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# STRUCTURED BACKWARD ERRORS AND PSEUDOSPECTRA OF STRUCTURED MATRIX PENCILS*

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**Abstract.** Structured backward perturbation analysis plays an important role in the accuracy assessment of computed eigenelements of structured eigenvalue problems. We undertake a detailed structured backward perturbation analysis of approximate eigenelements of linearly structured matrix pencils. The structures we consider include, for example, symmetric, skew-symmetric, Hermitian, skew-Hermitian, even, odd, palindromic, and Hamiltonian matrix pencils. We also analyze structured backward errors of approximate eigenvalues and structured pseudospectra of structured matrix pencils.

**Key words.** structured backward error, pseudospectrum, structured matrix pencils

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**1. Introduction.** Backward perturbation analysis and condition numbers play an important role in the accuracy assessment of computed solutions of eigenvalue problems. Backward perturbation analysis determines the smallest perturbation for which a computed solution is an exact solution of the perturbed problem. On the other hand, condition numbers measure the sensitivity of solutions to small perturbations in the data of the problem. Thus, backward errors when combined with condition numbers provide approximate upper bounds on the errors in the computed solutions.

Structured eigenvalue problems occur in many applications (see, for example, [16, 21, 25] and the references therein). With a view to preserving structures and their associated properties, structured preserving algorithms for structured eigenproblems have been proposed in the literature (see, for example, [4, 5, 7, 11, 20, 21] and the references therein). Consequently, there is a growing interest in the structured perturbation analysis of structured eigenproblems (see, for example, [10, 13, 12, 24, 22, 6] for sensitivity analysis of structured eigenproblems).

The main purpose of this paper is to undertake a detailed structured backward perturbation analysis of approximate eigenelements of linearly structured matrix pencils. Needless to mention that structured backward errors when combined with structured condition numbers provide approximate upper bounds on the errors in the computed eigenelements. Hence, structured backward perturbation analysis plays an important role in the accuracy assessment of approximate eigenelements of structured pencils. Further, it also plays an important role in the selection of an optimum structured linearization of a structured matrix polynomial [1]. This assumes significance due to the fact that linearization is a standard approach to solving a polynomial eigenvalue problem (see, for example, [15] and the references therein).

We consider regular matrix pencils of the form $\mathrm{L}(\lambda) = A + \lambda B$, where $A$ and $B$ are square matrices of size $n$. We assume L to be linearly structured, that is, L to be an element of a real or a complex linear subspace $\mathbb{S}$ of the space of pencils. More

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specifically, we consider ten special classes of linearly structured pencils, namely, $T$-symmetric, $T$-skew-symmetric, $T$-odd, $T$-even, $T$-palindromic, $H$-Hermitian, $H$-skew-Hermitian, $H$-even and $H$-odd, and $H$-palindromic. These structures, defined in the next section, are prototypes of structured pencils which occur in many applications (see, [16, 21] and the references therein). We also consider $\mathbb{S}$ to be the space of pencils whose coefficient matrices are elements of Jordan and/or Lie algebras associated with the scalar product $(x, y) \mapsto y^T M x$ or $(x, y) \mapsto y^H M x$, where $M$ is unitary and $M^T = \pm M$ or $M^H = \pm M$. For example, when $M := \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, the Lie and Jordan algebras associated with the scalar product $(x, y) \mapsto y^H M x$ consist of Hamiltonian and skew-Hamiltonian matrices, respectively. The structures so considered encompass a wide variety of structured pencils and, in particular, include pencils whose coefficient matrices are Hamiltonian and skew-Hamiltonian. We show, however, that analyzing these wide classes of structured pencils ultimately boils down to analyzing one of the ten special classes of structured pencils considered above. Consequently, in this paper, we consider these ten special classes of structured pencils and investigate structured backward perturbation analysis of approximate eigenelements.

So, let $\mathbb{S}$ be the space of pencils having one of the ten structures. Let $\mathrm{L} \in \mathbb{S}$ and $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ with $x^H x = 1$. Then we define the structured backward error $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$ of $(\lambda, x)$ by

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) := \inf\{|\!|\!|\Delta\mathrm{L}|\!|\!| : \Delta\mathrm{L} \in \mathbb{S} \text{ and } \mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0\}.$$

Here the pencil norm $|\!|\!|\mathrm{L}|\!|\!|$ is given by $|\!|\!|\mathrm{L}|\!|\!| := \sqrt{\|A\|^2 + \|B\|^2}$, where $\mathrm{L}(z) = A + zB$ and $\|\cdot\|$ is either the spectral norm or the Frobenius norm on $\mathbb{C}^{n\times n}$. The main contributions of this paper are as follows.

Given $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ with $x^H x = 1$ and $\mathrm{L} \in \mathbb{S}$, we show that there is a pencil $\mathrm{K} \in \mathbb{S}$ such that $\mathrm{L}(\lambda)x + \mathrm{K}(\lambda)x = 0$. Consequently, $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) < \infty$. We determine $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$ and construct a pencil $\Delta\mathrm{L} \in \mathbb{S}$ such that $|\!|\!|\Delta\mathrm{L}|\!|\!| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$ and $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$. Moreover, we show that $\Delta\mathrm{L}$ is unique for the Frobenius norm on $\mathbb{C}^{n\times n}$, but there are infinitely many such $\Delta\mathrm{L}$ for the spectral norm on $\mathbb{C}^{n\times n}$. Further, for the spectral norm, we show how to construct all such $\Delta\mathrm{L}$. In either case, we show that if $\mathrm{K} \in \mathbb{S}$ is such that $\mathrm{L}(\lambda)x + \mathrm{K}(\lambda)x = 0$, then $\mathrm{K} = \Delta\mathrm{L} + (I - xx^H)^* \,\mathrm{N}(I - xx^H)$ for some $\mathrm{N} \in \mathbb{S}$, where $(I - xx^H)^*$ denotes the transpose or the conjugate transpose of $(I - xx^H)$ depending upon the structure defined by $\mathbb{S}$. Furthermore, we show that the unstructured backward error $\eta(\lambda, x, \mathrm{L})$ of $(\lambda, x)$ is a lower bound of $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$ and is attained by $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$ for certain $\lambda \in \mathbb{C}$. However, $\eta(\lambda, x, \mathrm{L}) \neq \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$ for most $\lambda \in \mathbb{C}$.

Next, we consider structured pseudospectra of structured matrix pencils. It is a well-known fact that pseudospectra of matrices and matrix pencils are powerful tools for sensitivity and perturbation analysis (see, [26] and the references therein). We consider structured and unstructured $\epsilon$-pseudospectra

$$\Lambda^{\mathbb{S}}_{\epsilon}(\mathrm{L}) := \left\{\lambda \in \mathbb{C} : \eta^{\mathbb{S}}(\lambda, \mathrm{L}) \leq \epsilon\right\} \text{ and } \Lambda_{\epsilon}(\mathrm{L}) := \{\lambda \in \mathbb{C} : \eta(\lambda, \mathrm{L}) \leq \epsilon\}$$

of L, where $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) := \min_{x^H x = 1} \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$ and $\eta(\lambda, \mathrm{L}) := \min_{x^H x=1} \eta(\lambda, x, \mathrm{L})$, respectively, are structured and unstructured backward errors of an approximate eigenvalue $\lambda$. When L is $T$-symmetric or $T$-skew-symmetric pencils, we show that $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = \eta(\lambda, \mathrm{L})$ for the spectral norm and $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = \sqrt{2}\,\eta(\lambda, \mathrm{L})$ for the Frobenius norm. Consequently, for these structures, we show that $\Lambda^{\mathbb{S}}_{\epsilon}(\mathrm{L}) = \Lambda_{\epsilon}(\mathrm{L})$ for the spectral norm and $\Lambda^{\mathbb{S}}_{\epsilon}(\mathrm{L}) = \Lambda_{\epsilon/\sqrt{2}}(\mathrm{L})$ for the Frobenius norm. For the rest of the structures, we show that there is a set $\Omega \subset \mathbb{C}$ such that $\Lambda_\epsilon^{\mathbb{S}}(\mathrm{L}) \cap \Omega = \Lambda_\epsilon(\mathrm{L}) \cap \Omega$. For example, $\Omega = \mathbb{R}$ when L is $H$-Hermitian or $H$-skew-Hermitian and $\Omega = i\mathbb{R}$ when L is $H$-even or $H$-odd. Often the spectrum of L is symmetric with respect to $\Omega$. When $\Omega$ does not contain an eigenvalue of L, it is of practical importance to determine the smallest perturbation $\Delta\mathrm{L} \in \mathbb{S}$ of L such that $\mathrm{L} + \Delta\mathrm{L}$ has an eigenvalue in $\Omega$. We show how to construct such a $\Delta\mathrm{L}$. Indeed, we show that the equality $\Lambda_\epsilon^{\mathbb{S}}(\mathrm{L}) \cap \Omega = \Lambda_\epsilon(\mathrm{L}) \cap \Omega$ plays a crucial role in the construction of such a $\Delta\mathrm{L}$.

The paper is organized as follows. In section 2, we define the ten special classes of structured pencils mentioned above. We also discuss some basic facts about spectral symmetry of structured pencils and, given $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ and a structured pencil L, we show that there exists a structured pencil K such that $\mathrm{L}(\lambda)x + \mathrm{K}(\lambda)x = 0$. In section 3, we undertake a detailed structured backward perturbation analysis when $\mathbb{C}^{n\times n}$ is equipped with the Frobenius norm. For each of the ten structures, we derive $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$ and a unique $\Delta\mathrm{L} \in \mathbb{S}$ such that $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$. In section 4, we undertake a detailed structured backward perturbation analysis for each of the ten classes of structured pencils when $\mathbb{C}^{n\times n}$ is equipped with the spectral norm. We show that the choice of a norm on $\mathbb{C}^{n\times n}$ plays a crucial role in the structured backward perturbation analysis. Finally, in section 5, we analyze structured pseudospectra of structured pencils.

**Notation.** We consider 2-norm on $\mathbb{C}^n$ defined by $\|x\|_2 := (x^H x)^{1/2}$, where $x^H$ is the conjugate transpose of $x$. We denote the set of $n$-by-$n$ matrices with real or complex entries by $\mathbb{C}^{n\times n}$. For $A \in \mathbb{C}^{n\times n}$, we denote the transpose of $A$ by $A^T$ and the conjugate transpose of $A$ by $A^H$. We consider spectral norm and the Frobenius norm on $\mathbb{C}^{n\times n}$. For $A \in \mathbb{C}^{n\times n}$, the spectral norm of $A$ is given by $\|A\|_2 := \max_{\|x\|_2=1} \|Ax\|_2$ and the Frobenius norm of $A$ is given by $\|A\|_F := (\mathrm{trace}(A^H A))^{1/2}$. We denote the smallest singular value of $A \in \mathbb{C}^{n\times n}$ by $\sigma_{\min}(A)$. The Moore–Penrose inverse of $A$ is denoted by $A^\dagger$. As usual, the conjugate of a complex number $z$ is denoted by $\overline{z}$. For a matrix $A$, $\overline{A}$ denotes the matrix whose entries are conjugate of that of $A$. The spectrum of $A \in \mathbb{C}^{n\times n}$ is denoted by $\Lambda(A)$.

## 2. Structured matrix pencils.

We consider $n$-by-$n$ matrix pencils of the form $\mathrm{L}(\lambda) := A + \lambda B$, where $A, B \in \mathbb{C}^{n\times n}$, and $\lambda \in \mathbb{C}$. Thus, the set of $n$-by-$n$ matrix pencils consists of affine transformations from $\mathbb{C}$ to $\mathbb{C}^{n\times n}$ which we denote by $\mathbb{A}^{n\times n}$. Hence, $\mathbb{A}^{n\times n}$ is a vector space which we endow with an appropriate norm $|\!|\!|\cdot|\!|\!|$ as follows. Let $\mathrm{L} \in \mathbb{A}^{n\times n}$ be given by $\mathrm{L}(\lambda) = A + \lambda B$. Then we define the pencil norm $|\!|\!|\mathrm{L}|\!|\!|$ by

$$|\!|\!|\mathrm{L}|\!|\!| := \left(\|A\|^2 + \|B\|^2\right)^{1/2}, \tag{2.1}$$

where $\|\cdot\|$ is either the spectral norm or the Frobenius norm on $\mathbb{C}^{n\times n}$. We refer to [3] for various other norms on $\mathbb{A}^{n\times n}$. It is evident that $\|\mathrm{L}(\lambda)\| \leq |\!|\!|\mathrm{L}|\!|\!| \; \|(1, \lambda)\|_2$.

The spectrum $\Lambda(\mathrm{L})$ of a regular pencil $\mathrm{L} \in \mathbb{A}^{n\times n}$ is given by

$$\Lambda(\mathrm{L}) := \{\lambda \in \mathbb{C} : \mathrm{rank}(\mathrm{L}(\lambda)) < n\}.$$

To be precise, $\Lambda(\mathrm{L})$ consists of finite eigenvalues of L. When $B$ is singular, the pencil L has an infinite eigenvalue. In this paper, we consider only finite eigenvalues of matrix pencils. By convention, if $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$, then $x$ is assumed to be nonzero, that is, $x \neq 0$. Treating $(\lambda, x)$ as an approximate eigenpair of L, we define the backward error of $(\lambda, x)$ by

$$\eta(\lambda, x, \mathrm{L}) := \inf\{|\!|\!|\Delta\mathrm{L}|\!|\!| : \Delta\mathrm{L} \in \mathbb{A}^{n\times n} \text{ and } \mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0\}.$$

We follow the convention that if L is given by $\mathrm{L}(\lambda) = A + \lambda B$, then the pencil $\Delta\mathrm{L}$ to be of the form $\Delta\mathrm{L}(\lambda) = \Delta A + \lambda \Delta B$. Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$. Then setting $r := -\mathrm{L}(\lambda)x$, we have

$$\eta(\lambda, x, \mathrm{L}) = \frac{\|r\|_2}{\|x\|_2 \|(1,\lambda)\|_2}.$$

Indeed, defining $\Delta A := \frac{r x^H}{x^H x (1+|\lambda|^2)}$ and $\Delta B := \frac{\overline{\lambda} r x^H}{x^H x(1+|\lambda|^2)}$, and considering the pencil $\Delta\mathrm{L}(z) = \Delta A + z\Delta B$, we have $|\!|\!|\Delta\mathrm{L}|\!|\!| = \|r\|_2 / \|x\|_2 \|(1,\lambda)\|_2$ and $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$.

Next, let $\mathbb{S}$ be a (real or complex) linear subspace of $\mathbb{A}^{n\times n}$. Pencils in $\mathbb{S}$ will be referred to as structured pencils. Let $\mathrm{L} \in \mathbb{S}$. Then treating $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^{n\times n}$ as an approximate eigenpair of L, we define the structured backward error of $(\lambda, x)$ by

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) := \inf\{|\!|\!|\Delta\mathrm{L}|\!|\!| : \Delta\mathrm{L} \in \mathbb{S} \text{ and } \mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0\}.$$

Obviously, we have $\eta(\lambda, x, \mathrm{L}) \leq \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$. Let L be given by $\mathrm{L}(z) = A + zB$. Then the ten special structures of L we consider in this paper are as follows.

- ***T*-symmetric:** $\mathrm{L}(\lambda)^T = \mathrm{L}(\lambda)$ for all $\lambda \in \mathbb{C}$, that is, $A^T = A$ and $B^T = B$.
- ***T*-skew-symmetric:** $\mathrm{L}(\lambda)^T = -\mathrm{L}(\lambda)$ for all $\lambda \in \mathbb{C}$, that is, $A^T = -A$ and $B^T = -B$.
- ***T*-even:** $\mathrm{L}(\lambda)^T = \mathrm{L}(-\lambda)$ for all $\lambda \in \mathbb{C}$, that is, $A^T = A$ and $B^T = -B$.
- ***T*-odd:** $\mathrm{L}(\lambda)^T = -\mathrm{L}(-\lambda)$ for all $\lambda \in \mathbb{C}$, that is, $A^T = -A$ and $B^T = B$.
- ***T*-palindromic:** $\mathrm{L}(\lambda)^T = \lambda\mathrm{L}(1/\lambda)$ for all $\lambda \neq 0$, that is, $B = A^T$.
- ***H*-Hermitian:** $\mathrm{L}(\lambda)^H = \mathrm{L}(\overline{\lambda})$ for all $\lambda \in \mathbb{C}$, that is, $A^H = A$ and $B^H = B$.
- ***H*-skew-Hermitian:** $\mathrm{L}(\lambda)^H = -\mathrm{L}(\overline{\lambda})$ for all $\lambda \in \mathbb{C}$, that is, $A^H = -A$ and $B^H = -B$.
- ***H*-even:** $\mathrm{L}(\lambda)^H = \mathrm{L}(-\overline{\lambda})$ for all $\lambda \in \mathbb{C}$, that is, $A^H = A$ and $B^H = -B$.
- ***H*-odd:** $\mathrm{L}(\lambda)^H = -\mathrm{L}(-\overline{\lambda})$ for all $\lambda \in \mathbb{C}$, that is, $A^H = -A$ and $B^H = B$.
- ***H*-palindromic:** $\mathrm{L}(\lambda)^H = \overline{\lambda}\,\mathrm{L}(1/\overline{\lambda})$ for all $\lambda \neq 0$, that is, $B = A^H$.

Let L be a regular pencil. We say that $(\lambda, x, y)$ is an eigentriple of L if $\lambda$ is an eigenvalue of L and $x$ and $y$, respectively, are right and left eigenvectors of L corresponding to $\lambda$; that is, $\mathrm{L}(\lambda)x = 0$ and $y^H\mathrm{L}(\lambda) = 0$. An eigentriple $(\lambda, x, y)$ is said to be normalized if $y^H y = x^H x = 1$. We consider only normalized eigentriples. Now, for ready reference, we collect some basic facts about eigenpairs of structured pencils in the following theorem.

THEOREM 2.1. *Let* $\mathrm{L} \in \mathbb{S}$ *be given by* $\mathrm{L}(z) = A + zB$. *Let* $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ *be an eigenpair of* L. *Then we have the following.*

| $\mathbb{S}$ | ***eigenvalue pairing*** | ***eigentriple*** | $x^TAx$ | $x^TBx$ |
|---|---|---|---|---|
| *T-symmetric* | $\lambda$ | $(\lambda, x, \overline{x})$ | *in* $\mathbb{C}$ | *in* $\mathbb{C}$ |
| *T-skew-symmetric* | $\lambda$ | $(\lambda, x, \overline{x})$ | 0 | 0 |
| *T-even* | $(\lambda, -\lambda)$ | $(\lambda, x, \overline{y}), (-\lambda, y, \overline{x})$ | 0 | 0 |
| *T-odd* | $(\lambda, -\lambda)$ | $(\lambda, x, \overline{y}), (-\lambda, y, \overline{x})$ | 0 | 0 *if* $\lambda \neq 0$ |
| *T-palindromic* | $(\lambda, 1/\lambda)$ | $(\lambda, x, \overline{y}), (1/\lambda, y, \overline{x})$ | 0 *if* $\lambda \neq -1$ | 0 *if* $\lambda \neq -1$ |
| | ***eigenvalue pairing*** | ***eigentriple*** | $x^HAx$ | $x^HBx$ |
| *H-Hermitian / H-skew-Hermitian* | $(\lambda, \overline{\lambda})$ | $(\lambda, x, y)$ $(\overline{\lambda}, y, x)$ | 0 *if* $\mathrm{im}\,\lambda \neq 0$ | 0 *if* $\mathrm{im}\,\lambda \neq 0$ |
| *H-even/ H-odd* | $(\lambda, -\overline{\lambda})$ | $(\lambda, x, y)$ $(-\overline{\lambda}, y, x)$ | 0 *if* $\mathrm{re}\,\lambda \neq 0$ | 0 *if* $\mathrm{re}\,\lambda \neq 0$ |
| *H-palindromic* | $(\lambda, 1/\overline{\lambda})$ | $(\lambda, x, y), (1/\overline{\lambda}, y, x)$ | 0 *if* $|\lambda| \neq 1$ | 0 *if* $|\lambda| \neq 1$ |

*Proof.* Note that when L is $T$-symmetric or $T$-skew-symmetric, we have $\mathrm{L}(\lambda)x = 0$ and $\overline{x}^H \mathrm{L}(\lambda) = 0$. Hence, $(\lambda, x, \overline{x})$ is an eigentriple of L. In particular, if L is $T$-skew-symmetric, then both $A$ and $B$ are skew-symmetric and, hence, $x^T Ax = 0 = x^T Bx$.

When L is $T$-even or $T$-odd, we have $\mathrm{L}(\lambda)^T = \mathrm{L}(-\lambda)$ or $\mathrm{L}(\lambda)^T = -\mathrm{L}(-\lambda)$. Hence, if $\mathrm{L}(\lambda)x = 0$ and $\mathrm{L}(-\lambda)y = 0$, then $\overline{x}^H \mathrm{L}(-\lambda) = 0$ and $\overline{y}^H \mathrm{L}(\lambda) = 0$. This shows $(\lambda, -\lambda)$ pairing of eigenvalues and that $(\lambda, x, \overline{y})$ and $(-\lambda, y, \overline{x})$ are eigentriples. When L is $T$-even, $B$ is skew-symmetric and, hence, $x^T Bx = 0$. Consequently, $x^T \mathrm{L}(\lambda)x = 0 \Rightarrow x^T Ax = 0$. Similarly, when L is $T$-odd, $A$ is skew-symmetric and, hence, $x^T Ax = 0$. Consequently, $x^T \mathrm{L}(\lambda)x = 0 \Rightarrow x^T Bx = 0$ whenever $\lambda \neq 0$. The proof is similar for $H$-Hermitian, $H$-skew-Hermitian, $H$-odd, and $H$-even pencils.

Now let L be $T$-palindromic given by $\mathrm{L}(z) = A + zA^T$. Suppose that $\lambda \neq 0$. Then $\mathrm{L}(\lambda)x = 0 \Rightarrow \overline{x}^H \mathrm{L}(1/\lambda) = 0$ which shows $(\lambda, 1/\lambda)$ pairing of eigenvalues. It also follows that $(\lambda, \overline{y}, x)$ is an eigentriple of L if and only if $(1/\lambda, \overline{x}, y)$ is an eigentriple of L. Note that $x^T \mathrm{L}(\lambda)x = 0 \Rightarrow x^T Ax + \lambda x^T Ax = 0$. Thus, if $\lambda \neq -1$, then $x^T Ax = 0$.

Similarly, when L is $H$-palindromic it follows that $(\lambda, y, x)$ is an eigentriple of L if and only if $(1/\overline{\lambda}, x, y)$ is an eigentriple of L. Now $x^H \mathrm{L}(\lambda)x \Rightarrow x^H Ax + \lambda\, \overline{x^H Ax} = 0 \Rightarrow |x^H Ax| = |\lambda|\ |x^H Ax| \Rightarrow (1 - |\lambda|)\ |x^H Ax| = 0$. Hence, for $|\lambda| \neq 1$, we have $x^H Ax = 0$. ∎

Next, we show that if $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ and $\mathrm{L} \in \mathbb{S}$, then there exists $\Delta\mathrm{L} \in \mathbb{S}$ such that $(\lambda, x)$ is an eigenpair of $\mathrm{L} + \Delta\mathrm{L}$, that is, $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$. Consequently, we have $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) < \infty$.

THEOREM 2.2. *Let* $\mathbb{S} \in \{T$*-symmetric*, $T$*-skew-symmetric*, $T$*-odd*, $T$*-even*, $H$*-Hermitian*, $H$*-skew-Hermitian*, $H$*-odd*, $H$*-even*$\}$ *and* $\mathrm{L} \in \mathbb{S}$ *be given by* $\mathrm{L}(z) = A + zB$. *Let* $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ *be such that* $x^H x = 1$. *Set* $r := -\mathrm{L}(\lambda)x$ *and define*

$$\Delta A := \begin{cases} -\overline{x}x^T Axx^H + \frac{1}{1+|\lambda|^2}\left[\overline{x}r^T + rx^H - 2\left(x^T r\right)\overline{x}x^H\right], & \textit{if } A = A^T, \\ -\frac{1}{1+|\lambda|^2}\left[\overline{x}r^T - rx^H\right], & \textit{if } A = -A^T, \end{cases}$$

$$\Delta B := \begin{cases} -\overline{x}x^T Bxx^H + \frac{\overline{\lambda}}{1+|\lambda|^2}\left[\overline{x}r^T + rx^H - 2\left(x^T r\right)\overline{x}x^H\right], & \textit{if } B = B^T, \\ -\frac{\overline{\lambda}}{1+|\lambda|^2}\left[\overline{x}r^T - rx^H\right], & \textit{if } B = -B^T, \end{cases}$$

*and*

$$\Delta A := \begin{cases} -xx^H Axx^H + \frac{1}{1+|\lambda|^2}\left[xr^H\left(I - xx^H\right) + \left(I - xx^H\right)rx^H\right], & \textit{if } \ A = A^H, \\ -xx^H Axx^H - \frac{1}{1+|\lambda|^2}\left[xr^H\left(I - xx^H\right) - \left(I - xx^H\right)rx^H\right], & \textit{if } \ A = -A^H. \end{cases}$$

$$\Delta B := \begin{cases} -xx^H Bxx^H + \frac{1}{1+|\lambda|^2}\left[\lambda xr^H\left(I - xx^H\right) + \overline{\lambda}\left(I - xx^H\right)rx^H\right], & \textit{if } \ B = B^H \\ -xx^H Bxx^H - \frac{1}{1+|\lambda|^2}\left[\lambda xr^H\left(I - xx^H\right) - \overline{\lambda}\left(I - xx^H\right)rx^H\right], & \textit{if } \ B = -B^H. \end{cases}$$

*Consider the pencil* $\Delta\mathrm{L}(z) = \Delta A + z\Delta B$. *Then* $\Delta\mathrm{L} \in \mathbb{S}$ *and* $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$.

*Proof.* The proof is computational and is easy to check. ∎

For palindromic pencils, we have the following result.

THEOREM 2.3. *Let* $\mathbb{S} \in \{T$*-palindromic*, $H$*-palindromic*$\}$ *and* $\mathrm{L} \in \mathbb{S}$ *be given by* $\mathrm{L}(z) = A + zA^*$, *where* $A^* = A^T$ *or* $A^* = A^H$. *Let* $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ *be such that*

$x^H x = 1$. *Set* $r := -\mathrm{L}(\lambda)x$ *and define*

$$\Delta A := \begin{cases} -\overline{x}x^T Axx^H + \frac{1}{1+|\lambda|^2}\left[\overline{\lambda}\overline{x}r^T\left(I - xx^H\right) + \left(I - \overline{x}x^T\right)rx^H\right], & \text{if } B = A^T, \\ -xx^H Axx^H + \frac{1}{1+|\lambda|^2}\left[\lambda xr^H\left(I - xx^H\right) + \left(I - xx^H\right)rx^H\right], & \text{if } B = A^H. \end{cases}$$

*Consider the pencil* $\Delta\mathrm{L}(z) = \Delta A + z(\Delta A)^*$. *Then* $\Delta\mathrm{L} \in \mathbb{S}$ *and* $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$.

*Proof.* The proof is computational and is easy to check. ∎

In section 3, we consider general classes of linearly structured pencils whose coefficient matrices are elements of certain Jordan and/or Lie algebras and show that for these pencils structured backward perturbation analysis ultimately reduces to that of one of the ten classes of structured pencils discussed above.

**3. Frobenius norm and structured backward errors.** Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$. Unless stated otherwise, we always assume that $x^H x = 1$. Let $\mathrm{L} \in \mathbb{S}$ be given by $\mathrm{L}(z) = A + zB$. In this section, we determine the structured backward error $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$ when $\mathbb{C}^{n\times n}$ is equipped with the Frobenius norm. Recall that the pencil norm defined in (2.1) is then given by $|||\mathrm{L}||| := \sqrt{\|A\|_F^2 + \|B\|_F^2} = \|[A\ B]\|_F$. Also recall that the unstructured backward error $\eta(\lambda, x, \mathrm{L})$ for the spectral norm as well as for the Frobenius norm on $\mathbb{C}^{n\times n}$ is given by $\eta(\lambda, x, \mathrm{L}) = \|\mathrm{L}(\lambda)x\|_2 / \|(1, \lambda)\|_2$.

THEOREM 3.1. *Let* $\mathbb{S}$ *be the space of* $T$*-symmetric pencils and let* $\mathrm{L} \in \mathbb{S}$ *be given by* $\mathrm{L}(z) = A + zB$. *Then for* $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$, *setting* $r := -\mathrm{L}(\lambda)x$, *we have*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \frac{\sqrt{2\|r\|_2^2 - |x^T r|^2}}{\|(1, \lambda)\|_2} \leq \sqrt{2}\,\eta(\lambda, x, \mathrm{L}).$$

*Define* $\Delta A := \frac{1}{1+|\lambda|^2}[\overline{x}r^T + rx^H - (r^T x)\overline{x}x^H]$ *and* $\Delta B := \frac{\overline{\lambda}}{1+|\lambda|^2}[\overline{x}r^T + rx^H - (r^T x)\,\overline{x}x^H]$ *and consider the pencil* $\Delta\mathrm{L}(z) = \Delta A + z\Delta B$. *Then* $\Delta\mathrm{L}$ *is* $T$*-symmetric,* $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$ *and* $|||\Delta\mathrm{L}||| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$.

*Proof.* By Theorem 2.2 there is a $\Delta\mathrm{L} \in \mathbb{S}$ such that $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$. Let $\Delta\mathrm{L}$ be given by $\Delta\mathrm{L}(z) = \Delta A + z\Delta B$. Then we have $(\Delta A + \lambda\Delta B)x = r$. Choose $Q_1 \in \mathbb{C}^{n\times(n-1)}$ such that $Q := [x,\ Q_1]$ is unitary. Then

$$\widetilde{\Delta A} := Q^T \Delta A Q = \begin{pmatrix} a_{11} & a_1^T \\ a_1 & A_1 \end{pmatrix},$$

$$\widetilde{\Delta B} := Q^T \Delta B Q = \begin{pmatrix} b_{11} & b_1^T \\ b_1 & B_1 \end{pmatrix},$$

$$Q^T r = \begin{pmatrix} x^T r \\ Q_1^T r \end{pmatrix},$$

where $A_1 = A_1^T$ and $B_1 = B_1^T$ are of size $n-1$. Since $\overline{Q}Q^T = I$, we have

$$\left(\overline{Q}\widetilde{\Delta A}Q^H + \lambda\overline{Q}\widetilde{\Delta B}Q^H\right)x = r \Rightarrow \left(\widetilde{\Delta A}Q^H + \lambda\widetilde{\Delta B}Q^H\right)x = Q^T r = \begin{pmatrix} x^T r \\ Q_1^T r \end{pmatrix}.$$

As $Q^H x = e_1$, the first column of the identity matrix, we have

$$\left(\widetilde{\Delta A} + \lambda\widetilde{\Delta B}\right)Q^H x = \begin{pmatrix} x^T r \\ Q_1^T r \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11} + \lambda b_{11} \\ a_1 + \lambda b_1 \end{pmatrix} = \begin{pmatrix} x^T r \\ Q_1^T r \end{pmatrix}.$$

This gives $a_{11} + \lambda b_{11} = x^T r$ and $a_1 + \lambda b_1 = Q_1^T r$ whose minimum norm solutions are

$$(a_1\quad b_1) = Q_1^T r \begin{pmatrix} 1 \\ \lambda \end{pmatrix}^{\dagger} \Rightarrow a_1 = \frac{1}{1+|\lambda|^2}Q_1^T r,\quad b_1 = \frac{\overline{\lambda}}{1+|\lambda|^2}Q_1^T r$$

and $(a_{11} \;\; b_{11}) = x^T r \begin{pmatrix} 1 \\ \lambda \end{pmatrix}^{\dagger} \Rightarrow a_{11} = \frac{1}{1+|\lambda|^2} x^T r, \;\; b_{11} = \frac{\overline{\lambda}}{1+|\lambda|^2} x^T r$. Hence, we have

$$\widetilde{\Delta A} = \begin{pmatrix} \frac{1}{1+|\lambda|^2} x^T r & \frac{1}{1+|\lambda|^2}\left(Q_1^T r\right)^T \\ \frac{1}{1+|\lambda|^2} Q_1^T r & A_1 \end{pmatrix}, \quad \widetilde{\Delta B} = \begin{pmatrix} \frac{\overline{\lambda}}{1+|\lambda|^2} x^T r & \frac{\overline{\lambda}}{1+|\lambda|^2}\left(Q_1^T r\right)^T \\ \frac{\overline{\lambda}}{1+|\lambda|^2} Q_1^T r & B_1 \end{pmatrix}.$$

This shows that the Frobenius norms of $\widetilde{\Delta A}$ and $\widetilde{\Delta B}$ are minimized when $A_1 = 0$ and $B_1 = 0$. Hence, $\|\Delta A\|_F^2 = \|\widetilde{\Delta A}\|_F^2 = |a_{11}|^2 + 2\,\|a_1\|_2^2$ and $\|\Delta B\|_F^2 = \|\widetilde{\Delta B}\|_F^2 = |b_{11}|^2 + 2\,\|b_1\|_2^2$. Note that $QQ^H = I \Rightarrow Q_1 Q_1^H = I - xx^H \Rightarrow \overline{Q}_1 Q_1^T = I - \overline{x}x^T$. Consequently, we have

$$|\!|\!|\Delta \mathrm{L}|\!|\!| = \left(\|\Delta A\|_F^2 + \|\Delta B\|_F^2\right)^{1/2} = \frac{\sqrt{|x^T r|^2 + 2\,\|(I - \overline{x}x^T)r\|_2^2}}{\|(1,\lambda)\|_2} = \frac{\sqrt{2\|r\|_2^2 - |x^T r|^2}}{\|(1,\lambda)\|_2}.$$

Next, we have

$$\begin{aligned} \Delta A = \overline{Q}\widetilde{\Delta A} Q^H &= \frac{1}{1+|\lambda|^2}\overline{x}x^T r x^H + \frac{1}{1+|\lambda|^2}\left[\overline{x} r^T Q_1 Q_1^H + \overline{Q}_1 Q_1^T r x^H\right] + \overline{Q}_1 A_1 Q_1^H \\ &= \frac{1}{1+|\lambda|^2}\left[\overline{x} r^T + r x^H - \left(r^T x\right)\overline{x}x^H\right] + \overline{Q}_1 A_1 Q_1^H, \\ \Delta B = \overline{Q}\widetilde{\Delta B} Q^H &= \frac{\overline{\lambda}}{1+|\lambda|^2}\overline{x}x^T r x^H + \frac{1}{1+|\lambda|^2}\left[\overline{\lambda x} r^T Q_1 Q_1^H + \overline{\lambda Q}_1 Q_1^T r x^H\right] + \overline{Q}_1 B_1 Q_1^H \\ &= \frac{\overline{\lambda}}{1+|\lambda|^2}\left[\overline{x} r^T + r x^H - \left(r^T x\right)\overline{x}x^H\right] + \overline{Q}_1 B_1 Q_1^H \end{aligned}$$

from which we obtain the desired pencil by setting $A_1 = 0$ and $B_1 = 0$. This completes the proof. ◻

Observe that if $Y$ is symmetric and $Yx = 0$, then $Y = (I - xx^H)^T Z (I - xx^H)$ for some symmetric matrix $Z$. Consequently, we have $\overline{Q_1} A_1 Q_1^H = (I - xx^H)^T Z_1 (I - xx^H)$ and $\overline{Q_1} B_1 Q_1^H = (I - xx^H)^T Z_2 (I - xx^H)$ for some symmetric matrices $Z_1$ and $Z_2$. Hence, from the proof of Theorem 3.1 we have following.

COROLLARY 3.2. *Let* $\mathrm{L}$ *be a* $T$*-symmetric pencil and* $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$. *Set* $r := -\mathrm{L}(\lambda)x$. *Let* $\mathrm{K}$ *be a* $T$*-symmetric pencil. Then* $\mathrm{L}(\lambda)x + \mathrm{K}(\lambda)x = 0$ *if and only if* $\mathrm{K}(z) = \Delta\mathrm{L}(z) + (I - xx^H)^T \mathrm{N}(z)(I - xx^H)$ *for some* $T$*-symmetric pencil* $\mathrm{N}$, *where* $\Delta\mathrm{L}$ *is the* $T$*-symmetric pencil given in Theorem* 3.1.

Next, we consider $T$-skew-symmetric pencils.

THEOREM 3.3. *Let* $\mathbb{S}$ *be the space of* $T$*-skew-symmetric pencils and let* $\mathrm{L} \in \mathbb{S}$ *be given by* $\mathrm{L}(z) = A + zB$. *Let* $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ *and* $r := -\mathrm{L}(\lambda)x$. *Then* $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \sqrt{2}\,\|r\|_2/\|(1,\lambda)\|_2 = \sqrt{2}\,\eta(\lambda, x, \mathrm{L})$. *Further, for the* $T$*-skew-symmetric pencil* $\Delta\mathrm{L}$ *given in Theorem* 2.2, *we have* $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$ *and* $|\!|\!|\Delta\mathrm{L}|\!|\!| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$.

*Proof.* As $A$ and $B$ are skew-symmetric, from the proof of Theorem 3.1, we have

$$\begin{aligned} \widetilde{\Delta A} &= Q^T \Delta A Q = \begin{pmatrix} 0 & a_1^T \\ -a_1 & A_1 \end{pmatrix}, \\ \widetilde{\Delta B} &= Q^T \Delta B Q = \begin{pmatrix} 0 & b_1^T \\ -b_1 & B_1 \end{pmatrix}, \\ Q^T r &= \begin{pmatrix} x^T r \\ Q_1^T r \end{pmatrix}, \end{aligned}$$

where $A_1$ and $B_1$ are skew-symmetric matrices of size $n-1$. Consequently, as before, we have $(\widetilde{\Delta A}+\lambda\widetilde{\Delta B})Q^H x = \begin{pmatrix} x^T r \\ Q_1^T r \end{pmatrix}$ which gives $\begin{pmatrix} 0 \\ -a_1-\lambda b_1 \end{pmatrix} = \begin{pmatrix} x^T r \\ Q_1^T r \end{pmatrix}$. Note that $x^T r = 0$ and the smallest norm solution of $-a_1 - \lambda b_1 = Q_1^T r$ is given by

$$(a_1 \;\; b_1) = Q_1^T r \begin{pmatrix} -1 \\ -\lambda \end{pmatrix}^{\dagger} \Rightarrow a_1 = -\frac{1}{1+|\lambda|^2} Q_1^T r, \;\; b_1 = -\frac{\overline{\lambda}}{1+|\lambda|^2} Q_1^T r.$$

Hence, we have

$$\Delta A = \overline{Q}\begin{pmatrix} 0 & -\frac{1}{1+|\lambda|^2}\left(Q_1^T r\right)^T \\ \frac{1}{1+|\lambda|^2} Q_1^T r & A_1 \end{pmatrix} Q^H,$$
$$\Delta B = \overline{Q}\begin{pmatrix} 0 & -\frac{\overline{\lambda}}{1+|\lambda|^2}\left(Q_1^T r\right)^T \\ \frac{\overline{\lambda}}{1+|\lambda|^2} Q_1^T r & B_1 \end{pmatrix} Q^H.$$

Setting $A_1 = 0$ and $B_1 = 0$ we obtain $\Delta \mathrm{L}$ such that $|||\Delta \mathrm{L}||| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \sqrt{2}\|r\|_2 / \|(1,\lambda)\|_2$.

Since $\overline{Q}_1 Q_1^T = I - \overline{x}x^T$, we have

$$\Delta A = -\frac{1}{1+|\lambda|^2}\left[\overline{x}r^T - rx^H\right] + \overline{Q}_1 A_1 Q_1^H \text{ and } \Delta B = -\frac{\overline{\lambda}}{1+|\lambda|^2}\left[\overline{x}r^T - rx^H\right] + \overline{Q}_1 B_1 Q_1^H.$$

Setting $A_1 = B_1 = 0$ we obtain the $T$-skew-symmetric pencil $\Delta\mathrm{L}$ given in Theorem 2.2. ∎

Using the fact that if $Y$ is skew-symmetric and $Yx = 0$ then $Y = (I - xx^H)^T Z (I - xx^H)$ for some skew-symmetric matrix $Z$, we obtain an analogue of Corollary 3.2 for $T$-skew-symmetric pencils.

Next, we derive structured backward errors for $T$-even and $T$-odd pencils.

THEOREM 3.4. *Let $\mathbb{S} \in \{T\text{-even}, T\text{-odd}\}$ and $\mathrm{L} \in \mathbb{S}$ be given by $\mathrm{L}(z) = A + zB$. Let $(\lambda, x) \in \mathbb{C}\times\mathbb{C}^n$ and $r := -\mathrm{L}(\lambda)x$. Then we have*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \sqrt{|x^T A x|^2 + \frac{2\|r\|_2^2 - 2|x^T r|^2}{1+|\lambda|^2}} = \frac{\sqrt{2\|r\|_2^2 + (|\lambda|^2 - 1)|x^T r|^2}}{\|(1,\lambda)\|_2}$$

*when* L *is $T$-even and*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \begin{cases} \sqrt{|x^T B x|^2 + \frac{2\|r\|_2^2 - 2|x^T r|^2}{1+|\lambda|^2}} = \dfrac{\sqrt{2\|r\|_2^2 + (|\lambda|^{-2} - 1)|x^T r|^2}}{\|(1,\lambda)\|_2}, & \text{if } \lambda \neq 0. \\ \sqrt{2}\,\eta(\lambda, x, \mathrm{L}), & \text{if } \lambda = 0, \end{cases}$$

*when* L *is $T$-odd. The pencil $\Delta\mathrm{L} \in \mathbb{S}$ given in Theorem 2.2 satisfies $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$ and $|||\Delta\mathrm{L}||| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$.*

*Proof.* First, assume that L is $T$-even. Then noting that $A = A^T$ and $B = -B^T$, the proof follows from similar arguments as those employed for $T$-symmetric and $T$-skew-symmetric pencils. Indeed, considering a unitary matrix $Q := [x, \; Q_1]$, we have

$$\widetilde{\Delta A} := Q^T \Delta A Q = \begin{pmatrix} a_{11} & a_1^T \\ a_1 & A_1 \end{pmatrix},$$
$$\widetilde{\Delta B} := Q^T \Delta B Q = \begin{pmatrix} 0 & b_1^T \\ -b_1 & B_1 \end{pmatrix},$$
$$Q^T r = \begin{pmatrix} x^T r \\ Q_1^T r \end{pmatrix},$$

where $A_1 = A_1^T$ and $B_1 = -B_1^T$ are of size $n-1$. Consequently, we have $(\widetilde{\Delta A} + \lambda\widetilde{\Delta B})Q^H x = \begin{pmatrix} x^T r \\ Q_1^T r \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11} \\ a_1 - \lambda b_1 \end{pmatrix} = \begin{pmatrix} x^T r \\ Q_1^T r \end{pmatrix}$. This gives $a_{11} = -x^T A x$. The smallest norm solution of $a_1 + \lambda b_1 = Q_1^T r$ is given by

$$(a_1 \;\; b_1) = Q_1^T r \begin{pmatrix} 1 \\ -\lambda \end{pmatrix}^{\dagger} \Rightarrow a_1 = \frac{1}{1+|\lambda|^2} Q_1^T r, \quad b_1 = -\frac{\overline{\lambda}}{1+|\lambda|^2} Q_1^T r.$$

Consequently, we have

$$\Delta A = \overline{Q} \begin{pmatrix} -x^T A x & \left(\frac{1}{1+|\lambda|^2} Q_1^T r\right)^T \\ \frac{1}{1+|\lambda|^2} Q_1^T r & A_1 \end{pmatrix} Q^H,$$
$$\Delta B = \overline{Q} \begin{pmatrix} 0 & \left(-\frac{\overline{\lambda}}{1+|\lambda|^2} Q_1^T r\right)^T \\ \frac{\overline{\lambda}}{1+|\lambda|^2} Q_1^T r & B_1 \end{pmatrix} Q^H.$$

Setting $A_1 = B_1 = 0$ and using the fact that $\overline{Q}_1 Q_1^T = I - \overline{x} x^T$, we obtain the pencil $\Delta \mathrm{L}$ such that

$$|||\Delta \mathrm{L}||| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \sqrt{|x^T A x|^2 + \frac{2\|r\|_2^2 - 2|x^T r|^2}{1+|\lambda|^2}}.$$

Now simplifying expressions for $\Delta A$ and $\Delta B$, we obtain

$$\Delta A = -\overline{x} x^T A x x^H + \frac{1}{1+|\lambda|^2}\left[\overline{x} r^T + r x^H - 2\left(x^T r\right)\overline{x} x^H\right] + \overline{Q}_1 A_1 Q_1^H,$$
$$\Delta B = -\frac{\overline{\lambda}}{1+|\lambda|^2}\left[\overline{x} r^T - r x^H\right] + \overline{Q}_1 B_1 Q_1^H.$$

Setting $A_1 = B_1 = 0$ we obtain the $T$-even pencil $\Delta \mathrm{L}$ given in Theorem 2.2.

When L is $T$-odd, the results follow by interchanging the role of $A$ and $B$. ∎

It follows from Theorem 3.4 that for a $T$-even pencil, we have $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) \leq \sqrt{2}\,\eta(\lambda, x, \mathrm{L})$ when $|\lambda| \leq 1$ and $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) \leq \|(1,\lambda)\|_2\,\eta(\lambda, x, \mathrm{L})$ when $|\lambda| > 1$. Similarly, for a $T$-odd pencil, we have $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) \leq \sqrt{2}\,\eta(\lambda, x, \mathrm{L})$ when $|\lambda| \geq 1$ and $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) \leq \|(1,\lambda^{-1})\|_2\,\eta(\lambda, x, \mathrm{L})$ when $\lambda \neq 0$ and $|\lambda| < 1$.

We mention that an analogue of Corollary 3.2 holds for $T$-even and $T$-odd pencils as well. Now, we consider a $T$-palindromic pencil $\mathrm{L}(z) = A + zA^T$.

THEOREM 3.5. *Let $\mathbb{S}$ be the space of $T$-palindromic pencils and $\mathrm{L} \in \mathbb{S}$ be given by $\mathrm{L}(z) = A + zA^T$. Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ and $r := -\mathrm{L}(\lambda)x$. Then we have*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \begin{cases} \sqrt{2}\sqrt{|x^T A x|^2 + \frac{\|r\|_2^2 - |x^T r|^2}{1+|\lambda|^2}} = \sqrt{2}\,\frac{\sqrt{\|r\|_2^2 - 2\mathsf{re}\lambda\,|x^T A x|^2}}{\|(1,\,\lambda)\|_2}, & \text{if } \lambda \neq -1, \\ \sqrt{2}\;\eta(\lambda, x, \mathrm{L}), & \text{if } \lambda = -1. \end{cases}$$

*In particular, we have $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \sqrt{2}\;\eta(\lambda, x, \mathrm{L})$, if $\lambda \in i\mathbb{R}$.*

*Now define*

$$\Delta A = \begin{cases} \frac{1}{1+|\lambda|^2}\left[\overline{\lambda}\overline{x} r^T\left(I - x x^H\right) + \left(I - \overline{x} x^T\right) r x^H\right], & \text{if } \lambda = -1, \\ -\overline{x} x^T A x x^H + \frac{1}{1+|\lambda|^2}\left[\overline{\lambda}\overline{x} r^T\left(I - x x^H\right) + \left(I - \overline{x} x^T\right) r x^H\right], & \text{if } \lambda \neq -1, \end{cases}$$

*and consider the pencil* $\Delta L(z) = \Delta A + z(\Delta A)^T$. *Then* $L(\lambda)x + \Delta L(\lambda)x = 0$ *and* $\|\Delta L\| = \eta^{\mathbb{S}}(\lambda, x, L)$.

*Proof.* By Theorem 2.3, there exists a $T$-palindromic pencil $\Delta L(z) = \Delta A + z\Delta A^T$ such that $(L(\lambda) + \Delta L(\lambda))x = 0$. Let $Q_1 \in \mathbb{C}^{n\times(n-1)}$ be such that $Q := [x \quad Q_1]$ is unitary. Then

$$\widetilde{\Delta A} := Q^T \Delta A Q = \begin{pmatrix} a_{11} & a_1^T \\ b_1 & A_1 \end{pmatrix}, \quad Q^T r = \begin{pmatrix} x^T r \\ Q_1^T r \end{pmatrix}.$$

Now, if $\lambda \neq -1$ then by Theorem 2.1, we have $x^T(\Delta A + A)x = 0 \Rightarrow x^T \Delta A x = -x^T A x$. Hence, we have $a_{11} = -x^T A x$. When $\lambda = -1$, we have $\lambda a_{11}^T + a_{11} = x^T r = 0$ for any $a_{11}$. Since the aim is to minimize the Frobenius norm of $\Delta A$, we set $a_{11} = 0$.

Next, the minimum norm solution of $a_1 \lambda + b_1 = Q_1^T r$ is given by

$$\begin{pmatrix} a_1 & b_1 \end{pmatrix} = Q_1^T r \begin{pmatrix} \lambda \\ 1 \end{pmatrix}^{\dagger} \Rightarrow a_1 = \frac{\overline{\lambda} Q_1^T r}{1+|\lambda|^2}, \quad b_1 = \frac{Q_1^T r}{1+|\lambda|^2}.$$

Therefore, when $\lambda = -1$, we have

$$\Delta A = \overline{Q} \begin{pmatrix} 0 & \left(\frac{\overline{\lambda} Q_1^T r}{1+|\lambda|^2}\right)^T \\ \frac{Q_1^T r}{1+|\lambda|^2} & A_1 \end{pmatrix} Q^H.$$

Setting $A_1 = 0$, we obtain $\eta^{\mathbb{S}}(\lambda, x, L) = \|\Delta L\| = \sqrt{2}\|r\|_2/\sqrt{1+|\lambda|^2} = \sqrt{2}\,\eta(\lambda, x, L)$. Since $Q_1 Q_1^H = I - xx^H \Rightarrow \overline{Q}_1 Q_1^T = I - \overline{x}x^T$, simplifying the expression for $\Delta A$, we obtain

$$\Delta A = \frac{1}{1+|\lambda|^2}\left[\overline{\lambda}\overline{x}r^T\left(I - xx^H\right) + \left(I - \overline{x}x^T\right) r x^H\right] + \overline{Q}_1 A_1 Q_1^H.$$

When $\lambda \neq -1$, we have

$$\Delta A = \overline{Q} \begin{pmatrix} -x^T A x & \left(\frac{\overline{\lambda} Q_1^T r}{1+|\lambda|^2}\right)^T \\ \frac{Q_1^T r}{1+|\lambda|^2} & A_1 \end{pmatrix} Q^H.$$

Setting $A_1 = 0$, we obtain

$$\eta^{\mathbb{S}}(\lambda, x, L) = \|\Delta L\| = \sqrt{2|x^T A x|^2 + \frac{2\|[I - \overline{x}x^T]r\|_2^2}{1+|\lambda|^2}} = \sqrt{2}\sqrt{|x^T A x|^2 + \frac{\|r\|_2^2 - |x^T r|^2}{1+|\lambda|^2}}$$

from which the result follows. Since $|x^T r|^2 = |x^T A x|^2(1+|\lambda|^2)$ when $\lambda \in i\mathbb{R}$, we have $\eta^{\mathbb{S}}(\lambda, x, L) = \sqrt{2}\|r\|_2/\|(1,\lambda)\|_2$, for $\lambda \in i\mathbb{R}$. Again, simplifying the expression for $\Delta A$, we obtain $\Delta A = -\overline{x}x^T A x x^H + \frac{1}{1+|\lambda|^2}[\overline{\lambda}\overline{x}r^T(I - xx^H) + (I - \overline{x}x^T)rx^H] + \overline{Q}_1 A_1 Q_1^H$. This completes the proof. ∎

Observe that from Theorem 3.5 we have $\eta^{\mathbb{S}}(\lambda, x, L) \leq \sqrt{2}\,\eta(\lambda, x, L)$ when $\mathsf{re}\lambda > 0$ and $\eta^{\mathbb{S}}(\lambda, x, L) \leq \|(1, \sqrt{|\mathsf{re}\lambda|}/|1+\lambda|)\|_2\, \eta(\lambda, x, L)$ when $\lambda \neq -1$ and $\mathsf{re}\lambda < 0$.

Note that if $Y \in \mathbb{C}^{n\times n}$ is such that $Yx = 0$ and $Y^T x = 0$, then $Y = (I - xx^H)^T Z(I - xx^H)$ for some matrix $Z$. Hence, from the proof of Theorem 3.5, we obtain an analogue of Corollary 3.2 for $T$-palindromic pencil. Indeed, if K is a $T$-palindromic pencil such that $L(\lambda)x + K(\lambda)x = 0$, then $K(z) = \Delta L(z) + (I - xx^H)^T N(z)(I - xx^H)$ for some $T$-palindromic pencil N, where $\Delta L$ is given in Theorem 3.5.

Now we turn to $H$-Hermitian, $H$-skew-Hermitian, $H$-even, $H$-odd, and $H$-palindromic pencils.

THEOREM 3.6. *Let* $\mathbb{S} \in \{H\text{-Hermitian}, H\text{-skew-Hermitian}\}$ *and* $\mathrm{L} \in \mathbb{S}$ *be given by* $\mathrm{L}(z) = A + zB$. *For* $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$, *set* $r := -\mathrm{L}(\lambda)x$. *Then we have*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \begin{cases} \dfrac{\sqrt{2\|r\|_2^2 - |x^H r|^2}}{\|(1,\lambda)\|_2} \leq \sqrt{2}\,\eta(\lambda, x, \mathrm{L}) & \text{if } \lambda \in \mathbb{R}, \\ \sqrt{|x^H A x|^2 + |x^H B x|^2 + \dfrac{2\|r\|_2^2 - 2|x^H r|^2}{1+|\lambda|^2}}, & \text{if } \lambda \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

*In particular, we have* $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \|r\|_2 = \sqrt{2}\,\eta(\lambda, x, \mathrm{L})$, *if* $\lambda = \pm i$.

*When* $\lambda \in \mathbb{R}$, *define*

$$\Delta A := \begin{cases} \frac{1}{1+\lambda^2}\left[xr^H + rx^H - \left(r^H x\right) xx^H\right], & \text{if } A = A^H \\ \frac{1}{1+\lambda^2}\left[rx^H - xr^H + \left(r^H x\right) xx^H\right], & \text{if } A = -A^H \end{cases}$$

$$\Delta B := \begin{cases} \frac{\lambda}{1+\lambda^2}\left[xr^H + rx^H - \left(r^H x\right) xx^H\right], & \text{if } B = B^H \\ \frac{\lambda}{1+\lambda^2}\left[rx^H - xr^H + \left(r^H x\right) xx^H\right], & \text{if } B = -B^H \end{cases}$$

*and consider the pencil* $\Delta\mathrm{L}(z) = \Delta A + z\Delta B$. *Then* $\Delta\mathrm{L} \in \mathbb{S}$, $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$, *and* $\|\Delta\mathrm{L}\| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$.

*When* $\lambda \in \mathbb{C} \setminus \mathbb{R}$, *the* $H$*-Hermitian/*$H$*-skew-Hermitian pencil* $\Delta\mathrm{L}$ *given in Theorem* 2.2 *satisfies* $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$ *and* $\|\Delta\mathrm{L}\| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$.

*Proof.* Suppose that $\mathrm{L}(z) = A + zB$ is $H$-Hermitian so that $A = A^H$ and $B = B^H$. By Theorem 2.2 there exists $H$-Hermitian pencil $\Delta\mathrm{L}(z) = \Delta A + z\Delta B$ such that $(\Delta A + \lambda\Delta B)x = r$. Again, choosing a unitary matrix $Q := [x,\ Q_1]$, we have

$$\widetilde{\Delta A} := Q^H \Delta A Q = \begin{pmatrix} a_{11} & a_1^H \\ a_1 & A_1 \end{pmatrix},$$
$$\widetilde{\Delta B} := Q^H \Delta B Q = \begin{pmatrix} b_{11} & b_1^H \\ b_1 & B_1 \end{pmatrix},$$
$$Q^H r = \begin{pmatrix} x^H r \\ Q_1^H r \end{pmatrix},$$

where $A_1 = A_1^H$ and $B_1 = B_1^H$ are of size $n-1$. This gives

$$(\widetilde{\Delta A} + \lambda\widetilde{\Delta B})Q^H x = \begin{pmatrix} x^H r \\ Q_1^H r \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11} + \lambda b_{11} \\ a_1 + \lambda b_1 \end{pmatrix} = \begin{pmatrix} x^H r \\ Q_1^H r \end{pmatrix}.$$

The minimum norm solution of $a_1 + \lambda b_1 = Q_1^H r$ is given by

$$(a_1 \;\; b_1) = Q_1^H r \begin{pmatrix} 1 \\ \lambda \end{pmatrix}^{\dagger} \Rightarrow a_1 = \frac{1}{1+|\lambda|^2} Q_1^H r, \quad b_1 = \frac{\overline{\lambda}}{1+|\lambda|^2} Q_1^H r.$$

For the equation $a_{11} + \lambda b_{11} = x^H r$, two cases arise.

*Case*-I: When $\lambda \in \mathbb{R}$, the minimum norm solution is given by

$$(a_{11} \;\; b_{11}) = x^H r \begin{pmatrix} 1 \\ \lambda \end{pmatrix}^{\dagger} \Rightarrow a_{11} = \frac{1}{1+\lambda^2} x^H r \in \mathbb{R}, \quad b_{11} = \frac{\lambda}{1+\lambda^2} x^H r \in \mathbb{R}.$$

Hence, we have

$$\Delta A = Q \begin{pmatrix} \frac{1}{1+\lambda^2} x^H r & \frac{1}{1+\lambda^2} \left(Q_1^H r\right)^H \\ \frac{1}{1+\lambda^2} Q_1^H r & A_1 \end{pmatrix} Q^H,$$
$$\Delta B = Q \begin{pmatrix} \frac{\lambda}{1+\lambda^2} x^H r & \left(\frac{\lambda}{1+\lambda^2} Q_1^H r\right)^H \\ \frac{\lambda}{1+\lambda^2} Q_1^H r & B_1 \end{pmatrix} Q^H.$$

Setting $A_1 = B_1 = 0$ and using the fact that $Q_1 Q_1^H = I - xx^H$, we have

$$\eta^{\mathbb{S}}(x, \lambda, \mathrm{L}) = |||\Delta \mathrm{L}||| = \frac{\sqrt{2\|r\|_2^2 - |x^H r|^2}}{\|(1, \lambda)\|_2}.$$

Now simplifying the expressions for $\Delta A$ and $\Delta B$, we have

$$\begin{aligned} \Delta A &= \frac{1}{1+\lambda^2} x x^H r x^H + \frac{1}{1+\lambda^2} \left[x r^H Q_1 Q_1^H + Q_1 Q_1^H r x^H\right] + Q_1 A_1 Q_1^H \\ &= \frac{1}{1+\lambda^2} \left[x r^H + r x^H - \left(r^H x\right) x x^H\right] + Q_1 A_1 Q_1^H, \\ \Delta B &= \frac{\lambda}{1+\lambda^2} x x^H r x^H + \frac{\lambda}{1+\lambda^2} \left[x r^H Q_1 Q_1^H + Q_1 Q_1^H r x^H\right] + Q_1 B_1 Q_1^H \\ &= \frac{\lambda}{1+\lambda^2} \left[x r^H + r x^H - \left(r^H x\right) x x^H\right] + Q_1 B_1 Q_1^H. \end{aligned}$$

Hence, the results follow.

*Case*-II: Suppose that $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then by Theorem 2.1, we have $x^H(A+\Delta A)x = 0$ and $x^H(B + \Delta B)x = 0$. Hence, we have $a_{11} = -x^H A x$ and $b_{11} = -x^H B x$. Consequently,

$$\Delta A = Q \begin{pmatrix} -x^H A x & \left(\frac{1}{1+|\lambda|^2} Q_1^H r\right)^H \\ \frac{1}{1+|\lambda|^2} Q_1^H r & A_1 \end{pmatrix} Q^H,$$
$$\Delta B = Q \begin{pmatrix} -x^H B x & \left(\frac{\lambda}{1+|\lambda|^2} Q_1^H r\right)^H \\ \frac{\lambda}{1+|\lambda|^2} Q_1^H r & B_1 \end{pmatrix} Q^H.$$

Setting $A_1 = B_1 = 0$, we obtain

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = |||\Delta \mathrm{L}||| = \sqrt{|x^H A x|^2 + |x^H B x|^2 + \frac{2\|(I - xx^H) r\|_2^2}{1+|\lambda|^2}}.$$

Hence, the result follows.

Now simplifying the expressions for $\Delta A$ and $\Delta B$, we have

$$\Delta A = -x x^H A x x^H + \frac{1}{1+|\lambda|^2} \left[x r^H \left(I - x x^H\right) + \left(I - x x^H\right) r x^H\right] + Q_1 A_1 Q_1^H,$$
$$\Delta B = -x x^H B x x^H + \frac{1}{1+|\lambda|^2} \left[\lambda x r^H \left(I - x x^H\right) + \overline{\lambda} \left(I - x x^H\right) r x^H\right] + Q_1 B_1 Q_1^H.$$

Setting $A_1 = B_1 = 0$, we obtain the $H$-Hermitian pencil $\Delta \mathrm{L}$ given in Theorem 2.2.

The proof is similar for the case when L is $H$-skew-Hermitian. ∎

Needless to mention that an analogue of Corollary 3.2 holds for $H$-Hermitian/$H$-skew-Hermitian pencils.

THEOREM 3.7. *Let* $\mathbb{S} \in \{H\text{-even, } H\text{-odd}\}$ *and* $\mathrm{L} \in \mathbb{S}$ *be given by* $\mathrm{L}(z) = A + zB$. *For* $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$, *set* $r := -\mathrm{L}(\lambda)x$. *Then we have*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \begin{cases} \dfrac{\sqrt{2\|r\|_2^2 - |x^H r|^2}}{\|(1,\, \lambda)\|_2} \leq \sqrt{2}\,\eta(\lambda, x, \mathrm{L}) & \text{if } \lambda \in i\mathbb{R}, \\ \sqrt{|x^H A x|^2 + |x^H B x|^2 + \dfrac{2\|r\|_2^2 - 2|x^H r|^2}{1 + |\lambda|^2}}, & \text{if } \lambda \in \mathbb{C} \setminus i\mathbb{R}. \end{cases}$$

*In particular, we have* $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \|r\|_2 = \sqrt{2}\ \eta(\lambda, x, \mathrm{L}),$ *if* $\lambda = \pm 1$.

*When* $\lambda \in i\mathbb{R}$, *define*

$$\Delta A := \begin{cases} \frac{1}{1+|\lambda|^2}\left[x r^H + r x^H - \left(r^H x\right) x x^H\right], & \text{if } A = A^H \\ \frac{1}{1+|\lambda|^2}\left[r x^H - x r^H + \left(r^H x\right) x x^H\right], & \text{if } A = -A^H \end{cases}$$

$$\Delta B := \begin{cases} \frac{-\lambda}{1+|\lambda|^2}\left[r x^H - x r^H + \left(r^H x\right) x x^H\right], & \text{if } B = B^H \\ \frac{-\lambda}{1+|\lambda|^2}\left[r x^H + x r^H - \left(r^H x\right) x x^H\right], & \text{if } B = -B^H \end{cases}$$

*and consider the pencil* $\Delta\mathrm{L}(z) = \Delta A + z\Delta B$. *Then* $\Delta\mathrm{L} \in \mathbb{S}$, $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$, *and* $\|\Delta\mathrm{L}\| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$.

*When* $\lambda \in \mathbb{C} \setminus i\mathbb{R}$, *the* $H$*-even/*$H$*-odd pencil* $\Delta\mathrm{L}$ *given in Theorem* 2.2 *satisfies* $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$ *and* $\|\Delta\mathrm{L}\| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$.

*Proof.* First, suppose that $\mathrm{L}(z) = A + zB$ is $H$ even. Then $A = A^H$ and $B = -B^H$. By Theorem 2.2 there exists $H$-even pencil $\Delta\mathrm{L}(z) = \Delta A + z\Delta B$ such that $\Delta\mathrm{L}(\lambda)x = r$. Now choosing a unitary matrix $Q := [x, Q_1]$ and noting that $\Delta A = \Delta A^H$, $\Delta B = -\Delta B^H$, we have

$$\Delta A := Q \begin{pmatrix} a_{11} & a_1^H \\ a_1 & A_1 \end{pmatrix} Q^H \text{ and } \Delta B = Q \begin{pmatrix} b_{11} & b_1^H \\ -b_1 & B_1 \end{pmatrix} Q^H,$$

where $A_1 = A_1^H$ and $B_1 = -B_1^H$ are matrices of size $n-1$. Then $\Delta\mathrm{L}(\lambda)x = r$ gives $\begin{pmatrix} a_{11} + \lambda b_{11} \\ a_1 - \lambda b_1 \end{pmatrix} = \begin{pmatrix} x^H r \\ Q_1^H r \end{pmatrix}$. The minimum norm solution of $a_1 - \lambda b_1 = Q_1^H r$ is given by

$$(a_1 \;\; b_1) = Q_1^H r \begin{pmatrix} 1 \\ -\lambda \end{pmatrix}^{\dagger} \Rightarrow a_1 = \frac{1}{1+|\lambda|^2} Q_1^H r, \quad b_1 = -\frac{\overline{\lambda}}{1+|\lambda|^2} Q_1^H r.$$

For the solution of $a_{11} + \lambda b_{11} = x^H r$ two cases arise. When $\lambda \in i\mathbb{R}$, the minimum norm solution is given by

$$(a_{11} \;\; b_{11}) = x^H r \begin{pmatrix} 1 \\ \lambda \end{pmatrix}^{\dagger} \Rightarrow a_{11} = \frac{1}{1+|\lambda|^2} x^H r \in \mathbb{R}, \quad b_{11} = \frac{\overline{\lambda}}{1+|\lambda|^2} x^H r \in i\mathbb{R}.$$

When $\lambda \in \mathbb{C} \setminus i\mathbb{R}$, by Theorem 2.1, $x^H(A + \Delta A)x = 0 = x^H(B + \Delta B)x \Rightarrow a_{11} =$

$-x^HAx$ and $b_{11} = -x^HBx$. Consequently, we have

$$\Delta A = Q\begin{pmatrix} \frac{1}{1+|\lambda|^2}x^Hr & \left(\frac{1}{1+|\lambda|^2}Q_1^Hr\right)^H \\ \frac{1}{1+|\lambda|^2}Q_1^Hr & A_1 \end{pmatrix}Q^H,$$

$$\Delta B = Q\begin{pmatrix} \frac{\overline{\lambda}}{1+|\lambda|^2}x^Hr & \left(-\frac{\overline{\lambda}}{1+|\lambda|^2}Q_1^Hr\right)^H \\ \frac{\overline{\lambda}}{1+|\lambda|^2}Q_1^Hr & B_1 \end{pmatrix}Q^H$$

when $\lambda \in i\mathbb{R}$ and

$$\Delta A = Q\begin{pmatrix} -x^HAx & \left(\frac{1}{1+|\lambda|^2}Q_1^Hr\right)^H \\ \frac{1}{1+|\lambda|^2}Q_1^Hr & A_1 \end{pmatrix}Q^H,$$

$$\Delta B = Q\begin{pmatrix} -x^HBx & \left(-\frac{\overline{\lambda}}{1+|\lambda|^2}Q_1^Hr\right)^H \\ \frac{\overline{\lambda}}{1+|\lambda|^2}Q_1^Hr & B_1 \end{pmatrix}Q^H$$

when $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. Hence, the desired results follow. Finally, reversing the role of $A$ and $B$ we obtain the results for the case when $\mathrm{L}(z) = A + zB$ is $H$-odd. ∎

We have the following result for $H$-palindromic pencils.

THEOREM 3.8. *Let $\mathbb{S}$ be the space of $H$-palindromic pencils and $\mathrm{L} \in \mathbb{S}$ be given by $\mathrm{L}(z) = A + zA^H$. Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ and $r := -\mathrm{L}(\lambda)x$. Then we have*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \begin{cases} \sqrt{2}\sqrt{|x^HAx|^2 + \dfrac{\|r\|_2^2 - |x^Hr|^2}{1+|\lambda|^2}} & \text{if } |\lambda| \neq 1, \\ \sqrt{\|r\|_2^2 - \frac{1}{2}|x^Hr|^2}, & \text{if } |\lambda| = 1. \end{cases}$$

*Now define*

$$\Delta A := \begin{cases} \frac{1}{1+|\lambda|^2}\left[rx^H + \lambda xr^H\left(I - xx^H\right)\right], & \text{if } |\lambda| = 1, \\ -xx^HAxx^H + \frac{1}{1+|\lambda|^2}\left[\lambda xr^H\left(I - xx^H\right) + \left(I - xx^H\right)rx^H\right], & \text{if } |\lambda| \neq 1, \end{cases}$$

*and consider $\Delta\mathrm{L}(z) := \Delta A + z(\Delta A)^H$. Then $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$ and $|||\Delta\mathrm{L}||| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$.*

*Proof.* Let $Q := [x, \;\; Q_1]$ be unitary. Then $\widetilde{\Delta A} := Q^H\Delta AQ = \begin{pmatrix} a_{11} & a_1^H \\ b_1 & A_1 \end{pmatrix}$ and $Q^Hr = \begin{pmatrix} x^Hr \\ Q_1^Hr \end{pmatrix}$. Hence, $\Delta\mathrm{L}(\lambda)x = r$ gives $\begin{pmatrix} \lambda a_{11}^H + a_{11} \\ \lambda a_1 + b_1 \end{pmatrix} = \begin{pmatrix} x^Hr \\ Q_1^Hr \end{pmatrix}$. If $|\lambda| \neq 1$, then by Theorem 2.1, we have $x^H(\Delta A + A)x = 0 \Rightarrow x^H\Delta Ax = -x^HAx$. Hence, we have $a_{11} = -x^HAx$. On the other hand, when $|\lambda| = 1$, the minimum norm solution is given by

$$(\overline{a}_{11} \;\; a_{11}) = x^Hr\begin{pmatrix} \lambda \\ 1 \end{pmatrix}^{\dagger} = \left(\frac{\overline{\lambda}x^Hr}{1+|\lambda|^2} \;\; \frac{x^Hr}{1+|\lambda|^2}\right).$$

Note that when $|\lambda| = 1$ we have $\overline{x^Hr} = \overline{\lambda}x^Hr$. Next, the minimum solution of $a_1\lambda + b_1 = Q_1^Hr$ is given by $(a_1, \;\; b_1) = Q_1^Hr\begin{pmatrix} \lambda \\ 1 \end{pmatrix}^{\dagger} = \left(\frac{\overline{\lambda}Q_1^Hr}{1+|\lambda|^2} \;\; \frac{Q_1^Hr}{1+|\lambda|^2}\right)$. Consequently, when

$|\lambda| \neq 1$, we have

$$\Delta A = Q \begin{pmatrix} -x^H Ax & \left(\frac{\overline{\lambda} Q_1^H r}{1+|\lambda|^2}\right)^H \\ \frac{Q_1^H r}{1+|\lambda|^2} & A_1 \end{pmatrix} Q^H.$$

Setting $A_1 = 0$, we obtain

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = |||\Delta \mathrm{L}||| = \sqrt{2|x^H Ax|^2 + \frac{2\|[I - xx^H]r\|_2^2}{1+|\lambda|^2}} = \sqrt{2}\sqrt{|x^H Ax|^2 + \frac{\|r\|_2^2 - |r^H x|^2}{1+|\lambda|^2}}.$$

Using the fact that $Q_1 Q_1^H = I - xx^H$, we have

$$\Delta A = -xx^H Axx^H + \frac{1}{1+|\lambda|^2}\left[\lambda xr^H\left(I - xx^H\right) + \left(I - xx^H\right) rx^H\right] + Q_1 A_1 Q_1^H.$$

Setting $A_1 = 0$, the result follows.

For the case when $|\lambda| = 1$, we have

$$\Delta A = Q \begin{pmatrix} \frac{x^H r}{1+|\lambda|^2} & \left(\frac{\overline{\lambda} Q_1^H r}{1+|\lambda|^2}\right)^H \\ \frac{Q_1^H r}{1+|\lambda|^2} & A_1 \end{pmatrix} Q^H.$$

Again, setting $A_1 = 0$ we obtain

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = |||\Delta \mathrm{L}||| = \sqrt{\|r\|_2^2 - \frac{1}{2}|x^H r|^2}.$$

Since $Q_1 Q_1^H = (I - xx^H)$, simplifying the expression for $\Delta A$, we obtain

$$\Delta A := \frac{1}{1+|\lambda|^2}\left[rx^H + \lambda xr^H\left(I - xx^H\right)\right] + Q_1 A_1 Q_1^H.$$

Hence, the proof. □

*Remark* 3.9. Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ with $x^H x = 1$ and $\mathbb{S} \in \{T$-symmetric, $T$-skew-symmetric, $T$-odd, $T$-even, $T$-palindromic, $H$-Hermitian, $H$-skew-Hermitian, $H$-odd, $H$-even, $H$-palindromic$\}$. For $\mathrm{L} \in \mathbb{S}$, consider the set

$$\mathbb{S}(\lambda, x, \mathrm{L}) := \{\mathrm{K} \in \mathbb{S} : \mathrm{L}(\lambda)x + \mathrm{K}(\lambda)x = 0\}.$$

Then $\mathbb{S}(\lambda, x, \mathrm{L}) \neq \emptyset$ and there exists a unique $\Delta \mathrm{L} \in \mathbb{S}(\lambda, x, \mathrm{L})$ such that

$$\min\{|||\mathrm{K}||| : \mathrm{K} \in \mathbb{S}(\lambda, x, \mathrm{L})\} = |||\Delta \mathrm{L}||| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L}).$$

Further, each pencil in $\mathbb{S}(\lambda, x, \mathrm{L})$ is of the form $\Delta \mathrm{L} + (I - xx^H)^* \, \mathrm{Z}(I - xx^H)$ for some $\mathrm{Z} \in \mathbb{S}$, where $*$ is either the transpose or the conjugate transpose depending upon the structure defined by $\mathbb{S}$. In other words, we have $\mathbb{S}(\lambda, x, \mathrm{L}) = \Delta \mathrm{L} + (I - xx^H)^* \, \mathbb{S}\,(I - xx^H)$.

We mention that the results obtained above are easily extended to the case of pencils having more general structures. Indeed, let $M$ be a unitary matrix such that $M^T = M$ or $M^T = -M$. Consider the Jordan algebra $\mathbb{J} := \{A \in \mathbb{C}^{n\times n} : M^{-1}A^T M = A\}$ and the Lie algebra $\mathbb{L} := \{A \in \mathbb{C}^{n\times n} : M^{-1}A^T M = -A\}$ associated with the scalar product $(x, y) \mapsto y^T M x$. Consider a pencil $\mathrm{L}(z) = A + zB$, where $A$ and $B$

are in $\mathbb{J}$ and/or in $\mathbb{L}$. Then the pencil $M\mathrm{L}$ given by $M\mathrm{L}(z) = MA + zMB$ is either $T$-symmetric, $T$-skew-symmetric, $T$-even, or $T$-odd. Hence, replacing $A, B$, and $r$ by $MA, MB$, and $Mr$, respectively, in the above results, we obtain corresponding results for the pencil L.

Similarly, when $M$ is unitary and $M = M^H$ or $M = -M^H$, we consider the Jordan algebra $\mathbb{J} := \{A \in \mathbb{C}^{n\times n} : M^{-1}A^H M = A\}$ and the Lie algebra $\mathbb{L} := \{A \in \mathbb{C}^{n\times n} : M^{-1}A^H M = -A\}$ associated with the scalar product $(x, y) \mapsto y^H Mx$. Now, let $\mathrm{L}(z) = A + zB$ be a pencil where $A$ and $B$ are in $\mathbb{J}$ and/or in $\mathbb{L}$. Then the pencil $M\mathrm{L}(z) = MA + zMB$ is either $H$-Hermitian, $H$-skew-Hermitian, $H$-even, or $H$-odd. Hence, replacing $A, B$, and $r$ by $MA, MB$, and $Mr$, respectively, in the above results, we obtain corresponding results for the pencil L. In particular, when $M := J$, where $J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathbb{C}^{2n\times 2n}$, the Jordan algebra $\mathbb{J}$ consists of skew-Hamiltonian matrices and the Lie algebra $\mathbb{L}$ consists of Hamiltonian matrices. So, for example, considering the pencil $\mathrm{L}(z) := A+zB$, where $A$ is Hamiltonian and $B$ is skew-Hamiltonian, we see that the pencil $J\mathrm{L}(z) = JA + zJB$ is $H$-even. Hence, extending the results obtained for $H$-even pencil to the case of L, we have the following.

THEOREM 3.10. *Let $\mathbb{S}$ be the space of pencils of the form $\mathrm{L}(z) = A + zB$, where $A$ is Hamiltonian and $B$ is skew-Hamiltonian. For $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$, set $r := -\mathrm{L}(\lambda)x$. Then we have*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \begin{cases} \dfrac{\sqrt{2\|r\|_2^2 - |x^H Jr|^2}}{\|(1,\, \lambda)\|_2} \leq \sqrt{2}\, \eta(\lambda, x, \mathrm{L}) & \text{if } \lambda \in i\mathbb{R}, \\ \sqrt{|x^H JAx|^2 + |x^H JBx|^2 + \dfrac{2\|r\|_2^2 - 2|x^H Jr|^2}{1 + |\lambda|^2}}, & \text{if } \lambda \in \mathbb{C} \setminus i\mathbb{R}. \end{cases}$$

We mention that Remark 3.9 remains valid for structured pencils in $\mathbb{S}$ whose coefficient matrices are elements of Jordan and/or Lie algebras associated with a scalar product considered above. In such a case the $*$ in $(I - xx^H)^*$ is the adjoint induced by the scalar product that defines the Jordan and Lie algebras.

**4. Spectral norm and structured backward errors.** Considering Frobenius norm on $\mathbb{C}^{n\times n}$, in the previous section, we have obtained structured backward error of an approximate eigenpair. In this section, we derive structured backward errors when $\mathbb{C}^{n\times n}$ is equipped with the spectral norm. Recall that the norm of a pencil $\mathrm{L}(z) = A + zB$ as defined in (2.1) is then given by $|||\mathrm{L}||| := (\|A\|_2^2 + \|B\|_2^2)^{1/2}$. Derivations of structured backward errors of approximate eigenpairs turn out to be much more difficult when $\mathbb{C}^{n\times n}$ is equipped with the spectral norm than in the case when $\mathbb{C}^{n\times n}$ is equipped with the Frobenius norm. We mention that for certain structures (e.g., $T$-symmetric and $T$-skew-symmetric) it is indeed possible to use structured mapping theorems given in [18, 2] to derive structured backward errors of approximate eigenpairs. However, for most structures (e.g., even, odd, palindromic, Hermitian, skew-Hermitian), the structured mapping theorems are not of much help for deriving structured backward errors. We overcome this difficulty by employing Davis–Kahan–Weinberger solutions of norm preserving dilation problem for Hilbert space operators.

The Davis–Kahan–Weinberger (DKW, in short) solutions of norm-preserving dilations of matrices can be stated as follows (for a more general version of the DKW theorem, see [9]).

THEOREM 4.1 (Davis–Kahan–Weinberger, [9]). *Let $A, B, C$ satisfy $\left\|\begin{pmatrix} A \\ B \end{pmatrix}\right\|_2 = \mu$ and $\left\|\begin{pmatrix} A & C \end{pmatrix}\right\|_2 = \mu$. Then there exists $D$ such that $\left\|\begin{pmatrix} A & C \\ B & D \end{pmatrix}\right\|_2 = \mu$. Indeed, those $D$*

*which have this property are exactly those of the form*

$$D = -KA^H L + \mu\left(I - KK^H\right)^{1/2} Z \left(I - L^H L\right)^{1/2},$$

*where* $K^H := (\mu^2 \mathrm{I} - A^H A)^{-1/2} B^H,\ L := (\mu^2 \mathrm{I} - AA^H)^{-1/2} C$*, and* $Z$ *is an arbitrary contraction, that is,* $\|Z\|_2 \leq 1$.

We now use the DKW theorem with $Z = 0$ and derive structured backward error of an approximate eigenpair. Recall that for $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$, our standing assumption is that $x^H x = 1$.

THEOREM 4.2. *Let* $\mathbb{S} \in \{T\text{-symmetric}, T\text{-skew-symmetric}\}$ *and* $\mathrm{L} \in \mathbb{S}$ *be given by* $\mathrm{L}(z) := A + zB$. *Let* $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ *and* $r := -\mathrm{L}(\lambda)x$. *Then we have*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \frac{\|r\|_2}{\|(1, \lambda)\|_2} = \eta(\lambda, x, \mathrm{L}).$$

*Now define*

$$\Delta A := \begin{cases} \frac{1}{1+|\lambda|^2}\left[\overline{x} r^T + r x^H - \left(r^T x\right)\overline{x} x^H - \frac{\overline{x^T r}\,\left(I - \overline{x} x^T\right) r r^T \left(I - x x^H\right)}{\|r\|_2^2 - |x^T r|^2}\right], & \textit{if } A = A^T, \\ -\frac{1}{1+|\lambda|^2}\left[\overline{x} r^T - r x^H\right], & \textit{if } A = -A^T. \end{cases}$$

$$\Delta B := \begin{cases} \frac{\overline{\lambda}}{1+|\lambda|^2}\left[\overline{x} r^T + r x^H - \left(r^T x\right)\overline{x} x^H - \frac{\overline{x^T r}\,\left(I - \overline{x} x^T\right) r r^T \left(I - x x^H\right)}{\|r\|_2^2 - |x^T r|^2}\right], & \textit{if } B = B^T, \\ -\frac{\overline{\lambda}}{1+|\lambda|^2}\left[\overline{x} r^T - r x^H\right], & \textit{if } B = -B^T, \end{cases}$$

*and consider the pencil* $\Delta\mathrm{L}(z) := \Delta A + z\Delta B$. *Then* $\Delta\mathrm{L} \in \mathbb{S},\ \mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$, *and* $\|\Delta\mathrm{L}\| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$.

*Proof.* Suppose that L is $T$-symmetric. Then from the proof of Theorem 3.1, we have

$$\Delta A = \overline{Q}\begin{pmatrix} \frac{x^T r}{1+|\lambda|^2} & \frac{1}{1+|\lambda|^2}\left(Q_1^T r\right)^T \\ \frac{1}{1+|\lambda|^2}\left(Q_1^T r\right) & A_1 \end{pmatrix} Q^H,$$

$$\Delta B = \overline{Q}\begin{pmatrix} \frac{\overline{\lambda}\, x^T r}{1+|\lambda|^2} & \frac{\overline{\lambda}}{1+|\lambda|^2}\left(Q_1^T r\right)^T \\ \frac{\overline{\lambda}}{1+|\lambda|^2}\left(Q_1^T r\right) & B_1 \end{pmatrix} Q^H,$$

such that $\Delta\mathrm{L}(\lambda)x + \mathrm{L}(\lambda)x = 0$. Now, for $\mu_{\Delta A} := \frac{\|r\|_2}{1+|\lambda|^2}$ and $\mu_{\Delta B} := \frac{|\lambda|\ \|r\|_2}{1+|\lambda|^2}$, by the DKW Theorem 4.1, we have $A_1 = -\frac{\overline{x^T r}\,(Q_1^T r)(Q_1^T r)^T}{(1+|\lambda|^2)\,(\|r\|_2^2 - |x^T r|^2)}$ and $B_1 = -\frac{\overline{\lambda}\,\overline{x^T r}\,(Q_1^T r)(Q_1^T r)^T}{(1+|\lambda|^2)\,(\|r\|_2^2 - |x^T r|^2)}$. This gives $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = (\|\Delta A\|_2^2 + \|\Delta B\|_2^2)^{1/2} = \frac{\|r\|_2}{\|(1,\lambda)\|_2}$. Simplifying expressions for $\Delta A$ and $\Delta B$, we obtain the desired results.

When L is $T$-skew-symmetric, from the proof of Theorem 3.3, we have

$$\Delta A = \overline{Q}\begin{pmatrix} 0 & -\frac{\left(Q_1^T r\right)^T}{1+|\lambda|^2} \\ \frac{1}{1+|\lambda|^2} Q_1^T r & A_1 \end{pmatrix} Q^H,\ \Delta B = \overline{Q}\begin{pmatrix} 0 & -\frac{\overline{\lambda}\left(Q_1^T r\right)^T}{1+|\lambda|^2} \\ \frac{\overline{\lambda}}{1+|\lambda|^2} Q_1^T r & B_1 \end{pmatrix} Q^H,$$

such that $\Delta\mathrm{L}(\lambda)x + \mathrm{L}(\lambda)x = 0$. Now, for $\mu_{\Delta A} := \frac{\|r\|_2}{1+|\lambda|^2}$ and $\mu_{\Delta B} := \frac{|\lambda|\ \|r\|_2}{1+|\lambda|^2}$, by the DKW Theorem 4.1, we obtain $A_1 = 0 = B_1$. Consequently, we have $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) =$

$(\|\Delta A\|_2^2 + \|\Delta B\|_2^2)^{1/2} = \|r\|_2/\|(1,\lambda)\|_2$. Simplifying the expressions for $\Delta A$ and $\Delta B$, we obtain the desired results. ∎

*Remark* 4.3. If $|x^T r| = \|r\|_2$, then $\|Q_1^T r\|_2 = 0$. In such a case, considering $A_1 = 0 = B_1$ we obtain the desired results.

Next, we consider $T$-even and $T$-odd pencils. Recall that for $z \in \mathbb{C}$, $\mathrm{sign}(z) := \overline{z}/|z|$ when $z \neq 0$ and $\mathrm{sign}(z) := 1$ when $z = 0$.

THEOREM 4.4. *Let* $\mathbb{S} \in \{T\text{-even}, T\text{-odd}\}$ *and* $\mathrm{L} \in \mathbb{S}$ *be given by* $\mathrm{L}(z) := A + zB$. *Let* $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ *and* $r := -\mathrm{L}(\lambda)x$. *Then we have*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \begin{cases} \sqrt{|x^T A x|^2 + \dfrac{\|r\|_2^2 - |x^T r|^2}{1+|\lambda|^2}} = \dfrac{\sqrt{\|r\|_2^2 + |\lambda|^2 |x^T r|^2}}{\|(1,\,\lambda)\|_2}, & \text{if L is } T\text{-even}, \\ \sqrt{|x^T B x|^2 + \frac{\|r\|_2^2 - |x^T r|^2}{1+|\lambda|^2}} = \dfrac{\sqrt{\|r\|_2^2 + |\lambda|^{-2}\, |x^T r|^2}}{\|(1,\,\lambda)\|_2}, & \text{if L is } T\text{-odd},\ \lambda \neq 0, \\ \eta(\lambda, x, \mathrm{L}), & \text{if L is } T\text{-odd}, \lambda = 0. \end{cases}$$

*Now, define*

$$\Delta A := \begin{cases} -\,\overline{x} x^T A x x^H + \frac{1}{1+|\lambda|^2}\left[\overline{x} r^T + r x^H - 2\left(x^T r\right)\overline{x} x^H\right] \\ +\,\frac{\overline{x^T A x}\,\left(I - \overline{x} x^T\right) r r^T \left(I - x x^H\right)}{\|r\|_2^2 - |x^T r|^2}, \quad \text{if } A = A^T, \\ \frac{1}{1+|\lambda|^2}\left[r x^H - \overline{x} r^T\right], \ \text{if } A = -A^T. \end{cases}$$

$$\Delta B := \begin{cases} -\,\overline{x} x^T B x x^H + \frac{\overline{\lambda}}{1+|\lambda|^2}\left[\overline{x} r^T + r x^H - 2\left(x^T r\right)\overline{x} x^H\right] \\ -\,\frac{\mathrm{sign}(\lambda)^2\,\overline{x^T B x}\,\left(I - \overline{x} x^T\right) r r^T \left(I - x x^H\right)}{\|r\|_2^2 - |x^T r|^2}, \quad \text{if } B = B^T, \\ -\,\frac{\overline{\lambda}}{1+|\lambda|^2}\left[\overline{x} r^T - r x^H\right], \ \text{if } B = -B^T, \end{cases}$$

*and consider the pencil* $\Delta\mathrm{L}(z) := \Delta A + z\Delta B$. *Then* $\Delta\mathrm{L} \in \mathbb{S}$, $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$ *and* $\|\Delta\mathrm{L}\| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$.

*Proof.* Suppose that L is $T$-even. Then from the proof of Theorem 3.4, we have

$$\Delta A = \overline{Q}\begin{pmatrix} -x^T A x & \frac{\left(Q_1^T r\right)^T}{1+|\lambda|^2} \\ \frac{Q_1^T r}{1+|\lambda|^2} & A_1 \end{pmatrix} Q^H,\ \Delta B = \overline{Q}\begin{pmatrix} 0 & -\frac{\overline{\lambda}}{1+|\lambda|^2}\left(Q_1^T r\right)^T \\ \frac{\overline{\lambda}}{1+|\lambda|^2}\left(Q_1^T r\right) & B_1 \end{pmatrix} Q^H,$$

such that $\Delta\mathrm{L}(\lambda)x + \mathrm{L}(\lambda)x = 0$. Now, for

$$\mu_{\Delta A} := \sqrt{|x^T A x|^2 + \frac{\|r\|_2^2 - |x^T r|^2}{(1+|\lambda|^2)^2}} \text{ and } \mu_{\Delta B} := \sqrt{\frac{|\lambda|^2(\|r\|_2^2 - |x^T r|^2)}{(1+|\lambda|^2)^2}}$$

by the DKW Theorem 4.1, we have $A_1 = \frac{\overline{x^T A x}}{\|r\|_2^2 - |x^T r|^2}(Q_1^T r)(Q_1^T r)^T$ and $B_1 = 0$. This gives

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \sqrt{|x^T A x|^2 + \frac{\|r\|_2^2 - |x^T r|^2}{1+|\lambda|^2}} = \frac{\sqrt{\|r\|_2^2 + |\lambda|^2 |x^T r|^2}}{\|(1,\,\lambda)\|_2}.$$

Simplifying the expressions for $\Delta A$ and $\Delta B$, we obtain the desired results. When L is $T$-odd, the results follow by interchanging the role of $A$ and $B$. ∎

It follows that for a $T$-even pencil we have $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) \leq \|(1, \lambda)\|_2\, \eta(\lambda, x, \mathrm{L})$ whereas for a $T$-odd pencil we have $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) \leq \|(1, \lambda^{-1})\|_2\, \eta(\lambda, x, \mathrm{L})$ when $\lambda \neq 0$.

THEOREM 4.5. *Let* $\mathbb{S} \in \{H\text{-Hermitian}, H\text{-skew-Hermitian}\}$ *and* $\mathrm{L} \in \mathbb{S}$ *be given by* $\mathrm{L}(z) := A + zB$. *Let* $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ *and* $r := -\mathrm{L}(\lambda)x$. *Then we have*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \begin{cases} \eta(\lambda, x, \mathrm{L}), & \text{if } \lambda \in \mathbb{R}, \\ \sqrt{|x^H A x|^2 + |x^H B x|^2 + \dfrac{\|r\|_2^2 - |x^H r|^2}{1 + |\lambda|^2}}, & \text{if } \lambda \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

*When* $\lambda \in \mathbb{R}$, *define*

$$\Delta A := \begin{cases} \frac{1}{1+\lambda^2} \left[ x r^H + r x^H - \left(r^H x\right) x x^H - \frac{x^H r \;\left(I - x x^H\right) r r^H \left(I - x x^H\right)}{\|r\|_2^2 - |x^H r|^2} \right], & \text{if } A = A^H, \\ \frac{1}{1+\lambda^2} [r x^H - x r^H + (r^H x) x x^H + \frac{r^H x (I - x x^H) r r^H (I - x x^H)}{\|r\|_2^2 - |x^H r|^2}], & \text{if } A = -A^H. \end{cases}$$

$$\Delta B := \begin{cases} \frac{\lambda}{1+\lambda^2} \left[ x r^H + r x^H - \left(r^H x\right) x x^H - \frac{x^H r \;\left(I - x x^H\right) r r^H \left(I - x x^H\right)}{\|r\|_2^2 - |x^H r|^2} \right], & \text{if } B = B^H, \\ \frac{\lambda}{1+\lambda^2} \left[ r x^H - x r^H + \left(r^H x\right) x x^H + \frac{r^H x \;\left(I - x x^H\right) r r^H \left(I - x x^H\right)}{\|r\|_2^2 - |x^H r|^2} \right], & \text{if } B = -B^H. \end{cases}$$

*When* $\lambda \in \mathbb{C} \setminus \mathbb{R}$, *define*

$$\Delta A := \begin{cases} - x x^H A x x^H + \frac{1}{1+|\lambda|^2} \left[ x r^H \left(I - x x^H\right) + \left(I - x x^H\right) r x^H \right] \\ + \frac{x^H A x \;\left(I - x x^H\right) r r^H \left(I - x x^H\right)}{\|r\|_2^2 - |x^H r|^2}, & \text{if } A = A^H, \\ - x x^H A x x^H + \frac{1}{1+|\lambda|^2} \left[ \left(I - x x^H\right) r x^H - x r^H \left(I - x x^H\right) \right] \\ + \frac{x^H A x \left(I - x x^H\right) r r^H \left(I - x x^H\right)}{\|r\|_2^2 - |x^H r|^2}, & \text{if } A = -A^H. \end{cases}$$

$$\Delta B := \begin{cases} - x x^H B x x^H + \frac{1}{1+|\lambda|^2} \left[ \lambda x r^H \left(I - x x^H\right) + \overline{\lambda} \left(I - x x^H\right) r x^H \right] \\ + \frac{x^H B x \;\left(I - x x^H\right) r r^H \left(I - x x^H\right)}{\|r\|_2^2 - |x^H r|^2}, & \text{if } B = B^H, \\ - x x^H B x x^H - \frac{\lambda}{1+|\lambda|^2} x r^H \left(I - x x^H\right) + \frac{\overline{\lambda}}{1+|\lambda|^2} \left(I - x x^H\right) r x^H \\ + \frac{x^H B x \left(I - x x^H\right) r r^H \left(I - x x^H\right)}{\|r\|_2^2 - |x^H r|^2}, & \text{if } B = -B^H. \end{cases}$$

*Consider* $\Delta\mathrm{L}(z) := \Delta A + z \Delta B$. *Then* $\Delta\mathrm{L} \in \mathbb{S}$, $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$, *and* $\|\Delta\mathrm{L}\| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$.

*Proof.* First, suppose that L is $H$-Hermitian. Assume that $\lambda \in \mathbb{R}$. Then $x^H r \in \mathbb{R}$. Now from the proof of Theorem 3.6, we have

$$\Delta A = Q \begin{pmatrix} \frac{1}{1+\lambda^2} x^H r & \frac{1}{1+\lambda^2} \left(Q_1^H r\right)^H \\ \frac{1}{1+\lambda^2} Q_1^H r & A_1 \end{pmatrix} Q^H$$

and

$$\Delta B = Q \begin{pmatrix} \frac{\lambda}{1+\lambda^2} x^H r & \frac{\lambda}{1+\lambda^2} \left(Q_1^H r\right)^H \\ \frac{\lambda}{1+\lambda^2} Q_1^H r & B_1 \end{pmatrix} Q^H$$

such that $\Delta\mathrm{L}(\lambda)x + \mathrm{L}(\lambda)x = 0$. For $\mu_{\Delta A} := \frac{\|r\|_2}{1+\lambda^2}$ and $\mu_{\Delta B} := \frac{|\lambda|\,\|r\|_2}{1+\lambda^2}$ by the DKW Theorem 4.1, we have $A_1 = -\frac{x^H r\ (Q_1^H r)(Q_1^H r)^H}{(1+\lambda^2)\ (\|r\|_2^2 - |x^H r|^2)}$, $B_1 = -\frac{\lambda\ x^H r\ (Q_1^H r)(Q_1^H r)^H}{(1+\lambda^2)\ (\|r\|_2^2 - |x^H r|^2)}$. This gives $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = (\|\Delta A\|_2^2 + \|\Delta B\|_2^2)^{1/2} = \frac{\|r\|_2}{\|(1,\lambda)\|_2}$. Now simplifying the expressions for $\Delta A$ and $\Delta B$, we obtain the desired results.

Next, suppose that $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Then again from the proof of Theorem 3.6, we have

$$\Delta A = Q \begin{pmatrix} -x^H Ax & \frac{1}{1+|\lambda|^2}\left(Q_1^H r\right)^H \\ \frac{1}{1+|\lambda|^2} Q_1^H r & A_1 \end{pmatrix} Q^H,$$
$$\Delta B = Q \begin{pmatrix} -x^H Bx & \frac{\lambda}{1+|\lambda|^2}\left(Q_1^H r\right)^H \\ \frac{\overline{\lambda}}{1+|\lambda|^2} Q_1^H r & B_1 \end{pmatrix} Q^H.$$

For, $\mu_{\Delta A} := \sqrt{|x^H Ax|^2 + \frac{\|r\|_2^2 - |x^H r|^2}{(1+|\lambda|^2)^2}}$, $\mu_{\Delta B} := \sqrt{|x^H Bx|^2 + \frac{|\lambda|^2(\|r\|_2^2 - |x^H r|^2)}{(1+|\lambda|^2)^2}}$, by the DKW Theorem 4.1, we have

$$A_1 = \frac{x^H Ax}{\|r\|_2^2 - |x^H r|^2}\left(Q_1^H r\right)\left(Q_1^H r\right)^H \text{ and } B_1 = \frac{x^H Bx}{\|r\|_2^2 - |x^T r|^2}\left(Q_1^H r\right)\left(Q_1^H r\right)^H.$$

Hence, we have $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \sqrt{|x^H Ax|^2 + |x^H Bx|^2 + \frac{\|r\|_2^2 - |x^H r|^2}{1+|\lambda|^2}}$. Now, simplifying the expressions for $\Delta A$ and $\Delta B$, we obtain the desired results. The proof is similar for the case when L is $H$-skew-Hermitian. ∎

We mention that when $Q_1^H r = 0$, the desired results follow by considering $A_1 = 0 = B_1$.

THEOREM 4.6. *Let $\mathbb{S} \in \{H\text{-even}, H\text{-odd}\}$ and $\mathrm{L} \in \mathbb{S}$ be given by $\mathrm{L}(z) := A + zB$. Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ and $r := -\mathrm{L}(\lambda)x$. Then we have*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \begin{cases} \eta(\lambda, x, \mathrm{L}), & \text{if } \lambda \in i\mathbb{R}, \\ \sqrt{|x^H Ax|^2 + |x^H Bx|^2 + \dfrac{\|r\|_2^2 - |x^H r|^2}{1+|\lambda|^2}}, & \text{if } \lambda \in \mathbb{C}\setminus i\mathbb{R}. \end{cases}$$

*When $\lambda \in i\mathbb{R}$, define*

$$\Delta A := \begin{cases} \frac{1}{1+|\lambda|^2}\left[xr^H + rx^H - \left(r^H x\right)xx^H - \frac{x^H r\ \left(I - xx^H\right)rr^H\left(I - xx^H\right)}{\|r\|_2^2 - |x^H r|^2}\right], & \text{if } A = A^H, \\ \frac{1}{1+|\lambda|^2}\left[rx^H - xr^H + \left(r^H x\right)xx^H + \frac{r^H x\left(I - xx^H\right)rr^H\left(I - xx^H\right)}{\|r\|_2^2 - |x^H r|^2}\right], & \text{if } A = -A^H. \end{cases}$$

$$\Delta B := \begin{cases} \frac{1}{1+|\lambda|^2}\left[\overline{\lambda} rx^H + \lambda xr^H - \lambda\left(r^H x\right)xx^H + \frac{\lambda x^H r\ \left(I - xx^H\right)rr^H\left(I - xx^H\right)}{\|r\|_2^2 - |x^H r|^2}\right], & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } B = B^H, & \\ \frac{1}{1+|\lambda|^2}\left[\overline{\lambda} xx^H rx^H - \lambda xr^H\left(I - xx^H\right) + \overline{\lambda}\left(I - xx^H\right)rx^H \right. & \\ \qquad \left. + \frac{\lambda r^H x\left(I - xx^H\right)rr^H\left(I - xx^H\right)}{\|r\|_2^2 - |x^H r|^2}\right], \qquad \text{if } B = -B^H. & \end{cases}$$

*When* $\lambda \in \mathbb{C} \setminus i\mathbb{R}$, *define*

$$\Delta A := \begin{cases} -xx^H Axx^H + \frac{1}{1+|\lambda|^2}\left[xr^H\left(I-xx^H\right)+\left(I-xx^H\right)rx^H\right] \\ +\frac{x^H Ax\ \left(I-xx^H\right)rr^H\left(I-xx^H\right)}{\|r\|_2^2-|x^H r|^2}, & \textit{if } A = A^H, \\ -xx^H Axx^H + \frac{1}{1+|\lambda|^2}\left[\left(I-xx^H\right)rx^H - xr^H\left(I-xx^H\right)\right] \\ +\frac{x^H Ax\left(I-xx^H\right)rr^H\left(I-xx^H\right)}{\|r\|_2^2-|x^H r|^2}, & \textit{if } A = -A^H. \end{cases}$$

$$\Delta B := \begin{cases} -xx^H Bxx^H + \frac{1}{1+|\lambda|^2}\left[\lambda xr^H\left(I-xx^H\right)+\overline{\lambda}\left(I-xx^H\right)rx^H\right] \\ +\frac{x^H Bx\ \left(I-xx^H\right)rr^H\left(I-xx^H\right)}{\|r\|_2^2-|x^H r|^2}, & \textit{if } B = B^H, \\ -xx^H Bxx^H - \frac{\lambda xr^H\left(I-xx^H\right)}{1+|\lambda|^2} + \frac{\overline{\lambda}\left(I-xx^H\right)rx^H}{1+|\lambda|^2} + \frac{x^H Bx\left(I-xx^H\right)rr^H\left(I-xx^H\right)}{\|r\|_2^2-|x^H r|^2}, \\ & \textit{if } B = -B^H. \end{cases}$$

*Consider* $\Delta\mathrm{L}(z) := \Delta A + z\Delta B$. *Then* $\Delta\mathrm{L} \in \mathbb{S}$, $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$, *and* $\|\Delta\mathrm{L}\| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$.

*Proof.* First, suppose that L is $H$-even. Next, assume that $\lambda \in i\mathbb{R}$. Then it follows that $x^H r \in \mathbb{R}$. Now from the proof of Theorem 3.7, we have

$$\Delta A = Q\begin{pmatrix} \frac{1}{1+|\lambda|^2}x^H r & \frac{1}{1+|\lambda|^2}(Q_1^H r)^H \\ \frac{1}{1+|\lambda|^2}Q_1^H r & A_1 \end{pmatrix}Q^H,$$

$$\Delta B := Q\begin{pmatrix} \frac{\overline{\lambda}}{1+|\lambda|^2}x^H r & -\frac{\lambda}{1+|\lambda|^2}(Q_1^H r)^H \\ \frac{\overline{\lambda}}{1+|\lambda|^2}Q_1^H r & B_1 \end{pmatrix}Q^H$$

such that $\Delta\mathrm{L}(\lambda)x + \mathrm{L}(\lambda)x = 0$. For $\mu_{\Delta A} := \frac{\|r\|_2}{1+|\lambda|^2}$, $\mu_{\Delta B} := \frac{|\lambda|\ \|r\|_2}{1+|\lambda|^2}$, by the DKW Theorem 4.1 we have

$$A_1 = -\frac{x^H r\ \left(Q_1^H r\right)\left(Q_1^H r\right)^H}{(1+|\lambda|^2)\ (\|r\|_2^2 - |x^H r|^2)} \text{ and } B_1 = \frac{\lambda\ x^H r\ \left(Q_1^H r\right)\left(Q_1^H r\right)^H}{(1+|\lambda|^2)\ (\|r\|_2^2 - |x^H r|^2)}.$$

This gives $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = (\|\Delta A\|_2^2 + \|\Delta B\|_2^2)^{1/2} = \frac{\|r\|_2}{\|(1,\lambda)\|_2}$. Simplifying expressions for $\Delta A$ and $\Delta B$, we obtain the desired result.

Now suppose that $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. The again from the proof of Theorem 3.7, we have

$$\Delta A = Q\begin{pmatrix} -x^H Ax & \frac{1}{1+|\lambda|^2}\left(Q_1^H r\right)^H \\ \frac{1}{1+|\lambda|^2}Q_1^H r & A_1 \end{pmatrix}Q^H,$$

$$\Delta B := Q\begin{pmatrix} -x^H Bx & -\frac{\lambda}{1+|\lambda|^2}\left(Q_1^H r\right)^H \\ \frac{\overline{\lambda}}{1+|\lambda|^2}Q_1^H r & B_1 \end{pmatrix}Q^H.$$

For $\mu_{\Delta A} = \sqrt{|x^H Ax|^2 + \frac{\|r\|_2^2 - |x^H r|^2}{(1+|\lambda|^2)^2}}$ and $\mu_{\Delta B} = \sqrt{|x^H Bx|^2 + \frac{|\lambda|^2(\|r\|_2^2 - |x^H r|^2)}{(1+|\lambda|^2)^2}}$, by the DKW Theorem 4.1, we have

$$A_1 = \frac{x^H Ax}{\|r\|_2^2 - |x^H r|^2}\left(Q_1^H r\right)\left(Q_1^H r\right)^H \text{ and } B_1 = \frac{x^H Bx}{\|r\|_2^2 - |x^T r|^2}\left(Q_1^H r\right)\left(Q_1^H r\right)^H.$$

Consequently, we have $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \sqrt{|x^H Ax|^2 + |x^H Bx|^2 + \frac{\|r\|_2^2 - |x^H r|^2}{1+|\lambda|^2}}$. Now, simplifying the expressions for $\Delta A$ and $\Delta B$, we obtain the desired results.

When L is $H$-odd, the desired results follow by interchanging the role of $A$ and $B$. ∎

As before, the above results are easily extended to the case of general structured pencils where the coefficient matrices are elements of Jordan and/or Lie algebras. In particular, for the pencil $\mathrm{L}(z) := A + zB$, where $A$ is Hamiltonian and $B$ is skew-Hamiltonian, we have the following result.

THEOREM 4.7. *Let $\mathbb{S}$ be the space of pencils of the form $\mathrm{L}(z) = A + zB$, where $A$ is Hamiltonian and $B$ is skew-Hamiltonian. Let $\mathrm{L} \in \mathbb{S}$ and $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$. Set $r := -\mathrm{L}(\lambda)x$. Then we have*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \begin{cases} \eta(\lambda, x, \mathrm{L}), & \text{if } \lambda \in i\mathbb{R} \\ \sqrt{|x^H JAx|^2 + |x^H JBx|^2 + \dfrac{\|r\|_2^2 - |x^H Jr|^2}{1+|\lambda|^2}}, & \text{if } \lambda \in \mathbb{C} \setminus i\mathbb{R}. \end{cases}$$

Now we consider palindromic pencils.

THEOREM 4.8. *Let $\mathbb{S}$ be the space of $T$-palindromic pencils and $\mathrm{L} \in \mathbb{S}$ be given by $\mathrm{L}(z) := A + zA^T$. Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ and $r := -\mathrm{L}(\lambda)x$. Then we have*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \begin{cases} \sqrt{2}\, \sqrt{|x^T Ax|^2 + \dfrac{|\lambda|^2\ (\|r\|_2^2 - |x^T r|^2)}{(1+|\lambda|^2)^2}}, & \text{if } |\lambda| > 1, \\ \sqrt{2}\, \sqrt{|x^T Ax|^2 + \dfrac{\|r\|_2^2 - |x^T r|^2}{(1+|\lambda|^2)^2}}, & \text{if } |\lambda| \leq 1 \text{ and } \lambda \neq \pm 1, \\ \eta(\lambda, x, \mathrm{L}), & \text{if } \lambda = \pm 1. \end{cases}$$

*Now define*

$$\Delta A := \begin{cases} -\overline{x} x^T Axx^H + \frac{1}{1+|\lambda|^2}\left[\overline{\lambda}\overline{x} r^T \left(I - xx^H\right) + \left(I - \overline{x}x^T\right) rx^H\right] \\ + \frac{\overline{\lambda}\ \overline{x^T Ax}\ \left(I - \overline{x}x^T\right) rr^T \left(I - xx^H\right)}{|\lambda|^2\ (\|r\|_2^2 - |x^T r|^2)}, & \text{if } |\lambda| > 1, \\ -\overline{x} x^T Axx^H + \frac{1}{1+|\lambda|^2}\left[\overline{\lambda}\overline{x} r^T \left(I - xx^H\right) + \left(I - \overline{x}x^T\right) rx^H\right] \\ + \frac{\overline{\lambda}\ \overline{x^T Ax}\ \left(I - \overline{x}x^T\right) rr^T \left(I - xx^H\right)}{\|r\|_2^2 - |x^T r|^2}, & \text{if } |\lambda| \leq 1 \text{ and } \lambda \neq -1, \\ \frac{1}{1+|\lambda|^2}\left[\overline{\lambda}\overline{x} r^T \left(\mathrm{I} - xx^H\right) + \left(\mathrm{I} - \overline{x}x^T\right) rx^H\right], & \text{if } \lambda = -1. \end{cases}$$

*Consider the pencil $\Delta\mathrm{L}(z) := \Delta A + z(\Delta A)^T$. Then $\Delta\mathrm{L} \in \mathbb{S}$, $\mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$ and $\|\Delta\mathrm{L}\| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$.*

*Proof.* Suppose that $\lambda \neq -1$. Then from the proof of Theorem 3.5, we have

$$\Delta A = \overline{Q} \begin{pmatrix} -x^T Ax & \frac{\overline{\lambda}}{1+|\lambda|^2} \left(Q_1^T r\right)^T \\ \frac{1}{1+|\lambda|^2} Q_1^T r & A_1 \end{pmatrix} Q^H$$

such that $\Delta\mathrm{L}(\lambda)x + \mathrm{L}(\lambda)x = 0$. Now for

$$\mu_{\Delta A} := \begin{cases} \sqrt{|x^T Ax|^2 + \frac{|\lambda|^2 (\|r\|_2^2 - |x^T r|^2)}{(1+|\lambda|^2)^2}}, & \text{if } |\lambda| > 1, \\ \sqrt{|x^T Ax|^2 + \frac{\|r\|_2^2 - |x^T r|^2}{(1+|\lambda|^2)^2}}, & \text{if } |\lambda| \leq 1, \end{cases}$$

by the DKW Theorem 4.1, we have

$$A_1 = \begin{cases} \frac{\overline{\lambda}\ \overline{x^T Ax}}{|\lambda|^2\ (\|r\|_2^2 - |x^T r|^2)} Q_1^T r \left(Q_1^T r\right)^T, & \text{if } |\lambda| > 1, \\ \frac{\overline{\lambda}\ \overline{x^T Ax}}{\|r\|_2^2 - |x^T r|^2} Q_1^T r \left(Q_1^T r\right)^T, & \text{if } |\lambda| \le 1. \end{cases}$$

Consequently, we have

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \begin{cases} \sqrt{2}\ \sqrt{|x^T Ax|^2 + \frac{|\lambda|^2\ (\|r\|_2^2 - |x^T r|^2)}{(1+|\lambda|^2)^2}}, & \text{if } |\lambda| > 1, \\ \sqrt{2}\ \sqrt{|x^T Ax|^2 + \frac{\|r\|_2^2 - |x^T r|^2}{(1+|\lambda|^2)^2}}, & \text{if } |\lambda| \le 1. \end{cases}$$

Now simplifying the expression for $\Delta A$, we obtain the desired results.

Next, suppose that $\lambda = -1$. Then again from the proof of Theorem 3.5, we have

$$\Delta A = \overline{Q} \begin{pmatrix} 0 & \frac{\overline{\lambda}}{1+|\lambda|^2} \left(Q_1^T r\right)^T \\ \frac{1}{1+|\lambda|^2} Q_1^T r & A_1 \end{pmatrix} Q^H. \text{ For } \mu_{\Delta A} := \frac{\|r\|_2}{1+|\lambda|^2},$$

by the DKW Theorem 4.1, we have $A_1 = 0$. Hence, $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \frac{1}{\sqrt{2}}\|r\|_2$. Simplifying the expression for $\Delta A$, we obtain the desired result. ∎

For $H$-palindromic pencils we have the following.

THEOREM 4.9. *Let* $\mathbb{S}$ *be the space of* $H$*-palindromic pencils and* $\mathrm{L} \in \mathbb{S}$ *be given by* $\mathrm{L}(z) := A + zA^H$. *Let* $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ *and* $r := -\mathrm{L}(\lambda)x$. *Then we have*

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \begin{cases} \sqrt{2}\ \sqrt{|x^H Ax|^2 + \dfrac{|\lambda|^2\ (\|r\|_2^2 - |x^H r|^2)}{(1+|\lambda|^2)^2}}, & \text{if } |\lambda| > 1, \\ \sqrt{2}\ \sqrt{|x^H Ax|^2 + \dfrac{\|r\|_2^2 - |x^H r|^2}{(1+|\lambda|^2)^2}}, & \text{if } |\lambda| < 1, \\ \eta(\lambda, x, \mathrm{L}), & \text{if } |\lambda| = 1. \end{cases}$$

*Now define*

$$\Delta A := \begin{cases} -xx^H Axx^H + \frac{1}{1+|\lambda|^2} \left[\lambda x r^H \left(I - xx^H\right) + \left(I - xx^H\right) r x^H\right] \\ + \frac{\overline{x^H Ax}\ \left(I - xx^H\right) rr^H \left(I - xx^H\right)}{\overline{\lambda}\,(\|r\|_2^2 - |x^H r|^2)}, & \text{if } |\lambda| > 1, \\ -xx^H Axx^H + \frac{1}{1+|\lambda|^2} \left[\lambda x r^H \left(I - xx^H\right) + \left(I - xx^H\right) r x^H\right] \\ + \frac{\lambda\ \overline{x^H Ax}\ \left(I - xx^H\right) rr^H \left(I - xx^H\right)}{\|r\|_2^2 - |x^H r|^2}, & \text{if } |\lambda| < 1, \\ \frac{1}{1+|\lambda|^2} \left[ r x^H + \lambda x r^H \left(\mathrm{I} - xx^H\right) - \frac{x^H r\ \left(I - xx^H\right) rr^H \left(I - xx^H\right)}{(\|r\|_2^2 - |x^H r|^2)} \right], & \text{if } |\lambda| = 1, \end{cases}$$

*and consider the pencil* $\Delta\mathrm{L}(z) := \Delta A + z(\Delta A)^H$. *Then* $\Delta\mathrm{L} \in \mathbb{S},\ \mathrm{L}(\lambda)x + \Delta\mathrm{L}(\lambda)x = 0$ *and* $|||\Delta\mathrm{L}||| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$.

*Proof.* First, suppose that $|\lambda| \ne 1$. Then from the proof of Theorem 3.8, we have

$$\Delta A = Q \begin{pmatrix} -x^H Ax & \frac{\lambda}{1+|\lambda|^2} \left(Q_1^H r\right)^H \\ \frac{1}{1+|\lambda|^2} Q_1^H r & A_1 \end{pmatrix} Q^H$$

such that $\Delta\mathrm{L}(\lambda)x + \mathrm{L}(\lambda)x = 0$. In this case, we have

$$\mu_{\Delta A} = \begin{cases} \sqrt{|x^H Ax|^2 + \frac{|\lambda|^2 (\|r\|_2^2 - |x^H r|^2)}{(1+|\lambda|^2)^2}}, & \text{if } |\lambda| > 1, \\ \sqrt{|x^H Ax|^2 + \frac{\|r\|_2^2 - |x^H r|^2}{(1+|\lambda|^2)^2}}, & \text{if } |\lambda| < 1. \end{cases}$$

Hence, by the DKW Theorem 4.1, we have

$$A_1 = \begin{cases} \frac{\lambda\ \overline{x^H A x}}{|\lambda|^2\ (\|r\|_2^2 - |x^H r|^2)} Q_1^H r (Q_1^H r)^H, & \text{if } |\lambda| > 1, \\ \frac{\lambda\ \overline{x^H A x}}{\|r\|_2^2 - |x^H r|^2} Q_1^H r (Q_1^H r)^H, & \text{if } |\lambda| < 1. \end{cases}$$

This gives

$$\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \begin{cases} \sqrt{2}\ \sqrt{|x^H A x|^2 + \frac{|\lambda|^2\ (\|r\|_2^2 - |x^H r|^2)}{(1+|\lambda|^2)^2}}, & \text{if } |\lambda| > 1, \\ \sqrt{2}\ \sqrt{|x^H A x|^2 + \frac{\|r\|_2^2 - |x^H r|^2}{(1+|\lambda|^2)^2}}, & \text{if } |\lambda| < 1. \end{cases}$$

Simplifying the expression for $\Delta A$, we obtain the desired result.

When $|\lambda| = 1$, again from the proof of Theorem 3.8, we have

$$\Delta A = Q \begin{pmatrix} \frac{x^H r}{1+|\lambda|^2} & \frac{\lambda}{1+|\lambda|^2} \left(Q_1^H r\right)^H \\ \frac{1}{1+|\lambda|^2} Q_1^H r & A_1 \end{pmatrix} Q^H.$$

Now, we have $\mu_{\Delta A} = \frac{\|r\|_2}{1+|\lambda|^2}$. Hence, by the DKW Theorem 4.1, we have

$$A_1 = -\frac{x^H r\ \left(I - x x^H\right) r r^H \left(I - x x^H\right)}{(1+|\lambda|^2)(\|r\|_2^2 - |x^H r|^2)}.$$

Consequently, we have $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \frac{\|r\|_2}{\sqrt{2}}$. Simplifying the expression for $\Delta A$, we obtain the desired result. $\square$

*Remark* 4.10. Let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ with $x^H x = 1$ and $\mathbb{S} \in \{T$-symmetric, $T$-skew-symmetric, $T$-odd, $T$-even, $T$-palindromic, $H$-Hermitian, $H$-skew-Hermitian, $H$-odd, $H$-even, $H$-palindromic$\}$. For $\mathrm{L} \in \mathbb{S}$, consider the set

$$\mathbb{S}(\lambda, x, \mathrm{L}) := \{\mathrm{K} \in \mathbb{S} : \mathrm{L}(\lambda)x + \mathrm{K}(\lambda)x = 0\}.$$

Then $\mathbb{S}(\lambda, x, \mathrm{L}) \neq \emptyset$ and $\min\{|\!|\!|\mathrm{K}|\!|\!| : \mathrm{K} \in \mathbb{S}(\lambda, x, \mathrm{L})\} = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$. Further,

$$\mathbb{S}_{\mathrm{opt}}(\lambda, x, \mathrm{L}) := \{\Delta\mathrm{L} \in \mathbb{S}(x, \lambda, \mathrm{L}) : |\!|\!|\Delta\mathrm{L}|\!|\!| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})\}$$

is an infinite set and is characterized by the DKW Theorem 4.1 by taking into account the nonzero contractions. Let $\Delta\mathrm{L} \in \mathbb{S}_{\mathrm{opt}}(\lambda, x, \mathrm{L})$. Then each pencil in $\mathbb{S}(\lambda, x, \mathrm{L})$ is of the form $\Delta\mathrm{L} + (I - xx^H)^*\, \mathrm{Z}(I - xx^H)$ for some $\mathrm{Z} \in \mathbb{S}$, where $*$ is either the transpose or the conjugate transpose depending upon the structure defined by $\mathbb{S}$. In other words, we have

$$\mathbb{S}(\lambda, x, \mathrm{L}) = \Delta\mathrm{L} + \left(I - xx^H\right)^*\ \mathbb{S}\ \left(I - xx^H\right).$$

Needless to mention that Remark 4.10 remains valid for structured pencils in $\mathbb{S}$ whose coefficient matrices are elements of Jordan and/or Lie algebras associated with a scalar product considered in the previous section. In such a case the $*$ in $(I - xx^H)^*$ is the adjoint induced by the scalar product that defines the Jordan and Lie algebras.

We now illustrate various structured and unstructured backward errors by numerical examples. We use MATLAB.7.0 for our computation. We generate $A$ and $B$ as follows:

```
>> randn('state',15), A = randn(50)+ i*randn(50); A = A ± A*;
>> randn('state',25), B = randn(50)+i*randn(50); B = B ± B*;
```

For $T$-palindromic/$H$-palindromic pencils, we generate $A$ and $B$ by

```
>> randn('state',15), A = randn(50)+ i*randn(50); B = A*;
```

Here $A^* = A^T$ or $A^* = A^H$. Finally, we compute $(\lambda, x)$ by

```
>> [V,D] = eig(A,B); λ = -D(2,2); x = V(:,2)/norm(V(:,2));
```

We denote by $\eta_F^{\mathbb{S}}(\lambda, x, \mathrm{L})$ and $\eta_2^{\mathbb{S}}(\lambda, x, \mathrm{L})$ the backward error $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$ when $\mathbb{C}^{n\times n}$ is equipped with the Frobenius norm and the spectral norm, respectively. Note that $\eta(\lambda, x, \mathrm{L})$ is the same for the spectral and the Frobenius norms. Then we have the following.

| $\mathbb{S}$ | $\eta(\lambda, x, \mathrm{L})$ | $\eta_F^{\mathbb{S}}(\lambda, x, \mathrm{L})$ | $\eta_2^{\mathbb{S}}(\lambda, x, \mathrm{L})$ |
|---|---|---|---|
| $T$-symm | 1.387705737323579e−014 | 1.959539856593202e−014 | 1.387705737323579e−014 |
| $T$-skew-symm | 1.796046101865378e−014 | 2.539992755905347e−014 | 1.796046101865378e−014 |
| $T$-even | 2.219610496439476e−014 | 3.211055813711074e−014 | 2.324926535413804e−014 |
| $T$-odd | 1.559070464273151e−014 | 2.204626223816091e−014 | 1.559075824083717e−014 |
| $T$-palindromic | 1.068704043320177e−014 | 1.512088705618463e−014 | 1.487010794022381e−014 |
| $H$-Herm | 2.076731533185186e−014 | 2.947235222707197e−014 | 2.106896507205170e−014 |
| $H$-skew-Herm | 1.714743310005108e−014 | 2.489567700503872e−014 | 1.811820338752170e−014 |
| $H$-even | 1.590165856939442e−014 | 2.299115486213681e−014 | 1.663718384482337e−014 |
| $H$-odd | 2.343032834027323e−014 | 3.472518481940936e−014 | 2.566511276851151e−014 |
| $H$-palindromic | 9.161344100487524e−015 | 1.298942035829892e−014 | 1.296310627878570e−014 |

Note that structured backward errors are bigger than or equal to unstructured backward errors but they are marginally so. On the other hand, structured condition numbers are less than or equal to unstructured condition numbers [13, 6]. Consequently, structured backward errors when combined with structured condition numbers provide almost the same approximate upper bounds on the errors in the computed eigenelements as do their unstructured counterparts. We mention that the MATLAB `eig` command does not ensure spectral symmetry in the computed eigenvalues.

**5. Structured pseudospectra of structured pencils.** Let $\mathrm{L} \in \mathbb{A}^{n\times n}$ be a regular pencil. For $\lambda \in \mathbb{C}$, the backward error of $\lambda$ as an approximate eigenvalue of L is given by $\eta(\lambda, \mathrm{L}) := \min\{\eta(\lambda, x, \mathrm{L}) : x \in \mathbb{C}^n \text{ and } \|x\|_2 = 1\}$. Since $\eta(\lambda, x, \mathrm{L}) = \|\mathrm{L}(\lambda)x\|_2/\|(1, \lambda)\|_2$, it follows that for the spectral norm as well as for the Frobenius norm on $\mathbb{C}^{n\times n}$, we have $\eta(\lambda, \mathrm{L}) := \sigma_{\min}(\mathrm{L}(\lambda))/\|(1, \lambda)\|_2$. Similarly, we define structured backward error of an approximate eigenvalue $\lambda$ of $\mathrm{L} \in \mathbb{S}$ by

$$\eta^{\mathbb{S}}(\lambda, \mathrm{L}) := \min\left\{\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) : x \in \mathbb{C}^n \text{ and } \|x\|_2 = 1\right\}.$$

Note that backward errors of approximate eigenvalues and pseudospectra of a pencil are closely related. For $\epsilon > 0$, the unstructured $\epsilon$-pseudospectrum of L, denoted by $\Lambda_\epsilon(\mathrm{L})$, is given by [3]

$$\Lambda_\epsilon(\mathrm{L}) = \bigcup_{|||\Delta\mathrm{L}|||\leq\epsilon} \{\Lambda(\mathrm{L} + \Delta\mathrm{L}) : \Delta\mathrm{L} \in \mathbb{A}^{n\times n}\}.$$

Obviously, we have $\Lambda_\epsilon(\mathrm{L}) = \{z \in \mathbb{C} : \eta(z, \mathrm{L}) \leq \epsilon\}$, assuming, for simplicity, that $\infty \notin \Lambda_\epsilon(\mathrm{L})$. For the sake of simplicity, for rest of this section, we make an implicit assumption that $\infty \notin \Lambda_\epsilon(\mathrm{L})$. We observe the following.

- Since $\eta(\lambda, \mathrm{L})$ is the same for the spectral norm and the Frobenius norm on $\mathbb{C}^{n\times n}$, it follows that $\Lambda_\epsilon(\mathrm{L})$ is the same for the spectral and the Frobenius norms.

Similarly, when $\mathrm{L} \in \mathbb{S}$, we define the structured $\epsilon$-pseudospectrum of L, denoted by $\Lambda_\epsilon^{\mathbb{S}}(\mathrm{L})$, by

$$\Lambda_\epsilon^{\mathbb{S}}(\mathrm{L}) := \bigcup_{|||\Delta \mathrm{L}||| \leq \epsilon} \{\Lambda(\mathrm{L} + \Delta\mathrm{L}) : \Delta\mathrm{L} \in \mathbb{S}\}.$$

Then it follows that $\Lambda_\epsilon^{\mathbb{S}}(\mathrm{L}) = \{z \in \mathbb{C} : \eta^{\mathbb{S}}(\lambda, \mathrm{L}) \leq \epsilon\}$.

THEOREM 5.1. *Let* $\mathbb{S} \in \{T$*-symmetric,* $T$*-skew-symmetric*$\}$ *and* $\mathrm{L} \in \mathbb{S}$. *Let* $\lambda \in \mathbb{C}$. *Then for the spectral norm on* $\mathbb{C}^{n\times n}$, *we have* $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = \eta(\lambda, \mathrm{L})$ *and* $\Lambda_\epsilon^{\mathbb{S}}(\mathrm{L}) = \Lambda_\epsilon(\mathrm{L})$. *For the Frobenius norm on* $\mathbb{C}^{n\times n}$, *we have* $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = \sqrt{2}\,\eta(\lambda, \mathrm{L})$ *and* $\Lambda_\epsilon^{\mathbb{S}}(\mathrm{L}) = \Lambda_{\epsilon/\sqrt{2}}(\mathrm{L})$ *when* L *is* $T$*-skew-symmetric, and* $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = \eta(\lambda, \mathrm{L})$ *and* $\Lambda_\epsilon^{\mathbb{S}}(\mathrm{L}) = \Lambda_\epsilon(\mathrm{L})$ *when* L *is* $T$*-symmetric.*

*Proof.* For the spectral norm, by Theorem 4.2, we have $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \eta(\lambda, x, \mathrm{L})$ for all $x$. Consequently, we have $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = \eta(\lambda, \mathrm{L})$. Hence, the result follows.

For the Frobenius norm, the result follows from Theorem 3.3 when L is $T$-skew-symmetric. So, suppose that L is $T$-symmetric. Then $\mathrm{L}(\lambda) \in \mathbb{C}^{n\times n}$ is symmetric. Consider the Takagi factorization $\mathrm{L}(\lambda) = U\Sigma U^T$, where $U$ is unitary and $\Sigma$ is a diagonal matrix containing singular values of $\mathrm{L}(\lambda)$ (appear in descending order). Set $\sigma := \Sigma(n, n)$ and $u := U(:, n)$. Then we have $\mathrm{L}(\lambda)\overline{u} = \sigma u$. Now define $\Delta A := -\frac{\sigma\, u u^T}{1+|\lambda|^2}, \Delta B := -\frac{\overline{\lambda}\,\sigma\, u u^T}{1+|\lambda|^2}$, and consider the pencil $\Delta\mathrm{L}(z) = \Delta A + z\Delta B$. Then $\Delta\mathrm{L}$ is $T$-symmetric and $\mathrm{L}(\lambda)\overline{u} + \Delta\mathrm{L}(\lambda)\overline{u} = 0$. Notice that, for the spectral norm and the Frobenius norm on $\mathbb{C}^{n\times n}$, we have $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) \leq |||\Delta\mathrm{L}||| = \sigma/\|(1, \lambda)\|_2 = \eta(\lambda, \mathrm{L})$ and, hence, $\Lambda_\epsilon(\mathrm{L}) = \Lambda_\epsilon^{\mathbb{S}}(\mathrm{L})$. This completes the proof. ∎

When L is $T$-symmetric, the above proof shows how to construct a $T$-symmetric pencil $\Delta\mathrm{L}$ such that $\lambda \in \Lambda(\mathrm{L} + \Delta\mathrm{L})$ and $|||\Delta\mathrm{L}||| = \eta^{\mathbb{S}}(\lambda, \mathrm{L})$. When L is $T$-skew-symmetric, using Takagi factorization of the complex skew-symmetric matrix $\mathrm{L}(\lambda)$, one can construct a $T$-skew-symmetric pencil $\Delta\mathrm{L}$ such that $\lambda \in \Lambda(\mathrm{L} + \Delta\mathrm{L})$ and $|||\Delta\mathrm{L}||| = \eta^{\mathbb{S}}(\lambda, \mathrm{L})$. Indeed, consider the Takagi factorization $\mathrm{L}(\lambda) = U\mathrm{diag}(d_1, \dots, d_m)$ $U^T$, where $U$ is unitary, $d_j := \begin{pmatrix} 0 & s_j \\ -s_j & 0 \end{pmatrix}$, $s_j \in \mathbb{C}$ is nonzero, and $|s_j|$ are singular values of $\mathrm{L}(\lambda)$. Here the blocks $d_j$ appear in descending order of magnitude of $|s_j|$. Note that $\mathrm{L}(\lambda)\overline{U} = U\mathrm{diag}(d_1, \dots, d_m)$. Let $u := U(:, n-1 : n)$. Then $\mathrm{L}(\lambda)\overline{u} = u d_m = u d_m u^T \overline{u}$. Now define

$$\Delta A := -\frac{u d_m u^T}{1 + |\lambda|^2}, \;\; \Delta B := -\frac{\overline{\lambda}\, u d_m u^T}{1 + |\lambda|^2}$$

and consider $\Delta\mathrm{L}(z) := \Delta A + z\Delta B$. Then $\Delta\mathrm{L}$ is $T$-skew-symmetric and $\mathrm{L}(\lambda)\overline{u} + \Delta\mathrm{L}(\lambda)\overline{u} = 0$. For the spectral norm on $\mathbb{C}^{n\times n}$, we have $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = |||\Delta\mathrm{L}||| = \sigma_{\min}(\mathrm{L}(\lambda))/\|(1, \lambda)\|_2 = \eta(\lambda, \mathrm{L})$ and for the Frobenius norm on $\mathbb{C}^{n\times n}$, we have $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = |||\Delta\mathrm{L}||| = \sqrt{2}\,\sigma_{\min}(\mathrm{L}(\lambda))/\|(1, \lambda)\|_2 = \sqrt{2}\,\eta(\lambda, \mathrm{L})$.

We denote the unit circle in $\mathbb{C}$ by $\mathbb{T}$, that is, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Then for $T$-even and $T$-odd pencils we have the following.

THEOREM 5.2. *Let* $\mathbb{S} \in \{T$*-even,* $T$*-odd*$\}$ *and* $\mathrm{L} \in \mathbb{S}$. *Let* $\lambda \in \mathbb{T}$. *Then for the Frobenius norm on* $\mathbb{C}^{n\times n}$, *we have* $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = \sqrt{2}\,\eta(\lambda, \mathrm{L})$ *and* $\Lambda_\epsilon^{\mathbb{S}}(\mathrm{L}) \cap \mathbb{T} = \Lambda_{\epsilon/\sqrt{2}}(\mathrm{L}) \cap \mathbb{T}$.

*Proof.* Let $\lambda \in \mathbb{T}$. Then by Theorem 3.4, we have $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \frac{\sqrt{2}\,\|\mathrm{L}(\lambda)x\|_2}{\|(1,\lambda)\|_2}$ for all $x$ such that $\|x\|_2 = 1$. Hence, taking minimum over $\|x\|_2 = 1$, we obtain the desired results. ∎

THEOREM 5.3. *Let* $\mathbb{S} \in \{H\text{-Hermitian}, H\text{-skew-Hermitian}\}$ *and* $\mathrm{L} \in \mathbb{S}$. *Let* $\lambda \in \mathbb{R}$. *Then for the spectral and the Frobenius norms on* $\mathbb{C}^{n\times n}$, *we have* $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = \eta(\lambda, \mathrm{L})$ *and* $\Lambda^{\mathbb{S}}_{\epsilon}(\mathrm{L}) \cap \mathbb{R} = \Lambda_{\epsilon}(\mathrm{L}) \cap \mathbb{R}$. *Also when* $\lambda = \pm i$, *for the Frobenius norm, we have* $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = \sqrt{2}\,\eta(\lambda, \mathrm{L})$.

*Proof.* Note that $\mathrm{L}(\lambda)$ is either Hermitian or skew-Hermitian. Let $(\mu, u)$ be an eigenpair of the matrix $\mathrm{L}(\lambda)$ such that $|\mu| = \sigma_{\min}(\mathrm{L}(\lambda))$ and $u^H u = 1$. Then $\mathrm{L}(\lambda)u = \mu u$. Define $\Delta A := -\frac{\mu\, uu^H}{1+|\lambda|^2}$, $\Delta B := -\frac{\overline{\lambda}\,\mu\, uu^H}{1+|\lambda|^2}$, and consider the pencil $\Delta\mathrm{L}(z) = \Delta A + z\Delta B$. Then $\Delta\mathrm{L} \in \mathbb{S}$ and $\lambda \in \Lambda(\mathrm{L} + \Delta\mathrm{L})$. Further, for the spectral and the Frobenius norms, we have $|\!|\!|\Delta\mathrm{L}|\!|\!| = \sigma_{\min}(\mathrm{L}(\lambda))/\|(1, \lambda)\|_2$. Hence, the result follows. Finally, when $\lambda = \pm i$, the result follows from Theorem 3.6. ∎

THEOREM 5.4. *Let* $\mathbb{S} \in \{H\text{-even}, H\text{-odd}\}$ *and* $\mathrm{L} \in \mathbb{S}$. *Let* $\lambda \in i\mathbb{R}$. *Then for the spectral and the Frobenius norms on* $\mathbb{C}^{n\times n}$, *we have* $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = \eta(\lambda, \mathrm{L})$ *and* $\Lambda^{\mathbb{S}}_{\epsilon}(\mathrm{L}) \cap i\mathbb{R} = \Lambda_{\epsilon}(\mathrm{L}) \cap i\mathbb{R}$. *Also when* $\lambda = \pm 1$, *for the Frobenius norm, we have* $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = \sqrt{2}\,\eta(\lambda, \mathrm{L})$.

*Proof.* Note for $\lambda \in i\mathbb{R}$, the matrix $\mathrm{L}(\lambda)$ is again either Hermitian or skew-Hermitian. Hence, the result follows from the proof of Theorem 5.3. When $\lambda = \pm 1$, the result follows from Theorem 3.7. ∎

We mention that the above results are easily extended to the case of general structured pencils where the coefficients matrices are elements of Jordan and/or Lie algebras.

Finally, for $T$-palindromic and $H$-palindromic pencils we have the following result.

THEOREM 5.5. *Let* $\mathbb{S}$ *be the space of* $T$*-palindromic pencils and* $\mathrm{L} \in \mathbb{S}$. *Let* $\lambda \in i\mathbb{R}$. *Then for the Frobenius norm on* $\mathbb{C}^{n\times n}$, $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = \sqrt{2}\,\eta(\lambda, \mathrm{L})$ *and* $\Lambda^{\mathbb{S}}_{\epsilon}(\mathrm{L}) \cap i\mathbb{R} = \Lambda_{\epsilon/\sqrt{2}}(\mathrm{L}) \cap i\mathbb{R}$.

*Proof.* Let $\lambda \in i\mathbb{R}$. Then by Theorem 3.5, we have $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \sqrt{2}\,\|\mathrm{L}(\lambda)x\|_2/\|(1, \lambda)\|_2$ for all $x$ such that $\|x\|_2 = 1$. Hence, taking minimum over $\|x\|_2 = 1$, we obtain the desired results. ∎

THEOREM 5.6. *Let* $\mathbb{S}$ *be the space of* $H$*-palindromic matrix pencils and* $\mathrm{L} \in \mathbb{S}$. *Let* $\lambda \in \mathbb{T}$. *Then for the spectral and the Frobenius norms on* $\mathbb{C}^{n\times n}$, *we have* $\eta^{\mathbb{S}}(\lambda, \mathrm{L}) = \eta(\lambda, \mathrm{L})$ *and* $\Lambda^{\mathbb{S}}_{\epsilon}(\mathrm{L}) \cap \mathbb{T} = \Lambda_{\epsilon}(\mathrm{L}) \cap \mathbb{T}$.

*Proof.* Let L be given by $\mathrm{L}(\lambda) = A + \lambda A^H$. For $\lambda \in \mathbb{T}$, we have $\mathrm{L}(\lambda)^H = \overline{\lambda}\mathrm{L}(\lambda)$. This shows that $\mathrm{L}(\lambda)$ is a normal matrix. Let $(\mu, u)$ be an eigenpair of $\overline{\lambda}\mathrm{L}(\lambda)$ such that $|\mu| = \sigma_{\min}(\overline{\lambda}\mathrm{L}(\lambda)) = \sigma_{\min}(\mathrm{L}(\lambda))$. Define $\Delta A := -\frac{1}{2}\lambda\mu\, uu^H$ and consider the pencil $\Delta\mathrm{L}(z) = \Delta A + z(\Delta A)^H$. Noting the fact that $\overline{\lambda}\mathrm{L}(\lambda)u = \mu u$ and $\overline{\mu}u = (\overline{\lambda}\mathrm{L}(\lambda))^H u = \mu u$, we have $\mathrm{L}(\lambda)u + \Delta\mathrm{L}(\lambda)u = \lambda\mu\, u - \lambda\mu u = 0$. Further, we have $|\!|\!|\Delta\mathrm{L}|\!|\!| = |\mu|/\sqrt{2} = \sigma_{\min}(\mathrm{L}(\lambda))/\|(1, \lambda)\|_2 = \eta(\lambda, \mathrm{L})$. Hence, the results follow. ∎

For structured pencils, we have seen that $\Lambda^{\mathbb{S}}_{\epsilon}(\mathrm{L}) \cap \Omega = \Lambda_{\epsilon}(\mathrm{L}) \cap \Omega$ for appropriate $\Omega \subset \mathbb{C}$. We now show that this result plays an important role in solving certain distance problems associated with structured pencils. For illustration, we consider an $H$-even pencil $\mathrm{L}(z) = A + zB$. Then by Theorem 5.4, we have $\Omega = i\mathbb{R}$, that is, $\Lambda^{\mathbb{S}}_{\epsilon}(\mathrm{L}) \cap i\mathbb{R} = \Lambda_{\epsilon}(\mathrm{L}) \cap i\mathbb{R}$. The spectrum of L has Hamiltonian eigensymmetry, that is, the eigenvalues of L occur in $\lambda, -\overline{\lambda}$ pairs so that the eigenvalues are symmetric with respect to the imaginary axis $i\mathbb{R}$.

*Question:* Suppose that L is $H$-even and is of size $2n$. Suppose also that L has $n$ eigenvalues in the open left half complex plane and $n$ eigenvalues in the open right half complex plane. What is the smallest value of $|\!|\!|\Delta\mathrm{L}|\!|\!|$ such that $\Delta\mathrm{L}$ is $H$-even and $\mathrm{L} + \Delta\mathrm{L}$ has a purely imaginary eigenvalue?

Distance problems of this kind occur in many applications (see, for example, [8]). Let $\mathrm{d}(\mathrm{L})$ denote the smallest value of $|\!|\!|\Delta\mathrm{L}|\!|\!|$ such that $\mathrm{L} + \Delta\mathrm{L}$ has a purely imaginary eigenvalue. Then by Theorem 5.4, we have

$$\mathrm{d}(\mathrm{L}) = \inf_{t\in\mathbb{R}} \eta^{\mathbb{S}}(it,\mathrm{L}) = \min\left\{\epsilon : \Lambda_\epsilon^{\mathbb{S}}(\mathrm{L}) \cap i\mathbb{R} \neq \emptyset\right\} = \min\{\epsilon : \Lambda_\epsilon(\mathrm{L}) \cap i\mathbb{R} \neq \emptyset\} = \inf_{t\in\mathbb{R}} \eta(it,\mathrm{L}).$$

Hence, d(L) can be read off from the unstructured pseudospectra of L. Note that $\eta(z,\mathrm{L}) = \sigma_{\min}(A + zB)/\sqrt{1+|z|^2}$. Thus, if the infimum of $\eta(z,\mathrm{L})$ is attained at $\mu \in i\mathbb{R}$, then as in the proof of Theorem 5.4 we can construct an $H$-even pencil $\Delta\mathrm{L}$ such that $\mu$ is an eigenvalue of $\mathrm{L} + \Delta\mathrm{L}$ and that $|||\Delta\mathrm{L}||| = \eta(\mu,\mathrm{L}) = \mathrm{d}(\mathrm{L})$.

**6. Conclusions.** We have analyzed structured backward perturbations of ten special classes of structured pencils. Given a structured pencil $\mathrm{L} \in \mathbb{S}$ and an approximate eigenpair $(\lambda, x)$, we have determined the structured backward error $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$ and a structured pencil $\Delta\mathrm{L} \in \mathbb{S}$ such that $|||\Delta\mathrm{L}||| = \eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$. We have shown that such a $\Delta\mathrm{L}$ is unique when $\mathbb{C}^{n\times n}$ is equipped with the Frobenius norm. On the other hand, we have shown that there are infinitely many such $\Delta\mathrm{L}$ when $\mathbb{C}^{n\times n}$ is equipped with the spectral norm and that all such $\Delta\mathrm{L}$ are characterized by adopting Davis–Kahan–Weinberger solutions of norm-preserving dilation problem for structured matrices. More specifically, for the Frobenius norm on $\mathbb{C}^{n\times n}$, we have determined $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$ and a unique $\Delta\mathrm{L}$ for $T$-symmetric (Theorem 3.1), $T$-skew-symmetric (Theorem 3.3), $T$-even and $T$-odd (Theorem 3.4), $T$-palindromic (Theorem 3.5), $H$-Hermitian and $H$-skew-Hermitian (Theorem 3.6), $H$-even and $H$-odd (Theorem 3.7), and $H$-palindromic (Theorem 3.8) pencils. On the other hand, when $\mathbb{C}^{n\times n}$ is equipped with the spectral norm, we have shown that $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L}) = \eta(\lambda, x, \mathrm{L})$ for $T$-symmetric and $T$-skew-symmetric pencils (Theorem 4.2), and have determined $\eta^{\mathbb{S}}(\lambda, x, \mathrm{L})$ and a $\Delta\mathrm{L}$ for $T$-even and $T$-odd (Theorem 4.4), $H$-Hermitian and $H$-skew-Hermitian (Theorem 4.5), $H$-even and $H$-odd (Theorem 4.6), $T$-palindromic (Theorem 4.8), and $H$-palindromic (Theorem 4.9) pencils. We have shown that structured and unstructured pseudospectra are the same for $T$-symmetric and $T$-skew-symmetric pencils. For the rest of the structures, we have shown that $\Lambda_\epsilon^{\mathbb{S}}(\mathrm{L}) \cap \Omega = \Lambda_\epsilon(\mathrm{L}) \cap \Omega$ for some $\Omega \subset \mathbb{C}$. We have also shown that the equality $\Lambda_\epsilon^{\mathbb{S}}(\mathrm{L}) \cap \Omega = \Lambda_\epsilon(\mathrm{L}) \cap \Omega$ plays an important role in constructing solution of certain distance problems.

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