial condition $x_{0s}$ and $x_{0f}$, respectively. Then, $\text{col}(\bar{x}, \bar{y})$ is a lossless trajectory of $\Sigma$ in the sense of Definition 2.4.

**Proof.** First we need to show that $\text{col}(\bar{x}, \bar{y})$ is a trajectory of $\Sigma$. This follows from linearity principle (1) of Lemmas 4.2 and 4.3. Next to prove that $\text{col}(\bar{x}, \bar{y})$ is a lossless trajectory, we evaluate the right-hand side of equation (10) for the trajectory $\text{col}(\bar{x}, \bar{y})$.

$$\frac{d}{dt}(x^T K_s)_{\text{col}(\bar{x},\bar{y})} = x^T (A^T K + KA)x + 2\tilde{y}.$$

Using the fact that $\bar{x} = V_1\Lambda e^{T_1\beta} \xi$, $\bar{y} = W_1\Omega \alpha$, and Statements 1, 3, and 5 in Lemma 4.1 below, we infer that

$$\frac{d}{dt}(x^T K_s)_{\text{col}(\bar{x},\bar{y})} = 2\tilde{y}.$$

Thus, $\text{col}(\bar{x}, \bar{y})$ is lossless. $\square$

Theorem 4.4 is crucial in the sense that it provides an open loop control strategy to attain lossless trajectories of the system. Given an initial condition $x_0$, if one can ensure that the system is subjected to an input $u$ as defined in Theorem 4.4, then it is possible to attain lossless trajectories for the system. Of course, like any other open loop strategy, this strategy is susceptible to external disturbances. Hence a closed loop control strategy is important and this is what we investigate next.

### 5. FEEDBACK CONTROLLER DESIGN

In this section we design state-feedback controllers that confine the set of trajectories of a regularly/singularly passive SISO system to its lossless trajectories. We claim that a PD state-feedback control law of the form $u = F_p x + F_d \frac{dx}{dt}$ achieves this confinement. Using the nonsingularity of $X_{1\Lambda}$ of Theorem 3.1, we define $F_p$ and $F_d \in \mathbb{R}^{1 \times n}$ as follows:

$$F_p := [V_1\Lambda \ 0 \ g_1 \cdots \ g_{n-1}]X_{1\Lambda}^{-1}$$

$$F_d := [0 \ I_{n_s} \ -g_0 \cdots \ -g_{n-1}]X_{1\Lambda}^{-1},$$

where $V_{1\Lambda}$ is obtained as defined in Theorem 3.1 and $g_i \in \mathbb{R}$ for $i \in [0, 1, \ldots, n_s - 1]$. The closed loop system obtained on application of $u = F_p x + F_d \frac{dx}{dt}$ to $\Sigma$ is as follows:

$$E_c \frac{dx}{dt} = A_c x \quad \text{and} \quad y = C x,$$

where $E_c := (I_n - BF_d)$, $A_c := (A + BF_p)$.

We use the symbol $\Sigma\text{lossless}$ to represent the closed loop system in equation (21). This brings us to the main theorem.

**Theorem 5.1.** Consider a regularly/singularly passive SISO system $\Sigma$. Assume $F_p \in \mathbb{R}^{1 \times n}$ and $F_d \in \mathbb{R}^{1 \times n}$ to be as defined in equation (19) and equation (20), respectively with $\det(s(I_n - BF_d) - (A + BF_p)) \neq 0$. Let the closed loop system obtained on application of the PD state-feedback law $u = F_p x + F_d \frac{dx}{dt}$ to $\Sigma$ be as defined in equation (21). Then, for an arbitrary initial condition $x_0$, the corresponding trajectory of the closed loop system $\Sigma\text{lossless}$ is a lossless trajectory in the sense of Definition 2.4.

The proof follows along similar lines as (Bhawal and Pal, 2019, Theorem 2); hence we skip the proof here.

Although Theorem 5.1 gives a method to design PD state feedback controllers to restrict the system trajectories to its lossless ones, the existence of such controllers relies on the crucial condition $\det(s(I_n - BF_d) - (A + BF_p)) \neq 0$. The next lemma shows that one can always find $g_0, g_1, \ldots, g_{n_s-1}$ such that the condition is satisfied. This thus guarantees the ability to design the controllers formulated in Theorem 5.1.

**Lemma 5.2.** Let $F_p$ and $F_d$ be as defined in equation (19) and equation (20), respectively. Then, there exist $g_0, g_1, \ldots, g_{n_s-1} \in \mathbb{R}$ such that $\det(sE_c - A_c) \neq 0$, where $E_c, A_c$ are as defined in equation (21).

We skip the proof of this lemma, since its proof is similar to that in (Bhawal and Pal, 2019, Lemma 3).

Next we present an example to illustrate the main results of this paper, i.e., Theorem 3.1, Theorem 4.4, and Theorem 5.1.

**Example 5.3.** Consider the RC circuit with impedance function $G(s)$ as given in Fig. 1. A minimal iso representation of $\Sigma$ with states $x = [v_c_1 \ v_c_2 \ v_c_3]^T$, input $u = i$, and output $y = v$ is as follows.

$$\frac{d}{dt} x = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u, \ y = [1 1 1] x.$$

Since $D = 0$, the system $\Sigma$ is a singularly passive system. The zeros of the determinant of the corresponding Hamiltonian pencil $(sE - H)$ are

$$\{-\sqrt{4 + \sqrt{5}}, -\sqrt{4 - \sqrt{5}}, \sqrt{4 + \sqrt{5}}, \sqrt{4 - \sqrt{5}}\}.$$

A rank-minimizing solution of the KYP LMI:

Consider the following Lambda-set $\{-\sqrt{4 + \sqrt{5}}, -\sqrt{4 - \sqrt{5}}\}$. The eigenvectors of $(\delta, \delta')$ are given by the columns of the matrix

$$[v_{1,1}^T \ v_{2,1}^T \ v_{3,1}^T]^T = [-0.23 -0.70 0.69 -0.10 -0.08 -0.06 0.35]^T.$$  

$$-0.90 0.44 0.18 -0.13 -0.09 -0.07 0.29] .$$

Further $W = B = \text{col}(B, C') = [1 \ 1 \ 1 \ 1 \ 1 \ 1]^T$. Finally computing for $K$ using the method in Theorem 3.1 we have

$$K = \begin{bmatrix} 0.36 & 0.33 & 0.31 \\ 0.33 & 0.33 & 0.34 \\ 0.31 & 0.34 & 0.55 \end{bmatrix} .$$

On substituting this $K$ in the KYP LMI (3), we have

$$L(K) = \begin{bmatrix} -0.72 & -0.98 & -1.24 & 0 \\ -0.98 & -1.34 & -1.68 & 0 \\ -1.24 & -1.68 & -2.12 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leq 0 .$$

Clearly, $\text{rank}(L(K)) = 1$. Thus, $K$ is a rank-minimizing solution of the KYP LMI. Note that there exists no $K$ for which $\text{rank}(L(K)) = 0$, since the system $G(s)$ is not lossless to start with.

**Lossless trajectories**: The slow lossless trajectory $\text{col}(\bar{x}_s, \bar{u}_s, \bar{y}_s)$ of the RC circuit corresponding to an initial condition $x_0 = \begin{bmatrix} -0.23 & -0.90 \\ 0.69 & 0.18 \end{bmatrix}$, for $\beta_1, \beta_2 \in \mathbb{R}$, is

$$\bar{x}_s = \begin{bmatrix} -0.23e^{-\sqrt{4 + \sqrt{5}}t} & -0.90e^{-\sqrt{4 - \sqrt{5}}t} \\ -0.70e^{-\sqrt{4 + \sqrt{5}}t} & 0.44e^{-\sqrt{4 - \sqrt{5}}t} \\ 0.69e^{-\sqrt{4 - \sqrt{5}}t} & 0.18e^{-\sqrt{4 - \sqrt{5}}t} \end{bmatrix} ,$$

$$\bar{u}_s = 0.35e^{-\sqrt{4 + \sqrt{5}}t} \beta_1 + 0.29e^{-\sqrt{4 - \sqrt{5}}t} \beta_2 , \quad \bar{y}_s = C \bar{x}_s = -0.24e^{-\sqrt{4 + \sqrt{5}}t} \beta_1 - 0.28e^{-\sqrt{4 - \sqrt{5}}t} \beta_2 .$$

These slow lossless trajectories correspond to a Lambda-set $\Lambda \subseteq \mathbb{C}$. Recall that the initial condition $x_0$ here is from the space of regular initial conditions.
and hence, the lossless trajectories are exponential in nature. Next we look at lossless trajectories when the initial condition of the system is from the space of irregular initial conditions.

Let the initial condition of the system be \( x_{0e} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \alpha \), where \( \alpha \in \mathbb{R} \). Then, using Lemma 4.3, the fast lossless trajectory \( (x_t, u_t, y_t) \) of the RC circuit is

\[
\dot{x}_t = 0_{n1}, \quad \dot{u}_t = -\alpha \delta_t, \quad \text{and} \quad \dot{y}_t = 0.
\]

Note that in this case \( \frac{\partial}{\partial n}(x^T K x)_{col(\dot{x}_t, \dot{u}_t, \dot{y}_t)} = 0 = \dot{\bar{u}} \dot{\bar{y}} \). This confirms that \( (\bar{x}_t, \bar{u}_t, \bar{y}_t) \) is a lossless trajectory. Thus, corresponding to an initial condition \( x_0 = x_{0e} + x_{0r} = \begin{bmatrix} -0.23 & -0.90 & 1 \\ -0.70 & 0.44 & 1 \\ 0.69 & 0.18 & 1 \end{bmatrix} / \alpha \), the lossless trajectory of the system is given by

\[
\bar{x} = \bar{x}_e + \bar{x}_r = \begin{bmatrix} 0.23 e^{-\sqrt{4 + \sqrt{5}} \beta_1 - 0.90 e^{-\sqrt{4 - \sqrt{5}} \beta_2} \\ -0.70 e^{-\sqrt{4 + \sqrt{5}} \beta_1 + 0.44 e^{-\sqrt{4 - \sqrt{5}} \beta_2} \\ 0.69 e^{-\sqrt{4 + \sqrt{5}} \beta_1 + 0.18 e^{-\sqrt{4 - \sqrt{5}} \beta_2} \end{bmatrix},
\]

\[
\bar{u} = \bar{u}_e + \bar{u}_r = 0.35 e^{-\sqrt{4 + \sqrt{5}} \beta_1} + 0.29 e^{-\sqrt{4 - \sqrt{5}} \beta_2} - \alpha \delta,
\]

\[
\bar{y} = \bar{y}_e + \bar{y}_r = -0.24 e^{-\sqrt{4 + \sqrt{5}} \beta_1} - 0.28 e^{-\sqrt{4 - \sqrt{5}} \beta_2}.
\]

The presence of \( \delta \) in \( \bar{u} \) indicates that with the capacitors initially charged to \( x_0 \), if one discharges the capacitors very fast, i.e., in the limit of a sequence of exponential decays instantaneously\(^7\), then it is possible to extract the capacitors’ entire stored energy through the port. For \( \beta_1 = \beta_2 = 0 \) and \( \alpha \neq 0 \), if one does not discharge instantaneously, a nonzero and positive amount of dissipation at the resistor \( R \) is inevitable.

controller design to restrict trajectories to lossless ones:

Note that here \( \bar{u} \) indicates that with the capacitors initially charged to \( x_0 \), if one discharges the capacitors very fast, i.e., in the limit of a sequence of exponential decays instantaneously\(^7\), then it is possible to extract the capacitors’ entire stored energy through the port. For \( \beta_1 = \beta_2 = 0 \) and \( \alpha \neq 0 \), if one does not discharge instantaneously, a nonzero and positive amount of dissipation at the resistor \( R \) is inevitable.

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