Lecture -2

Modelling – Transfer Functions

Note Title 24-12-2009

Recall the 5 steps of control design

Requirements → Functional block diag → Schematic → Math. model → Reduction → Analysis/Design

In this lecture we conc. on
* Schematic → Mathematical modelling for transfer fn.

This step consists of 3 parts

1) Apply physical laws (Newtons laws, Kirchoff's laws) — get differential eqns (possibly non-linear)
2) Linearize the eqns.
3) Get transfer function

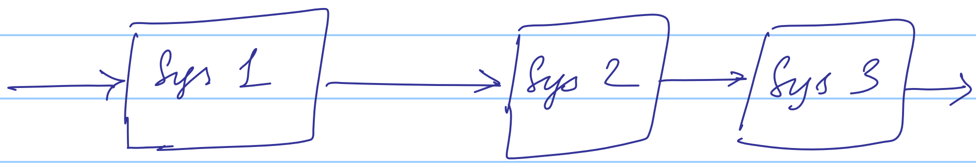
Q. What's wrong with differential eqns? OR Why do we need transfer function?

$$m_1 \frac{dx}{dt} + k_1 x(t) - F(t) = 0 \quad \text{[Spring mass system]}$$

* System parameters (m_1, k_1), output (x) and input (F) are intermixed.

We would like to have a block diagram like input/output model.

* Secondly, we would like to interconnect mathematical models easily



→ Difficult to do this with differential eqn description for each system.

* We need a convenient tool to solve differential eqns.

All these problems can be solved by using Laplace Transform.

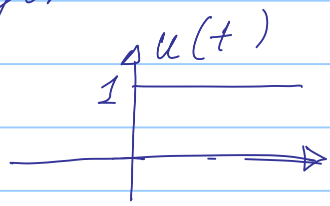
Laplace Transform Review

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

↑ Laplace Tr of $f(t)$

where $s = \sigma + j\omega$, is a complex var.

* The Laplace transform exists if the integral converges
* Discontinuities at the origin can be handled. e.g $\mathcal{L}[u(t)]$ exists.



Inverse Laplace Transform

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t)u(t)$$

where $u(t) = 1 \quad t > 0$
$ = 0 \quad t < 0$

Example: $f(t) = Ae^{-at}u(t)$

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty Ae^{-at}e^{-st}dt$$

$$= A\int_0^\infty e^{-(s+a)t}dt = \frac{A}{s+a}$$

* We will use these formulae RARELY. Instead, we will use (i) ready-made tables of standard functions. (Table 2.1) (ii) Laplace transform theorems (Table 2.2)

<u>Exercise</u>: Memorize Table 2.1 & 2.2

How to use formulae?

<u>Example</u>: $F(s) = \frac{1}{(s+3)^2}$ Find $f(t)$.

We use the following formulae:

* $t^n u(t) \longleftrightarrow \frac{n!}{s^{n+1}}$

* $\mathcal{L}\left[e^{-at} f(t)\right] = F(s+a) \quad \left\{\mathcal{L}[f(t)] = F(s)\right\}$

Hence $\mathcal{L}[t\,u(t)] = \frac{1}{s^2}$

and $\mathcal{L}\left[e^{-at}\{t\,u(t)\}\right] = \frac{1}{(s+a)^2}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} = e^{-3t}\,t\,u(t)$

Partial Fraction Expansion

Let $F(s) = \dfrac{N(s)}{D(s)}$

* If $ord(N(s)) < ord(D(s))$ PFE is possible
* If $ord(N(s)) \geq ord(D(s))$: divide

$N(s)$ by $D(s)$ successively until the remainder ord becomes less than $D(s)$.

E.g. $$F(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5}$$

$$= s + 1 + \frac{2}{s^2 + s + 5}$$

Using the Tables 2.1 & 2.2

$$f(t) = \dot{\delta}(t) + \delta(t) + \mathcal{L}^{-1}\left[\frac{2}{s^2 + s + 5}\right]$$

Case 1 : Denominator roots are real & distinct. (as in the example 1)

Assume $ord(N(s)) < ord(D(s))$

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_m)\cdots\cdots(s+p_n)}$$

$$= \frac{K_1}{s+p_1} + \frac{K_2}{s+p_2} + \cdots + \frac{K_m}{s+p_m} + \cdots + \frac{K_n}{s+p_n}$$

$\{K_1, K_2, \cdots, K_n\}$ are called residues. The formula for calculating them can be found as follows:

$$(s+p_m)F(s) = (s+p_m)\frac{K_1}{(s+p_1)} + (s+p_m)\frac{K_2}{s+p_2}$$

$$+ \cdots + K_m + \cdots + (s+p_m)\frac{K_n}{s+p_n}$$

Evaluate both sides at $s = -p_m$

Then:

$$K_m = \left.\frac{\cancel{(s+p_m)}N(s)}{(s+p_1)(s+p_2)\cdots\cancel{(s+p_m)}\cdots(s+p_n)}\right|_{s=-p_m}$$

Example: Sol^{n} of Linear ODEs

* Laplace Transforms can be used to solve linear differential equations. (by converting linear o.d.e's into algebraic eqns)

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32\,u(t)$$

Take the L.T. (assume the initial conditions zero, $y(0^-)=0$, $\dot{y}(0^-)=0$)

Use the following formula:

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0^-)$$

$$s^2 Y(s) + 12s\,Y(s) + 32\,Y(s) = \frac{32}{s}$$

$$Y(s) = \frac{32}{s(s^2+12s+32)} = \frac{32}{s(s+4)(s+8)}$$

Clearly, we can solve for $y(t)$ by taking inverse Laplace:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{32}{s(s+4)(s+8)}\right\}$$

$$= \mathcal{L}^{-1}\left[\frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+8}\right]$$

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s=0} = 1 \qquad K_3 = \frac{32}{s(s+4)} = 1$$

$$K_2 = \left.\frac{32}{s(s+8)}\right|_{s=-4} = -2$$

$$y(t) = \left(1 - 2e^{-4t} + e^{-8t}\right)u(t)$$

Case 2 : Roots of $D(s)$ are real and repeated.

$$F(s) = \frac{N(s)}{(s+p_1)^r (s+p_2)\cdots(s+p_n)}$$

$$= \frac{K_1}{(s+p_1)^r} + \frac{K_2}{(s+p_1)^{r-1}} + \cdots + \frac{K_r}{(s+p_1)}$$

$$+ \frac{K_{r+}}{(s+p_2)} + \cdots + \frac{K_{r+n-1}}{(s+p_n)}$$

General formula:

$$K_i = \frac{1}{(i-1)!} \frac{d^{i-1}\{F(s)(s+p_1)^r\}}{ds^{i-1}}\Bigg|_{s=-p_1} \qquad i = 1, 2, \cdots, r$$

The K_j $(j = r, \cdots, r+n-1)$ can be found as in Case 1.

Exercise : Derive this formula.

Case 3 : some Roots of $D(s)$ are complex or purely imaginary

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s^2+as+b)\ldots}$$

p_1 is real, (s^2+as+b) has complex roots

Then $F(s) = \frac{K_1}{s+p_1} + \frac{K_2 s + K_3}{s^2+as+b} + \ldots$

Example: $F(s) = \frac{3}{s(s^2+2s+5)}$

$$= \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+2s+5}$$

$K_1 = \frac{3}{5}$ (as in Case 1)

K_2 and K_3 are found by balancing coefficients

$$F(s) = \frac{3/5}{s} - \frac{3}{5}\,\frac{s+2}{s^2+2s+5}$$

For calculating $\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+2s+5}\right\}$

recall: $\mathcal{L}\left[Ae^{-at}\cos\omega t\right] = \frac{A(s+a)}{(s+a)^2+\omega^2}$

and $\mathcal{L}\left[Ae^{-at}\sin\omega t\right] = \frac{B\omega}{(s+a)^2+\omega^2}$

$$F(s) = \frac{3/5}{s} - \frac{3}{5}\,\frac{(s+1) + \frac{1}{2}\cdot 2}{(s+1)^2 + 2^2}$$

$$f(t) = \frac{3}{5}u(t) - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)u(t)$$

$$= 0.6\,u(t) - 0.671\,e^{-t}\cos(2t - \phi)$$

$$\phi = \tan^{-1}(0.5)$$

Exercise : Expand (s^2+2s+5) into $(s+1+j2)(s+1-j2)$ and use the formula of Case 1 to get the same $f(t)$.

Transfer Function :

Consider a n^{th} order linear, time-invariant differential eqn :

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1}\frac{d^{n-1}c(t)}{dt^{n-1}} + \cdots + a_0\, c(t)$$

$$= b_m \frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0\, r(t)$$

where $c(t)$ = output, $r(t)$ = input

$\{a_i, b_i\}$ ← system

Take the L.T. of both sides assuming all initial conditions zero

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s)$$

$$= b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s)$$

$$\Leftrightarrow \left(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0\right) C(s)$$

$$= \left(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0\right) R(s)$$

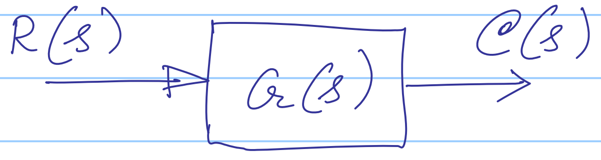
$$\Leftrightarrow \frac{C(s)}{R(s)} = G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

$G(s)$ = <u>transfer function</u>

$$\underbrace{C(s)}_{\text{output transform}} = \underbrace{G(s)}_{\text{transfer function}} \; \underbrace{R(s)}_{\text{Input transform}}$$

This multiplication can be represented

as the I/O block diag that we initially aimed for:



NOTE : Such a separation only exists if the original diff eq. was linear, constant coefficient (Time invariant) and ordinary d.e.

Hence T.F's exist only for linear time-invariant systems.

* Given $G(s)$ and $R(s)$ one can use inverse Laplace tr. to find the time response.

Example : Let $G(s) = \frac{1}{s+2}$

and let input $r(t) = u(t)$. Find $c(t)$.

$$R(s) = \frac{1}{s} \Rightarrow C(s) = \frac{1}{s+2} \cdot \frac{1}{s}$$

$$= \frac{1/2}{s} - \frac{1/2}{s+2}$$

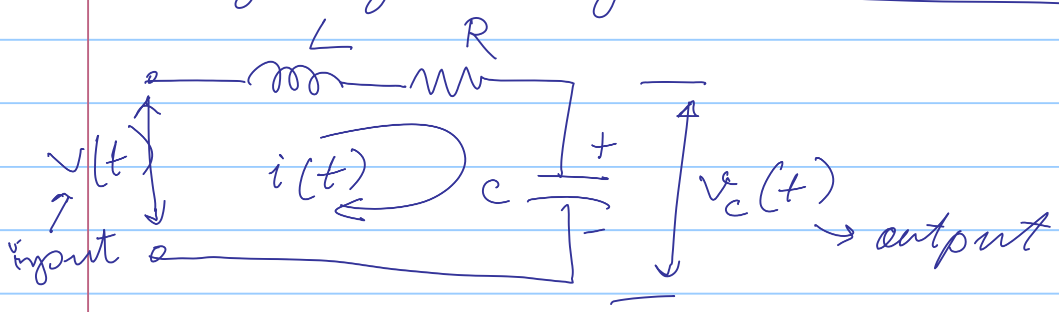
Then $c(t) = \mathcal{L}^{-1}[C(s)] = \left[\frac{1}{2} - \frac{1}{2}e^{-2t}\right]u(t)$

Exercise: (SKE 2.3). Find the output if the input is $u(t)$.

Transfer Functions of Physical Systems

Since we already know how to derive T.F. from o.d.es, we only need to write down odes for the given system.

Transfer functions for Electrical Networks



Q. Calculate the T.F. between v & v_c.

Use Kirchhoff's voltage law:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t) \quad -(1)$$

We want an ode involving $v_c(t)$ and $v(t)$ only.

$$v_c(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$$

Hence $i(t) = C\frac{dv_c}{dt}$; $\frac{di}{dt} = C\frac{dv_c^2}{dt^2}$

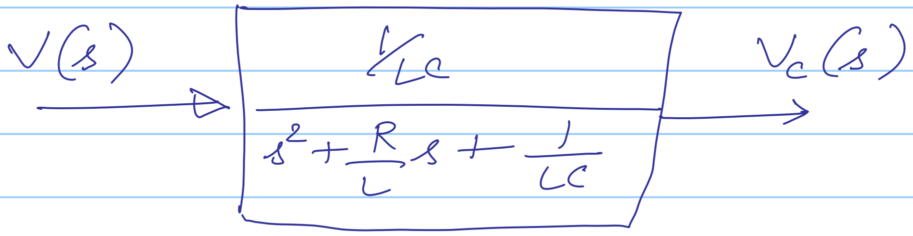
Using in (1),

$$LC\frac{dv_c^2}{dt^2} + RC\frac{dv_c}{dt} + v_c(t) = v(t)$$

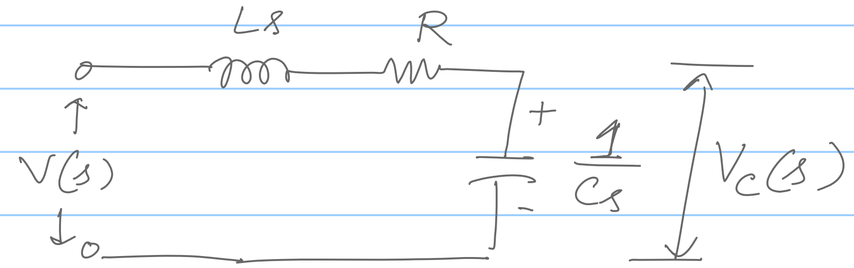
Take L.T. on both sides: (assume zero initial condition)

$$\left(LCs^2 + RCs + 1\right)V_c(s) = V(s)$$

or $$\frac{V_c(s)}{V(s)} = \left[\frac{1/LC}{s^2+\frac{R}{L}s+\frac{1}{LC}}\right] = G(s)$$

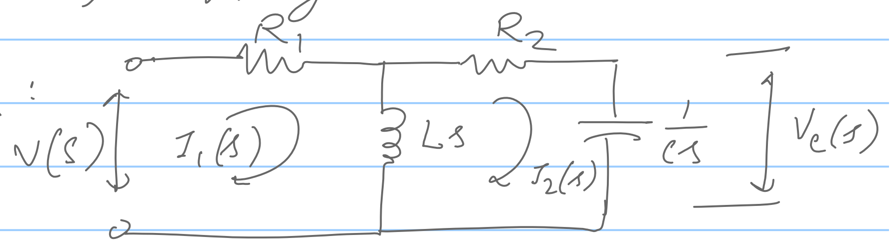


* The ckt. may be redrawn using L.T. quantities



Such L.T. "impedances" can be used to directly apply Kirchoff's laws:

Example :



Let Input = V(s)

Output = $I_2(s)$

The voltage eqns (in L.T. domain)

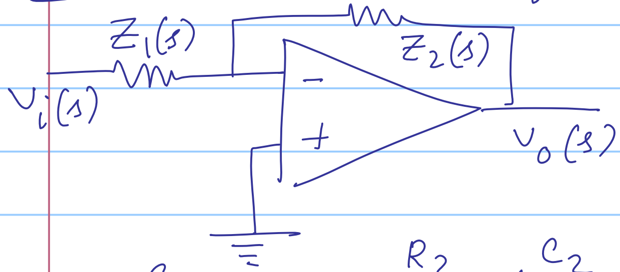
$$R_1 I_1(s) + Ls\left[I_1(s) - I_2(s)\right] = V(s)$$

$$Ls\left[I_2(s) - I_1(s)\right] + R_2 I_2(s) + \frac{1}{Cs} I_2(s) = 0$$

Eliminating $I_1(s)$: $G(s) = \dfrac{I_2(s)}{V(s)}$

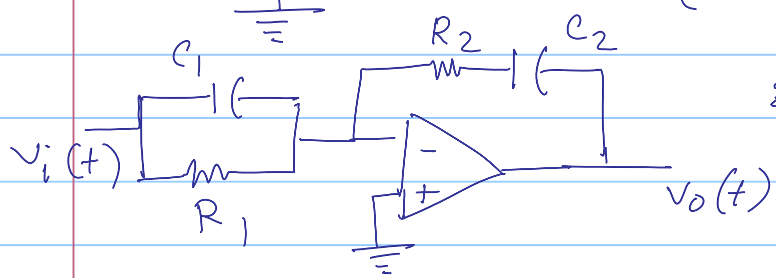
$$= \frac{LCs^2}{(R_1+R_2)LCs^2 + (R_1R_2C + L)s + R_1}$$

Example: Inverting Operational Ampl.



$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

(Exercise: show)



$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}}$$

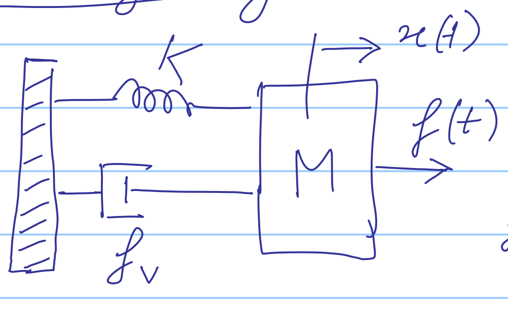
$$Z_2(s) = R_2 + \frac{1}{C_2 s}$$

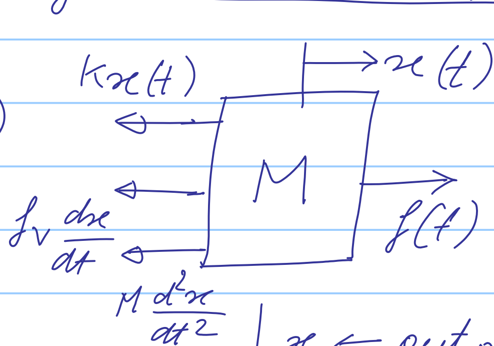
$$\frac{V_o(s)}{V_i(s)} = -\left(R_2 + \frac{1}{C_2 s}\right)\left(C_1 s + \frac{1}{R_1}\right)$$

$$= -\left[\frac{R_2}{R_1} + \frac{C_1}{C_2}\right] + \frac{1}{C_2 R_1}\cdot\frac{1}{s} + R_2 C_1 s$$

& CKT for PID controller.

Transfer function of Mechanical System





Use Newtons Laws:

$x \leftarrow$ output
$f \leftarrow$ input

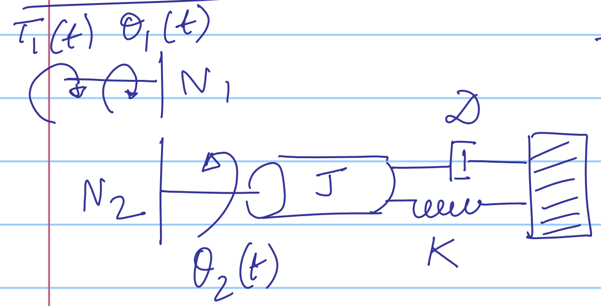
$$M\frac{d^2x(t)}{dt^2}+f_v\frac{dx(t)}{dt}+Kx(t)=f(t)$$

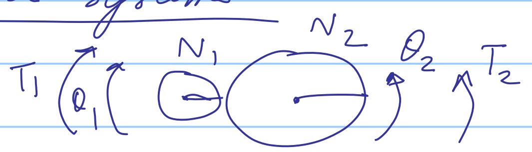
Take L.T. assuming zero initial condition:

$$Ms^2X(s)+f_v sX(s)+KX(s)=F(s)$$

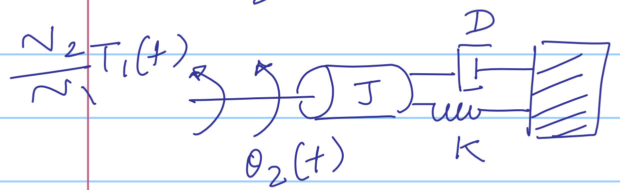
$$\frac{X(s)}{F(s)}=\frac{1}{Ms^2+f_v s+K}$$

Rotational Mechanical Systems





$$T_1 \longrightarrow \boxed{\frac{N_2}{N_1}} \longrightarrow T_2$$



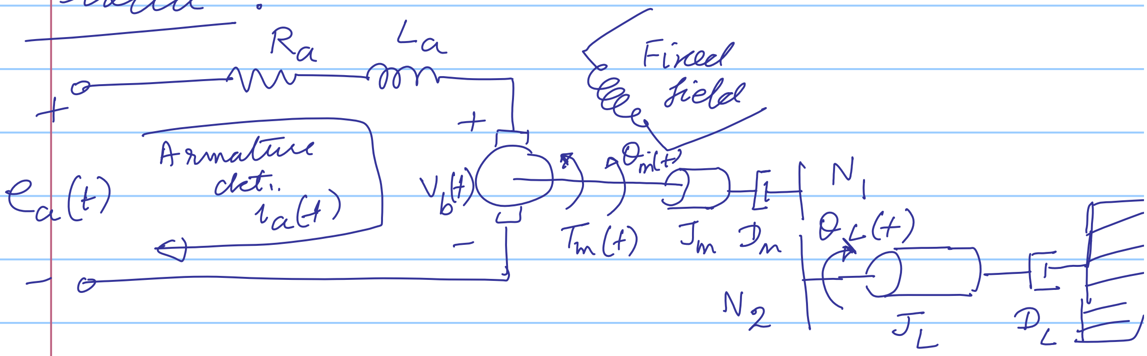
$$\theta_1 \longrightarrow \boxed{\frac{N_1}{N_2}} \longrightarrow \theta_2$$

$$J\frac{d^2\theta_2(t)}{dt^2}+D\frac{d\theta_2(t)}{dt}+K\theta_2(t)=T_1(t)\frac{N_2}{N_1}$$

$$\left(Js^2+Ds+K\right)\frac{N_1}{N_2}\theta_1(s)=T_1(s)\frac{N_2}{N_1}$$

$$G(s)=\frac{\theta_1(s)}{T_1(s)}=\frac{N_2^2}{N_1^2}\cdot\frac{1}{Js^2+Ds+K}$$

Transfer Function of DC motor with load :



$$e_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + K_b \frac{d\theta_m(t)}{dt} \quad \text{— (1)}$$

$$T_m(t) = K_t i_a(t) \quad \text{— (2)}$$

$$\frac{T_L}{T_m} = \frac{N_2}{N_1}, \qquad \frac{\theta_L}{\theta_m} = \frac{N_1}{N_2} \quad \text{(A)}$$

Effect of load

$$T_m(t) = J_m \frac{d^2\theta_m}{dt^2} + D_m \frac{d\theta_m}{dt} + \left(\frac{N_1}{N_2}\right) T_L(t) \quad \text{— (3)}$$

$$T_L(t) = J_L \frac{d^2\theta_L}{dt^2} + D_L \frac{d\theta_L}{dt}$$

$$= \left(\frac{N_1}{N_2}\right)\left[J_L \frac{d^2\theta_m}{dt^2} + D_L \frac{d\theta_m}{dt}\right] \quad \text{— (4)}$$

Using (4) in (3),

$$T_m(t) = \Big[\underbrace{J_m + \left(\frac{N_1}{N_2}\right)^2 J_L}_{J_{eq}}\Big]\frac{d^2\theta_m}{dt^2} + \Big[\underbrace{D_m + \left(\frac{N_1}{N_2}\right)^2 D_L}_{D_{eq}}\Big]\frac{d\theta_m}{dt}$$

$$= J_{eq} \frac{d^2\theta_m}{dt^2} + D_{eq} \frac{d\theta_m}{dt} \quad \text{— (5)}$$

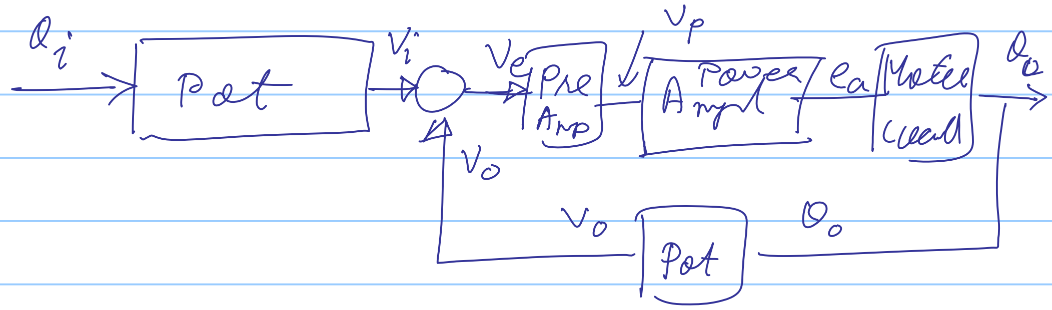
We want $E_a(s) \longrightarrow \boxed{G(s)} \longrightarrow \theta_L(s)$

Take L.T. and eliminate $I_a(s)$ and $T_m(s)$ from ①, ② and ⑤,

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_{eq})}{s\left[s + \frac{1}{J_{eq}}\left(D_{eq} + \frac{K_t K_b}{R_a}\right)\right]}$$

$$\frac{\theta_L(s)}{E_a(s)} = \frac{\frac{N_1}{N_2}\,\frac{K_t}{R_a J_m}}{s(s+\alpha)} = \frac{K}{s(s+\alpha)}$$

Example : Antenna Control



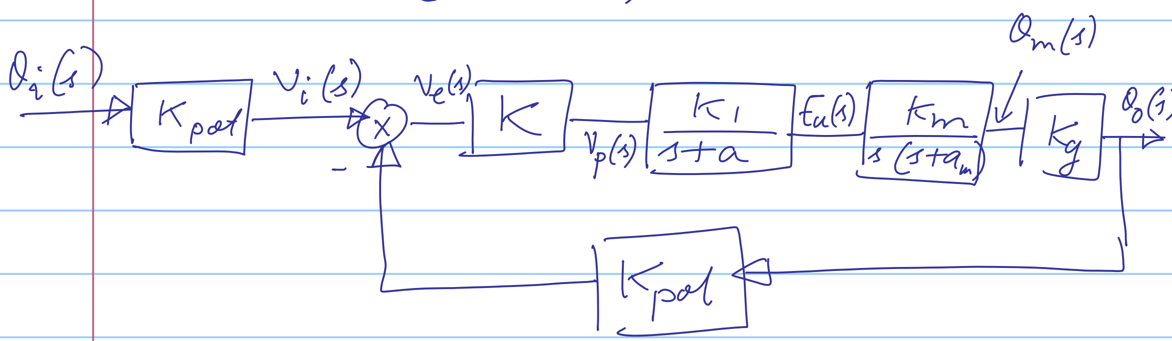
Pot. T.F. : $\dfrac{V_i(s)}{\theta_i(s)} = \dfrac{10}{10\pi} = \dfrac{1}{\pi}$ (dynamics neglected)

Pre-Amp : $\dfrac{V_p(s)}{V_e(s)} = K$ (dynamics neglected)

Power Ampl: $\dfrac{E_a(s)}{V_p(s)} = \dfrac{100}{s+100}$

Motor & load: $\dfrac{\theta_m(s)}{E_a(s)} = \dfrac{2.083}{s(s+1.71)}$

$$\frac{\theta_o(s)}{E_a(s)} = \frac{0.2083}{s(s+1.71)}$$



Homework:

2, 5, 6, 7, 10, 12, 14, 19, 20a, b, 44, 52, 54.