Lecture 3

Modelling : State Variables (3 hours)

Note Title 11-06-2008

We have learnt to model physical systems/ode's with transfer functions.

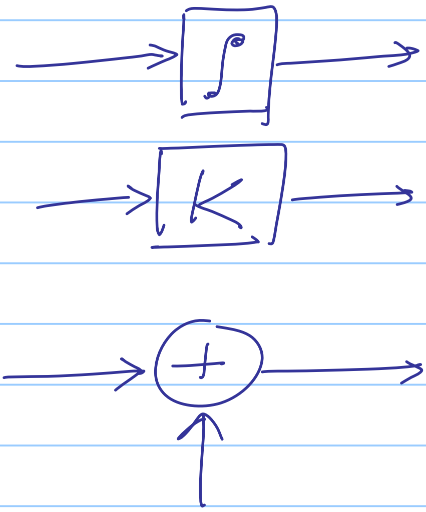
Here we learn how to represent systems by state variables.

Let us try to simulate a differential equation (linear time invariant) on an ANALOG Computer. Recall that analog comp. have only the following components.

1) Integrator

2) Amplifier

3) Adders



Consider the differential eqns:

$$\begin{cases} y^{(3)} + a_1 y^{(2)} + a_2 y^{(1)} + a_3 y \\ = b_0 u^{(3)} + b_1 u^{(2)} + b_2 u^{(1)} + b_3 u \end{cases}$$

where a_1, a_2, a_3 and b_1, b_2, b_3 are constant coefficients.

Step 1 : Express the highest $y(t)$ derivative in terms of all other quantities.

$$y^{(3)} = -a_1 \ddot{y} - a_2 \dot{y} - a_3 y + b_0 u^{(3)} + b_1 \ddot{u} + b_2 \dot{u} + b_3 u$$

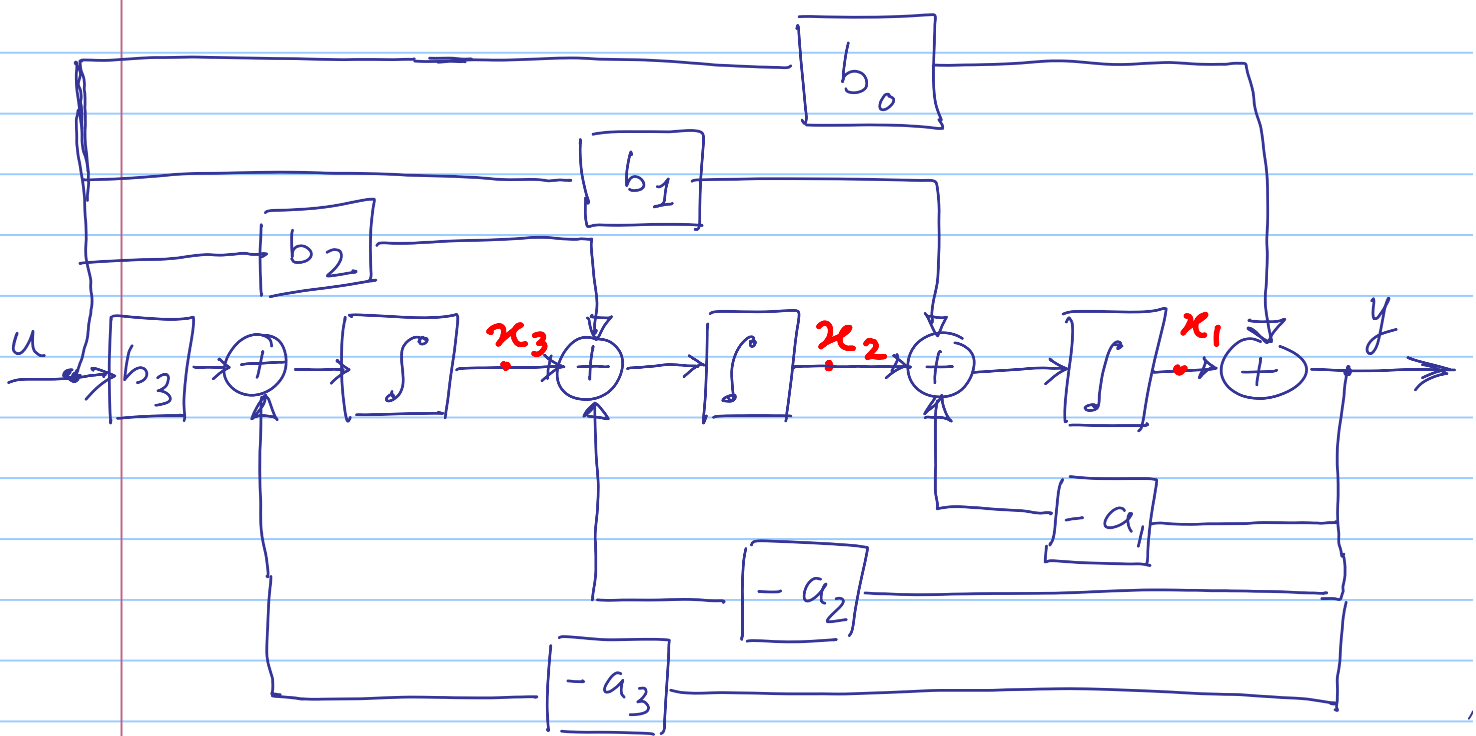
Step 2: Integrate both sides $n(=3)$ times

$$\iiint y^{(3)}dt = -a_1\iiint \ddot{y} - a_2\iiint \dot{y} - a_3\iiint y$$
$$+ b_0\iiint u^{(3)} + b_1\iiint \ddot{u} + b_2\iiint \dot{u} + b_3\iiint u$$

Step 3: Cancel all possible derivatives and integrals to leave only u and y.

$$y = -a_1\int y - a_2\iint y - a_3\iiint y + b_0 u$$
$$+ b_1\int u + b_2\iint \dot{u} + b_3\iiint u \quad \ldots\ldots (*)$$

Step 4: Draw $n(=3)$ integrators. Put y at right end and u at left end



Step 5: Connect according to $(*)$.

Now observe that the output of the integrators have some special features: (Name them x_1, x_2, x_3)

1) The wiring diagram above can be completely & compactly described in terms of x_1, x_2, x_3.

2) (x_1, x_2, x_3) are related to $u(t)$ and $y(t)$ through a set of first order differential equations

■ How to write a system of first order differential equations from the wiring diagram.

1) Name the integrator outputs
2) Write equations for $\dot{x}$, using only x's and u's.

$\dot{x}_1 = -a_1 y + x_2 + b_1 u \cdots\cdots$ ①

[Note that $y = x_1 + b_0 u$ (which is incidentally the output eqn.)] ②

Using ② in ①,

$$\dot{x}_1 = -a_1 x_1 + x_2 + (b_1 - a_1 b_0) u$$

Similarly, $\dot{x}_2 = -a_2 y + x_3 + b_2 u$

$$= -a_2 (x_1 + b_0 u) + x_3 + b_2 u$$

$$= -a_2 x_1 + x_3 + (b_2 - a_2 b_0) u$$

Similarly, $\dot{x}_3 = -a_3 x_1 + (b_3 - a_3 b_0) u$. ③

Eqns 1, 2 & 3 together are called state equations and x_1, x_2, x_3 are the states of the system.

3) How is y related to x's?

$$y = x_1 \quad \text{------} \quad (4)$$

4 is called the output equation.

In matrix form:

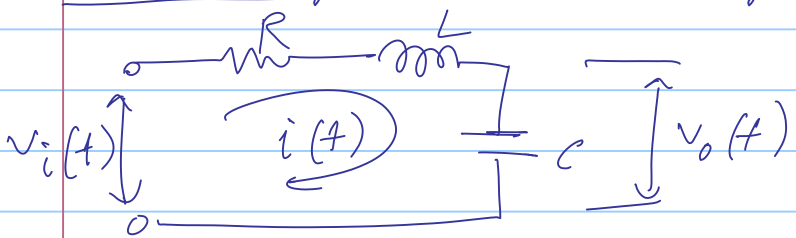
$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \leftarrow \text{state vector} \qquad \dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}$$

(A)

$$\dot{x}(t) = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) + b_0 u(t)$$

State Space Model of RLC ckt



Kirchoff's Laws: $L\dfrac{di}{dt} + Ri + \underbrace{\dfrac{1}{C}\int i\,dt}_{v_o(t)} = v_i(t)$

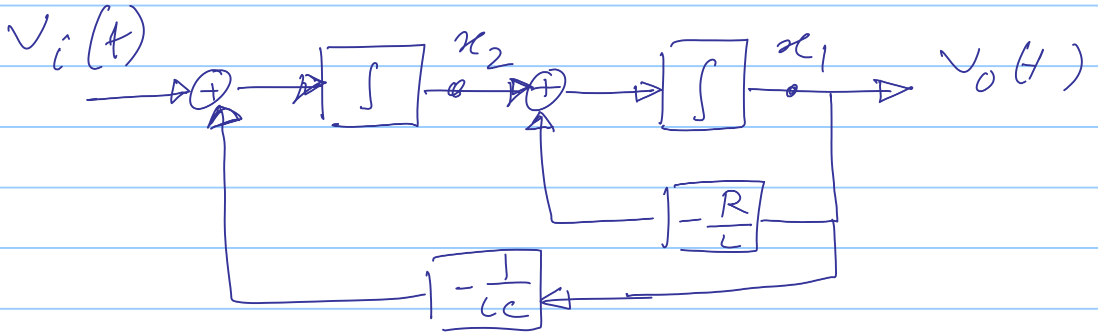
or $\dfrac{dv_o}{dt} = \dfrac{1}{C}\, i(t)$

Here $LC\frac{d^2v_0}{dt^2} + RC\frac{dv_0}{dt} + v_0(t) = v_i(t)$

Now we can apply our method

$$\iint v_0^{(2)} = -\frac{R}{L}\iint v_0^{(1)} - \frac{1}{LC}\iint v_0 + \iint v_i(t)$$

$$v_0(t) = -\frac{R}{L}\int v_0(t) - \frac{1}{LC}\iint v_0 + \iint v_i(t)$$



$$\left.\begin{aligned} \dot{x}_1 &= x_2 - \frac{R}{L}x_1 \\ \dot{x}_2 &= v_i(t) - \frac{1}{LC}x_1 \end{aligned}\right\} \qquad V_0(t) = x_1(t)$$

Matrix form: $\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 1 \\ -\frac{1}{LC} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_i(t)$

$$v_0(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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* State Equations are not unique

Example: $LC\frac{d^2v_0}{dt^2} + RC\frac{dv_0}{dt} + v_0(t) = v_i(t)$

Let $x_1(t) = v_0(t)$ $\quad x_2 = \frac{dv_0}{dt}$

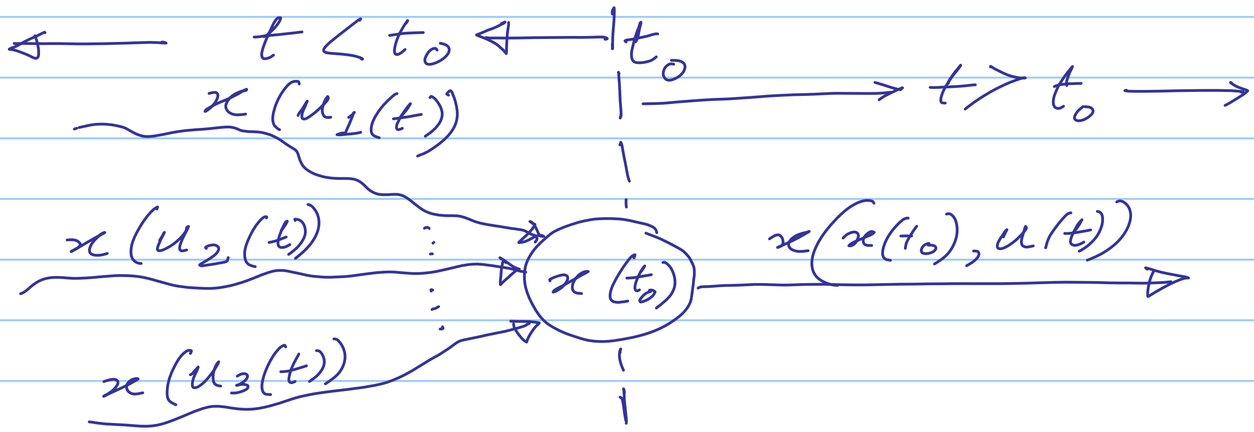
Then, $\dot{x}_1(t) = x_2$

$$\dot{x}_2(t) = \frac{d^2 v_0}{dt} = -\frac{R}{L} x_2 - \frac{1}{LC} x_1 + v_i(t)$$

Matrix form: $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{R}{L} & -\frac{1}{LC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_i(t)$$

$$v_0(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

DEFINITION: The state of a system at time t_0 is the amount of information at t_0 that together with $u(t)$ for $t \geq t_0$, determines uniquely every response in the system for all $t \geq t_0$.



Conclusions:

* A differential equation of order n can be represented as
  - n first order differential equations.
  - ≡ A wiring diagram for analog computer simulation
* A particular set of such first

order equations is called a realization.

* There may be different realizations of the same diff. equation

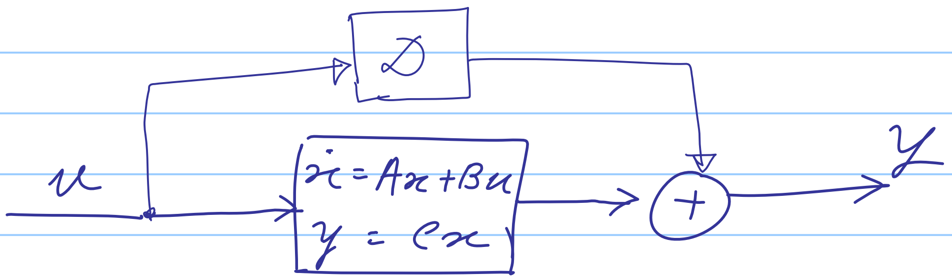
* A realization is of the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

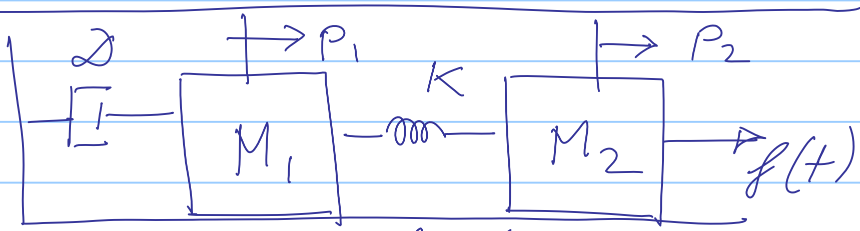
For a single-input single-output (SISO) system,

$A \in \mathbb{R}^{n \times n}$ $\quad C \in \mathbb{R}^{1 \times n}$

$B \in \mathbb{R}^{n \times 1}$ $\quad D \in \mathbb{R}^{1 \times 1}$



Example



No friction

Output = $x_1(t)$ $\quad$ Input $\rightarrow f(t)$

Newtons Laws yield: two differential equations:

$$\begin{cases} M_1 \dfrac{d^2 p_1}{dt^2} + D \dfrac{dp_1}{dt} + Kp_1 - Kp_2 = 0 \\ -Kp_1 + M_2 \dfrac{d^2 p_2}{dt^2} + Kp_2 = f(t) \end{cases}$$

* We can eliminate $p_2(t)$ and use our method. However, a simpler method would be to assume

$$x_1 = p_1 , \quad x_2 = \dot{p}_1 , \quad x_3 = p_2 , \quad x_4 = \dot{p}_2$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{K}{M_1} x_1 - \frac{D}{M_1} x_2 + \frac{K}{M_1} x_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{K}{M_2} x_1 - \frac{K}{M_2} x_3 + \frac{1}{M_2} f(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Connections to transfer function

Q) How to convert a given t.f. to s.s.?

T.F $\rightarrow$ Ordinary Diff. Eq. $\rightarrow$ S.S. realization

Q2) How to convert a S.S. realization to T.F.?

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Step 1: Take L.T. assuming zero initial conditions

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

Step 2: Eliminate $X(s)$

$$X(s) = (sI-A)^{-1}BU(s)$$

$$Y(s) = \left[C(sI-A)^{-1}B + D\right]U(s)$$

Hence the transfer function [SISO]

$$\frac{Y(s)}{U(s)} = C[sI-A]^{-1}B + D$$

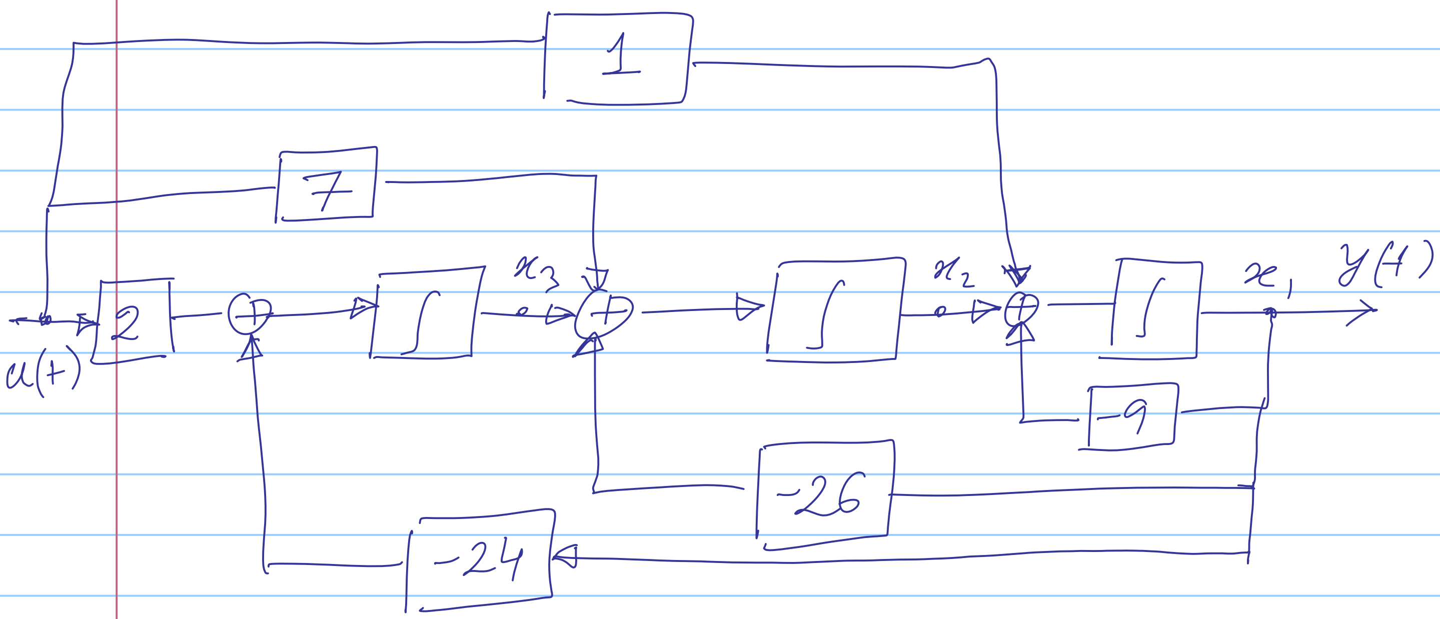
Example:
$$G(s) = \frac{s^2+7s+2}{s^3+9s^2+26s+24} = \frac{Y(s)}{U(s)}$$

$$y^{(3)}(t) + 9y^{(2)}(t) + 26y^{(1)}(t) + 24y(t) = u^{(2)}(t) + 7u^{(1)}(t) + 2u(t)$$

$$y^{(3)}(t) = -9y^{(2)}(t) - 26y^{(1)}(t) - 24y(t) + u^{(2)}(t) + 7u^{(1)}(t) + 2u(t)$$

Integrating both sides thrice:

$$y(t) = -9\int y(t) - 26\iint y(t) - 24\iiint y(t) + \int u(t) + 7\iint u(t) + 2\iiint u(t)$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

<u>Example:</u> Give. $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$

Find $G(s)$.

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

$$G(s) = C(sI-A)^{-1}B + \overset{=0}{D}$$

$$(sI-A) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{\operatorname{adj}(sI-A)}{\det(sI-A)} = \frac{\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1}$$

$$G(s) = \frac{10(s^2+3s+2)}{s^3+3s^2+2s+1}$$

Similarity Transformation

Q) How to get new realization from a given state space realization?

Given: $\dot{x} = Ax + Bu$, $y = Cx$ | $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$

Let T be a $n\times n$ real invertible matrix.

Define a new state vector:

$$\bar{x} = T^{-1}x(t) \quad \text{———} \quad (1)$$

Then, $\dot{\bar{x}} = T^{-1}\dot{x}(t) = T^{-1}[Ax + Bu]$

$= T^{-1}Ax + T^{-1}Bu$

$= T^{-1}AT\bar{x} + T^{-1}Bu$ (Using (1))

Similarly: $y = Cx = CT\bar{x}(t)$

Define, $\bar{A} = T^{-1}AT$, $\bar{B} = T^{-1}B$

$\bar{C} = CT$

Then, $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$

$y = \bar{C}\bar{x}$

Note that (i) we now have a

new realization relating the same $u(t)$ and $y(t)$

(ii) There can be infinitely many such realizations.

Invariance of $G(s)$ under Sim. Tr.

Q) Does the $G(s)$ change if we create a new S.S. realization using the above method.

$$\bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B}$$

$$= CT(sI - T^{-1}AT)^{-1}T^{-1}B$$

$$= CT(sT^{-1}T - T^{-1}AT)^{-1}T^{-1}B$$

$$= CT\left[T^{-1}(sI - A)T\right]^{-1}T^{-1}B$$

$$= CT\left\{T^{-1}(sI - A)^{-1}T\right\}T^{-1}B$$

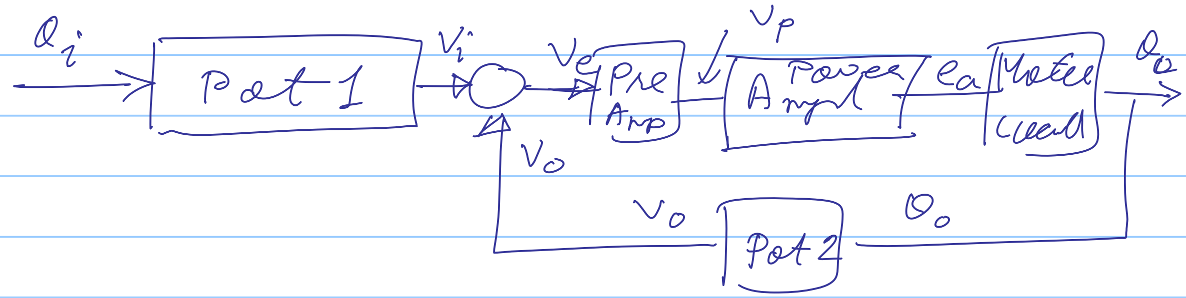
$$= C(sI - A)^{-1}B$$

$$= G(s)$$

Hence, the transfer function remains the same (invariant) under similarity tr.

$\Rightarrow$ The new realization still represents the same system.

Example: Antenna Control: State space model.



We have seen that Pot 1, Pot 2, Pre-Amp are just gains. So we form S.S. models for the Power-Ampl. and motor and load.

Power Ampl.: The diff. eq$^{\underline{n}}$ describing it is:

$$\frac{de_a}{dt} + 100 e_a = 100 v_p(t)$$

Observe that this is already in S.S. form

$$\dot{e}_a = -100 e_a + 100 v_p(t)$$

Output: $y = e_a$

$A = -100$, $B = 100$, $C = 1$, $D = 0$

Motor/Load: Recall (lecture 2) the diff eq$^{\underline{n}}$ relating the input and output.

$$e_a(t) = \left(\frac{R_a J_{eq}}{K_t}\right)\frac{d^2\theta_m}{dt} + \left(\frac{D_{eq} R_a}{K_t} + K_b\right)\frac{d\theta_m}{dt}$$

We can use any of the discussed methods:

Let $x_1 = \theta_m$, $x_2 = \frac{d\theta_m}{dt}$

Then: $\dot{x}_1 = x_2$

$$\dot{x}_2 = -\frac{1}{J_{eq}}\left(D_{eq} + \frac{K_t K_b}{R_a}\right) x_2 + \left(\frac{K_t}{R_a J_{eq}}\right) e_a(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{J_{eq}}\left(D_{eq} + \frac{K_t K_b}{R_a}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_t}{R_a J_{eq}} \end{bmatrix} e_a(t)$$

$$y = \begin{bmatrix} N_1/N_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \theta_L$$

Linearization:

Review: Taylor Series

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} \frac{(x-x_0)}{1!} + \left.\frac{d^2f}{dx^2}\right|_{x=x_0} \frac{(x-x_0)^2}{2!}$$

$$+ \; - - - - - - -$$

If x is near x_0, we can neglect higher order terms of $(x-x_0)$.

Then $f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x-x_0)$

The Taylor series can also be written for vector functions

Example : Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and

$f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$ be a vector function.

We can write $f(x)$ as a Taylor series about any $x = x_0$

Let $x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$, then for x near x_0, and neglecting higher order terms :

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0) \qquad \text{OR}$$

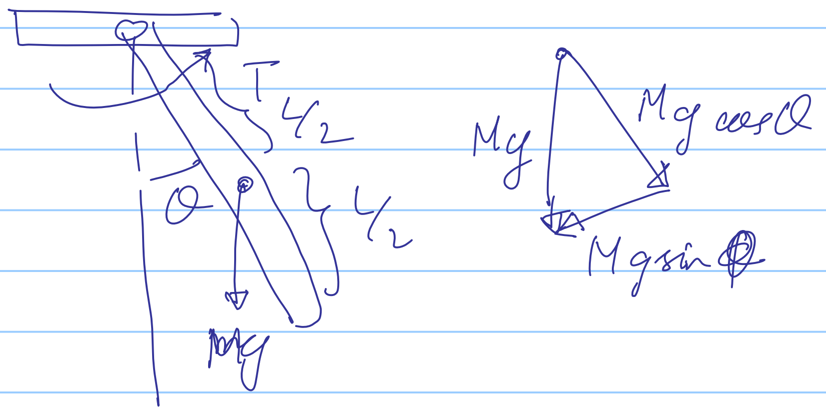
$$\begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} f_1(x_{10}, x_{20}) \\ f_2(x_{10}, x_{20}) \end{bmatrix} + \left.\begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \end{bmatrix}\right|_{\substack{x_1 = x_{10} \\ x_2 = x_{20}}} \begin{bmatrix} x_1 - x_{10} \\ x_2 - x_{20} \end{bmatrix}$$

Question : Can we approximate a non-linear system by a linear system ?

Ans: Yes, over a small region about some special point of operation.

Usually, the nonlinear system is linearized about its <u>equilibrium point</u>.

Example:



Torque Balance: $J\dfrac{d^2\theta}{dt^2} + Mg\dfrac{L}{2}\sin\theta = T$

A non-linear state space representation would be

$$x_1 = \theta(t) \qquad x_2(t) = \frac{d\theta(t)}{dt}$$

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{MgL}{2J}\sin x_1 + \frac{T}{J} \end{aligned}\right\} \text{1st order ODE's (Non-linear)}$$

If short hand, this can be written as:

$$\dot{x} = f(x) + Bu \qquad (1)$$

where $\quad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$ and

$$f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{MgL}{2} \sin x_1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1/J \end{bmatrix} \qquad u(t) = T(t)$$

For (1) ; $\quad \dot{x} = f(x) + Bu$

we define an <u>equilibrium point</u> as follows :

Assume <u>$u = 0$</u>, then x_0 is <u>an equilibrium</u> point for (1) if

$$\left.\frac{dx}{dt}\right|_{x = x_0} = 0 \quad \text{or alternatively,}$$

if $\quad f(x_0) = 0$.

<u>Explanation for vector functions:</u>

$x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$ is an eq. pt. for (1),

if for $u = 0$, $\quad \left.\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}\right|_{\substack{x_1 = x_{10} \\ x_2 = x_{20}}} = 0$

OR $\quad \begin{bmatrix} f_1(x_{10}, x_{20}) \\ f_2(x_{10}, x_{20}) \end{bmatrix} = 0$

Let us try to get a linear approximation of

$\dot{x} = f(x) + Bu$ about its

equilibrium point (say x_0)
(Then by def$^{\underline{n}}$ $f(x_0) = 0$)

We will use Taylor series expansion:

$$\dot{x} = f(x) + Bu$$

or $\dot{x} = f(x_0) + \left.\dfrac{df}{dx}\right|_{x=x_0} (x - x_0) + Bu$ — (2)

[Neglecting higher order terms]

But,

$\dfrac{d}{dt}(x - x_0) = \dfrac{d}{dt}(x)$ — (3) and $f(x_0) = 0$, — (4)

Using (3) and (4) in (2),

$$\frac{d(x - x_0)}{dt} = \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0) + Bu$$

Define $\delta x = x - x_0$; then

$$\boxed{\delta\dot{x} = \left.\frac{df}{dx}\right|_{x=x_0} \delta x + Bu}$$ (A)

Let us use this formula for our example.

First we need to find the eq. pt for

$$(*)\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{MgL}{2J}\sin x_1 + \dfrac{T}{J} \end{cases}$$

By definition, for $T = 0$, the eq. pt. (say x_{10}, x_{20}) should satisfy:

$$\left.\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}\right|_{\substack{x_1 = x_{10} \\ x_2 = x_{20}}} = 0 \quad (5)$$

Using (5) in $(*)$ (with $T=0$)

$$\left.\begin{aligned} x_{20} &= 0 \\ -\frac{MgL}{2}\sin x_{10} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{10} &= 0 \\ x_{20} &= 0 \end{aligned}$$

Hence, (by defn) the eq. pt. is

$$x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now we are ready to use formula $(**)$ for our example:

$$\frac{d}{dt}(x_1 - 0) = \frac{dx_2}{dx_1}\bigg|_{\substack{x_1=0\\x_2=0}}(x_1 - 0) + \frac{dx_2}{dx_2}\bigg|_{\substack{x_1=0\\x_2=0}}(x_2 - 0)$$

$$= (x_2 - 0)$$

$$\frac{d}{dt}(x_2 - 0) = \frac{d}{dx_1}\left[-\frac{MgL}{2J}\sin x_1\right]_{\substack{x_1=0\\x_2=0}}(x_1 - 0)$$
$$+ \frac{d}{dx_2}\left[-\frac{MgL}{2J}\sin x_1\right]_{\substack{x_1=0\\x_2=0}}(x_2 - 0)$$

$$= -\frac{MgL}{2J}\left[\cos 0\right](x_1 - 0) + T/J$$

$$= -\frac{MgL}{2J}(x_1 - 0) + T/J$$

Hence the linearized model is

$$\delta\dot{x}_1 = \delta x_2$$

$$\delta\dot{x}_2 = -\frac{MgL}{2J}\,\delta x_1 + T/J$$

where $\delta x_1 = x_1 - x_{10}$ $\quad \left(\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$

$\delta x_2 = x_2 - x_{20}$

Homework:

3, 6, 11, 12a, 13, 16, 22, 26