On the Sum Capacity of Multiaccess Block-Fading Channels with Individual Side Information

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***Abstract*—We consider the problem of finding optimal, fair and distributed power-rate strategies to achieve the sum capacity of the Gaussian multiple-access block-fading channel. The transmitters have access to only their own fading coefficients, while the receiver has access to all of the fading coefficients. We propose a distributed strategy called the 'midpoint' strategy which is optimal when the system cannot tolerate outage. In addition, we demonstrate a successive decoding scheme that can achieve this maximal sum-rate. In presence of outage, we show that the strategies based on a single threshold are suboptimal.**

I. Introduction

The multiple-access channel is a widely used model to understand the fundamental limits on information transmission in a 'many-to-one' communication scenario, such as the uplink channel of a cellular network. In the wireless regime, channel fading due to multipath, shadowing and inherent channel variability introduces interesting challenges in reliable communication. It is important to know whether the receiver (and transmitters) have access to measurements on the fading conditions, the delay and accuracy thereof.

Throughout this paper, we concentrate on the specific case of *individual* channel state information (CSI) at the transmitter, viz. each transmitter has instantaneous access to its own fading state causally, but that of no other. The receiver has complete channel state information. More specifically, we consider the block-fading case: the fading coefficients are constant over a block of channel uses, over which the codeword lasts. The transmitters, thus, are not allowed to take advantage of the ergodic nature of the fading process during coding, but may employ adaptive power and rates. This particular situation is motivated by systems involving occasional (opportunistic) access to a shared medium, such as in a cognitive radio or a sensor network with a star topology. Here, multiple users wish to communicate their data to the receiver over the awarded time slot in a fair but distributed fashion. If the slot duration is not fixed, as described in [1], the receiver may employ a beacon signal for synchronizing the rounds of communication.

There is considerable literature on multiaccess fading channels with instantaneous CSI. The Shannon capacity of a Gaussian MAC with CSI available only at the receiver is evaluated rigorously in [2]. The optimal power control strategies to achieve capacity for the case of complete channel state information at the transmitters (CSIT) are given in [3] and [4]. Coming to partial side information at the transmitters, [5] gives the capacity region of a fading MAC under very general notions of CSI at the transmitters. These notions can be specialized to nearly all practical scenarios including individual transmitter CSI. However, our work differs from [5] due to the block-fading assumption. The ergodic averaging inherently used in evaluating the Shannon capacity region in [5] turns out to be essential because of the absence of complete CSI. Alternate notions of capacity motivated by different practical scenarios have also been investigated: delay-limited capacity for the fading MAC is dealt with in [6] while [7] defines the notions of expected capacity and capacity with outage for information unstable single-user channels.

Our main results are summarized as follows:

- We introduce a fair, simple and distributed policy called the 'midpoint' strategy for the Gaussian multiple-access block-fading channel. The midpoint strategy is sum throughput-optimal for symmetrical users when outage cannot be tolerated.
- We also propose a low-complexity rate-splitting scheme that allows the midpoint strategy throughput to be achieved through successive decoding.
- When outage can be tolerated, we propose threshold-based policies which narrowly out-perform the midpoint strategy. We further show that schemes based on a fixed threshold are suboptimal compared to variable ones.

II. System Model

Consider M users communicating with a single receiver. These users transmit real-valued signals X_i, encountering real-valued fades H_i. If Y is the value of the received signal at a (discrete) time instant we have

$$Y = \sum_{i}^{M} H_i X_i + Z$$

where Z is an independent Gaussian noise process. The fading space $\mathcal{H}_i$ of the i-th user is the set of values taken by H_i, and the joint fading space $\mathcal{H}$ is the set of values taken by the joint fading state $\bar{H} = (H_1, H_2, \cdots, H_M)$. Similar vector quantities of user-wise parameters, like rate, power, channel state realization, will be denoted with a overbar symbol. We assume that the (stationary and ergodic) fading processes H_i are independent, and their distributions are known to all the transmitters and the receiver. In addition, we have *individual*

CSIT, i.e. each transmitter knows its own channel fading coefficient H_i but that of no other. The receiver knows all the fading coefficients. The transmitters have individual average power constraints P_i^{avg}, and have the freedom to adapt their rate (and power) according to their own channel conditions. This leads to the following notion of a power-rate strategy.

Definition 1. *A power-rate strategy is a collection of mappings* $(P_i, R_i) : \mathcal{H}_i \longmapsto \mathbb{R}^+ \times \mathbb{R}^+;\ i = 1, 2, \cdots, M$. *Thus, in the fading state* H_i, *the* i^{th} *user expends power* $P_i(H_i)$ *and employs a codebook of rate* $R_i(H_i)$.

This definition is reminiscent of power strategies in [4], but there are two key differences. Firstly, we incorporate individual transmitter CSI in the definition. Secondly, the rate is allowed to be adaptive due to the block-fading restriction. Considered block-wise the channel is a fixed-gain Gaussian multiaccess channel (MAC). Consequently we assume that the standard random Gaussian codebooks with ML decoding are employed to achieve capacity thereof. Let $C_{MAC}(\bar{h}, \bar{P}(\bar{h}))$ denote the capacity region of a Gaussian MAC with fixed channel gains $\bar{h} = h_1, \cdots, h_M$ and power allocations $\bar{P}(\bar{h}) = (P_1(h_1), \cdots, P_M(h_M))$. We know that,

$$C_{MAC}(\bar{h}, \bar{P}(\bar{h})) = \Bigg\{ \bar{R} : \forall S \subseteq \{1, 2, \cdots, M\} \\ \sum_{i \in S} R_i \leq \frac{1}{2} \log \left(1 + \sum_{i \in S} |h_i|^2 P_i(h_i) \right) \Bigg\} \quad (1)$$

Definition 2. *We call a power-rate strategy as* feasible *if it satisfies the average power constraints for each user i.e.* $\forall i \in \{1, 2, \cdots, M\}, \quad \mathbb{E}_{H_i} P_i(H_i) \leq P_i^{avg}$.

Definition 3. *A power-rate strategy is termed as* outage-free *if it never results in outage i.e.*

$$\forall \bar{h} \in \mathcal{H}, (R_1(h_1), \cdots, R_M(h_M)) \in C_{MAC}(\bar{h}, \bar{P}(\bar{h}))$$

The throughput achieved by a given power-rate strategy is, then:

$$R_{sum} = \mathbb{E}_{\bar{H}} \sum_{i=1}^{M} R_i(H_i) \mathbb{I}_{\{(R_1(H_1), \cdots, R_M(H_M)) \in C_{MAC}(\bar{H}, \bar{P}(\bar{H}))\}} \quad (2)$$

where $\mathbb{I}_A$ is the indicator function for the condition A.

Definition 4. *The sum capacity is the maximum (average) throughput achievable, i.e.* $C_{sum} = \max R_{sum}$ *where the maximum is taken over all feasible power-rate strategies.*

III. Optimal Strategies Without Outage

Consider a situation wherein it is required that the system never suffer outage. This would be of importance when the practical system under consideration involves occasional communication during arbitrarily allocated time slots, which are small in comparison with the channel coherence time. Coordination being difficult in such a setup, the challenge is to provide optimal, fair and distributed strategies for the system. We describe a distributed power-rate strategy called the *midpoint* strategy and a simple decoding scheme thereof. Prior to that, let us consider the simple strategy of time sharing among users, or plain time-division multiple-access (TDMA).

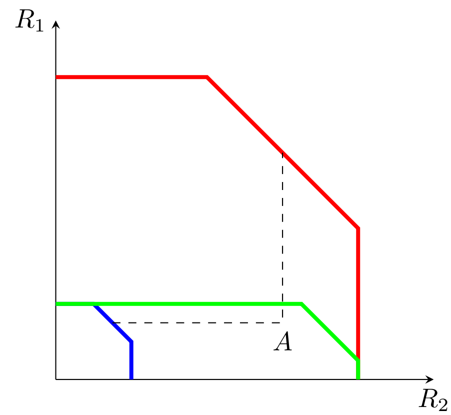

Fig. 1. The users 1 and 2 construct the innermost and outermost MAC capacity regions respectively. The intermediate pentagon is the instantiated MAC region and A denotes the operating point

A. Plain TDMA

In plain TDMA the transmitters employ a simple 'taking turns' policy. Each block is divided into sub-blocks with only one user transmitting in that sub-block. This requires some extra coordination such as agreeing on an ordering for the users. The channels for the users are now *orthogonal* and they may water-fill over their own sub-blocks to improve throughput. Thus, we obtain the power-rate strategy corresponding to plain TDMA as:

$$P_i(h_i) = \left(\frac{1}{\lambda_i} - \frac{1}{|h_i|^2} \right)^+ \\ R_i(h_i) = \frac{1}{2M} \log \left(1 + M|h_i|^2 P_i \right)$$

where λ_i is chosen such that $\mathbb{E}_{H_i} P_i(H_i) = P_i^{avg}$. The actual power employed by the user in its sub-block is $MP_i(H_i)$ and the full transmission rate supported thereby is chosen.

B. The Midpoint Rate Strategy

For simplicity, assume that the users have different fixed powers $P_1, P_2, \cdots, P_M$ for the given round of communication (for instance, after deciding on an arbitrary feasible power strategy). Each user assumes that all others are identical to itself and constructs the symmetrical MAC region based on this assumption. It then chooses the maximal equal-rates point for operation. Thus we have

$$R_i^{mid}(h_i) = \frac{1}{2M} \log \left(1 + M|h_i|^2 P_i \right). \quad (3)$$

Lemma 5. *The midpoint rate strategy is outage free, i.e.*

$$\forall \bar{h} \quad \bar{R}_i^{mid} \in C_{MAC}(\bar{h}, \bar{P}).$$

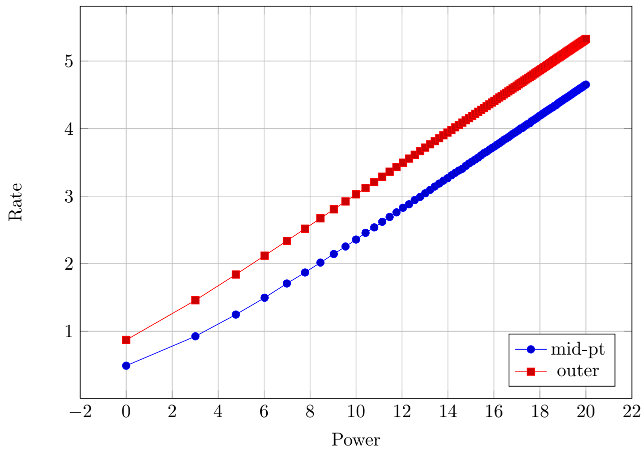

Fig. 2. The midpoint strategy is only a constant off the full CSI bound [3]

Proof: The lemma follows directly from the concavity of the logarithm function, i.e. $\forall S \subset \{1, 2, \cdots, M\}$:

$$\sum_{i \in S} R_i^{mid}(h_i) = \sum_{i \in S} \frac{1}{2M} \log\left(1 + M|h_i|^2 P_i\right) + \sum_{i=1}^{M-|S|} \frac{1}{2M} \log 1 \leq \frac{1}{2} \log\left(1 + \sum_{i \in S} |h_i|^2 P_i\right)$$

■

As $P_1, \cdots, P_M$ are arbitrary, the users' power strategies are now *completely independent* of each other. The best power strategy for each user would thus be to water-fill over its own channel and we obtain

$$P_i(h_i) = \left(\frac{1}{\lambda_i} - \frac{1}{|h_i|^2}\right)^+ \tag{4}$$

where λ_i is chosen such that $\mathbb{E}_{H_i} P_i(H_i) = P_i^{avg}$. When the users have the same average power and identical fading distributions, we call them a *symmetric user set*. We further define a **symmetric mid-point strategy**, in which each user in a symmetric user set employs the same power allocation scheme and chooses the corresponding symmetric mid-point rate. For a symmetric user set, we have the following result.

Theorem 6. *For a symmetric user set and any given outage-free strategy, there is a symmetric midpoint strategy which achieves at least as much throughput as the given strategy.*

Proof: See appendix. ■

Note here that the throughput achieved by the midpoint strategy is identical to that achieved by plain TDMA. We compare this in Figure 2 with the opportunistic TDMA possible with complete CSIT [3]. The advantage of plain TDMA is its simplicity in decoding, since only M single-user decoders are needed. However, the price for this is paid in the extra coordination required to set up an ordering for transmission between users. The midpoint strategy avoids this coordination, albeit at the cost of incurring joint decoding. We show in the next section that this cost can be ameliorated through rate-splitting and successive decoding.

C. Rate Splitting

We present an asymptotically optimal rate-splitting strategy that replaces the joint decoder with LM successive single-user decoders, where L is a parameter. This section is motivated by the work in [1] and their technique is useful in showing the achievability. However, [1] considers a rateless scheme with variable coding block-lengths between rounds of communication. The length of each round is determined by a feedback beacon link from the receiver, block or time slot, and there is no assumption of such a feedback link.

By a slight abuse of the notation, we denote the received signal power for user i, i.e. $P_i|h_i|^2$ as simply P_i, throughout this section. Assume that the users have different (received) powers $P_1, P_2 \ldots P_M$. For simplicity, we will assume that the additive noise is of unit variance. The values of P_i may change with each block of communication depending on the individual fading conditions. Each user is *unaware* of the fade values and transmit powers of the rest of the users and, consequently, the interference they may cause.

The encoding and decoding are done thus: each user splits itself into L virtual users and splits its power, perhaps unequally, among these users. Each user is to be visualized as a 'stack' of virtual users. For decoding, we use a successive cancellation based single-user decoder, which decodes one of the virtual users assuming all other virtual users as yet undecoded as Gaussian noise, see [8] for the details.

More specifically, transmitter i, having power P_i, splits its data stream in to L virtual users. This is done by allotting a power/rate pair (P_l^i, r_l^i) to the l^{th} virtual user, such that $\sum_l P_l^i = P_i$. The transmitter i assumes that all other users are also at (received) power P_i and imagines identical power splitting strategies across all users. It then chooses the rates r_l^i by considering all the other virtual users in the same and lower layers as interference, i.e.,

$$r_l^i = \frac{1}{2} \log\left(1 + \frac{P_l^i}{1 + (M-1)P_l^i + M\sum_{j=1}^{l-1} P_j^i}\right). \tag{5}$$

However, in the actual setting, the interference encountered from the other users are substantially different from that accounted for in the denominator of (5) and a layer by layer decoding may fail. Surprisingly, it turns out that this can be compensated by not strictly adhering to a layer by layer decoding. In particular, the receiver retains the freedom to decode the topmost hitherto undecoded layer of *any* transmitter, irrespective of the number of layers which were already decoded. We now show that this is sufficient for complete decoding.

Lemma 7. *Assuming layer-wise rate allocation as per (5), it is always possible to find a virtual user which can be decoded correctly, i.e. with arbitrarily small error probability.*

Proof: By induction: assume that layers (virtual users) above l_k have been decoded for the k^{th} transmitter. Choose:

$$k^* = \arg\max_k \sum_{j=1}^{l_k} P_j^k$$

The actual interference for this virtual user is given by:

$$\begin{aligned} 1 + \sum_{j=1}^{l_{k^*}-1} P_j^{k^*} + \sum_{k \neq k^*} \sum_{j=1}^{l_k} P_j^k & \\ &= 1 + \sum_{k=1}^{M} \sum_{j=1}^{l_k} P_j^k - P_{l_{k^*}}^{k^*} \\ &\leq 1 + M \sum_{j=1}^{l_{k^*}} P_j^{k^*} - P_{l_{k^*}}^{k^*} \\ &= 1 + \sum_{j=1}^{l_{k^*}-1} P_j^{k^*} + (M-1) \sum_{j=1}^{l_{k^*}} P_j^{k^*} \end{aligned}$$

The inequality follows directly from the choice of k^*. The RHS is the *expected* interference for the l_{k^*}th virtual user of the k^*th transmitter. Thus, as the actual interference is less than the expected interference, this virtual user can be correctly decoded. In other words, the user with the 'best' received SNR can always be chosen for decoding. ■

Theorem 8. *As $L \to \infty$ and $\forall j, l, \quad P_l^j \to 0$, the rate achieved by all the users equals their midpoint rate.*

Proof: Using 5, we have

$$R_i = \sum_{j=1}^{L} \frac{1}{2} \log \left(1 + \frac{P_l^i}{1 + (M-1)P_l^i + M \sum_{j=1}^{l-1} P_j^i} \right)$$

Under the given conditions, we can use the same method as in Lemma 1 of [1] to show that:

$$\begin{aligned} \lim_{L \to \infty} R_i &= \lim_{L \to \infty} \sum_{j=1}^{L} \frac{P_l^i}{1 + (M-1)P_l^i + M \sum_{j=1}^{l-1} P_j^i} \\ &= \frac{1}{2} \int_0^{P_i} \frac{dy}{1 + My} = \frac{1}{2M} \log (1 + MP_i) \end{aligned}$$

■

Computational results in [1] also show that only a nominal number of virtual users L suffice to yield good performance.

IV. Strategies with Outage

Thus far, we have seen that plain TDMA (or, equivalently, midpoint) is throughput-optimal without outage. However, a simple example demonstrates that this is not so when we allow outage. By sacrificing on some blocks (or rounds of communication) we may improve the overall throughput.

Consider 2 symmetrical users transmit over a fading channel with two states: H (or high) and L (or low). The fading coefficients are iid Bernoulli random variables for both the users, with $Pr(H) = \delta$. Suppose the users do not employ power control. If δ is small enough, any user who has access to a fading level of H, should not expect the other user to be also at H and in turn try a pessimistic mid-point strategy. On the contrary, the 'better' user should expect the other one to have a value L, which is more likely, and choose a rate of

$$R'(H, L) = \frac{1}{2} \log(1 + (H^2 + L^2)P) - \frac{1}{4} \log(1 + 2L^2 P), \tag{6}$$

where we adhered to the mid-point strategy for the fading value L. Certainly, the (H, H) fading states will result in outage, and the resulting throughput is

$$(1-\delta)\frac{1}{2} \log(1 + 2L^2 P) + 2(1-\delta)\delta R'(H, L), \tag{7}$$

which can be greater than that of the outage-free strategies at low values of δ.

Motivated by this, we move to the general case wherein the system can tolerate outage. We make some simplifying assumptions on the outage scenario. We consider only 2 completely symmetrical users (i.e. they have equal power constraints and fading marginals). The fading distributions are assumed to be iid Rayleigh. In addition, we ignore power control: the power is fixed to be $P(h) = P$. In this section, we detail two policies which outperform the midpoint strategy in terms of long-term throughput. It is shown that simple single-threshold policies are strictly suboptimal.

A. A Single-Threshold Strategy

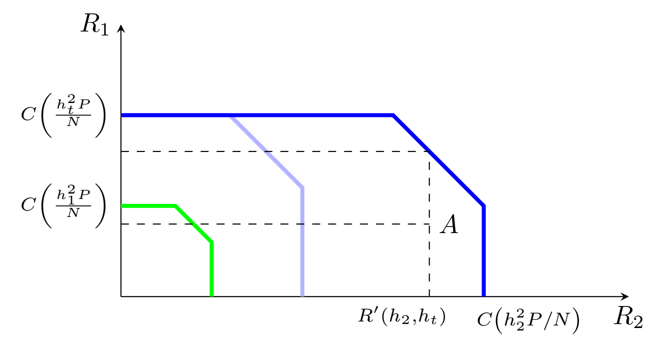

Fig. 3. Here, $C(x) \triangleq \frac{1}{2}\log(1+x)$. User 2, who is beyond the threshold, constructs the outer two pentagons assuming 1 to be at most as good as the intermediate region. User 1, who is within threshold constructs the innermost region, choosing its midpoint rate. A denotes the final operating point.

With iid Rayleigh fading, we can modify the above strategy to get good opportunistic access as follows:

$$R(h) = \begin{cases} \frac{1}{4} \log \left(1 + \frac{2|h|^2 P}{N}\right) & \text{for} \quad h \leq h_t \\ R'(h, h_t) & \text{otherwise}, \end{cases} \tag{8}$$

where $R'(\cdot, \cdot)$ is defined in (6). Here, beyond the threshold h_t, the transmitter assumes that the other transmitter is at most as good as h_t and operates on the boundary of the MAC constructed thereof (see Figure 3). Thus, the only time when outage occurs is when both the transmitters are beyond h_t. Since much of the probability mass is concentrated towards the 'bad' channel gains, the midpoint rates are retained in that region, while with good channels the transmitter takes a risk.

The throughput achieved by such a strategy can be computed according to (2). They maximal throughput by using the best threshold is plotted in Figure 5, which for the scales of our plot is indistinguishable from that of the **midpoint rate** strategy, suggesting the utility of the midpoint rate scheme.

B. A Single Threshold is Insufficient

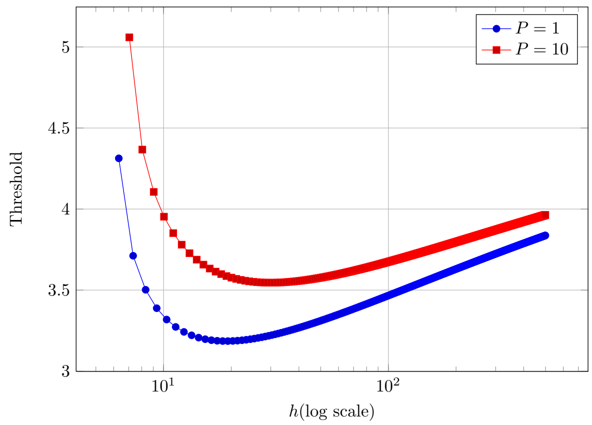

Fig. 4. Optimum variable threshold $H(h)$.

On closer scrutiny, one can strictly improve the single-threshold strategy. In an improved scheme, the transmitter is pessimistic when below the threshold h_t, similar to the previous scheme. However, when it experiences a better channel, say h_1, it attempts to choose the best threshold value $H_t(h_1)$ which maximizes the throughput when averaged over the realizations of the other user's fading coefficients. This involves maximization for each value of $h \geq h_t$ yielding a function $H_t(h)$ which depicts the optimal threshold assumption for that channel state. The rate strategy then gets modified as:

$$R(h) = \begin{cases} \frac{1}{4}\log\left(1+\frac{2|h|^2P}{N}\right) & \text{for} \quad h \leq h_t \\ R'(h, H_t(h)) & \text{otherwise} \end{cases}$$

where we have

$$H_t(h) = \min\left[h_t, \text{argmax}_{h'}\left\{(1-e^{-|h'|^2})R'(h,h')\right\}\right].$$

The function $H_t(h)$ is plotted in the Figure 4. The improvement of throughput by employing $H_t(h)$ is shown in Figure 5 as the dashed line, which shows the difference in throughput, when magnified 200 times to match the scale of the plot.

V. Conclusions and Future Work

The proposed midpoint strategy has straightforward generalizations to MIMO MAC systems, since its basis is the concavity property of the logarithm. Coupled with the proposed successive decoding, this strategy is a viable alternative for many practical systems wherein coordination is difficult to achieve due to large overhead. The notion of expected capacity matches the setup we consider here. For the case of none or complete CSIT, the expected capacity matches the Shannon capacity of the channel. However, with partial CSIT and, in particular, our case of individual CSIT, the characterization of the expected capacity is a line of work that can be pursued further. Similarly, the capacity with outage can be considered for this channel, generalizing on the conclusions of section IV.

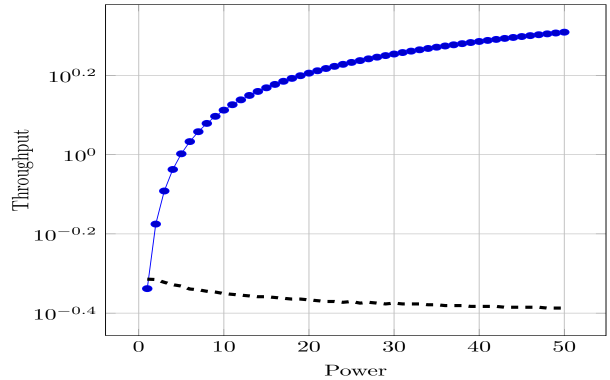

Fig. 5. Throughput comparison

Appendix A
Proof of Theorem 6

For a symmetric user set, the independent fading states H_i have the same distribution, say $p(h)$. Consider any outage-free power-rate strategy $(P_i(h_i), R_i(h_i))$. Let us define $P(h) = (1/M)\sum_{i=1}^{M} P_i(h)$. Since there is no outage by the above choice for any of the users, the average sum-throughput is

$$\begin{aligned}
\sum_{i=1}^{M}\mathbb{E}R_i(H_i) &= \sum_{i=1}^{M}\int_h R_i(h)p(h)dh \\
&= \int_h p(h)\left(\sum_{i=1}^{M}R_i(h)\right)dh \\
&\leq \int_h p(h)\left(0.5\log\left(1+|h|^2\sum_{i=1}^{M}P_i(h)\right)\right)dh \\
&= \int_h p(h)\left(0.5\log\left(1+M|h|^2P(h)\right)\right)dh
\end{aligned}$$

The first inequality follows from the sum-rate bound of a MAC, which should be necessarily satisfied for no outage. The right hand side is the expected sum-rate of the symmetric mid-point strategy for power allocation $P(h)$.

References

[1] U. Niesen, U. Erez, D. Shah, and G. Wornell, "Rateless codes for the gaussian multiple access channel," in *GLOBECOM '06*, 2006, pp. 1 –5.

[2] S. Shamai and A. D. Wyner, "Information-theoretic considerations for symmetric, cellular, multiple-access fading channels - part i," *IEEE Transactions on Information Theory*, vol. 43, no. 6, pp. 1877–1894, 1997.

[3] R. Knopp and P. Humblet, "Information capacity and power control in single-cell multiuser communications," in *ICC '95 Seattle*, pp. 331 –335.

[4] D. Tse and S. Hanly, "Multiaccess fading channels. i. polymatroid structure, optimal resource allocation and throughput capacities," *Information Theory, IEEE Transactions on*, vol. 44, no. 7, pp. 2796 –2815, Nov. 1998.

[5] A. Das and P. Narayan, "Capacities of time-varying multiple-access channels with side information," *Information Theory, IEEE Transactions on*, vol. 48, no. 1, pp. 4 –25, Jan. 2002.

[6] S. V. Hanly and D. N. C. Tse, "Multiaccess fading channels-part ii: Delay-limited capacities," *IEEE Transactions on Information Theory*, vol. 44, no. 7, pp. 2816–2831, 1998.

[7] M. Effros, A. J. Goldsmith, and Y. Liang, "Generalizing capacity: new definitions and capacity theorems for composite channels," *IEEE Transactions on Information Theory*, vol. 56, no. 7, pp. 3069–3087, 2010.

[8] B. Rimoldi and R. Urbanke, “A rate-splitting approach to the gaussian multiple-access channel,” *Information Theory, IEEE Transactions on*, vol. 42, no. 2, pp. 364 –375, Mar. 1996.