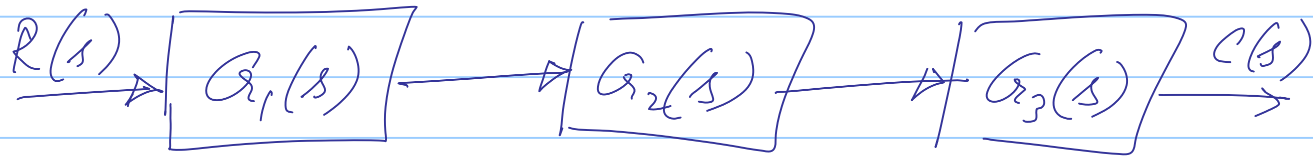
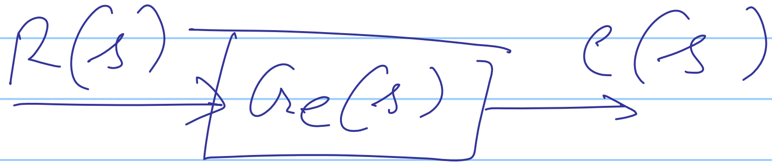
# Lecture 5: Equivalent Systems

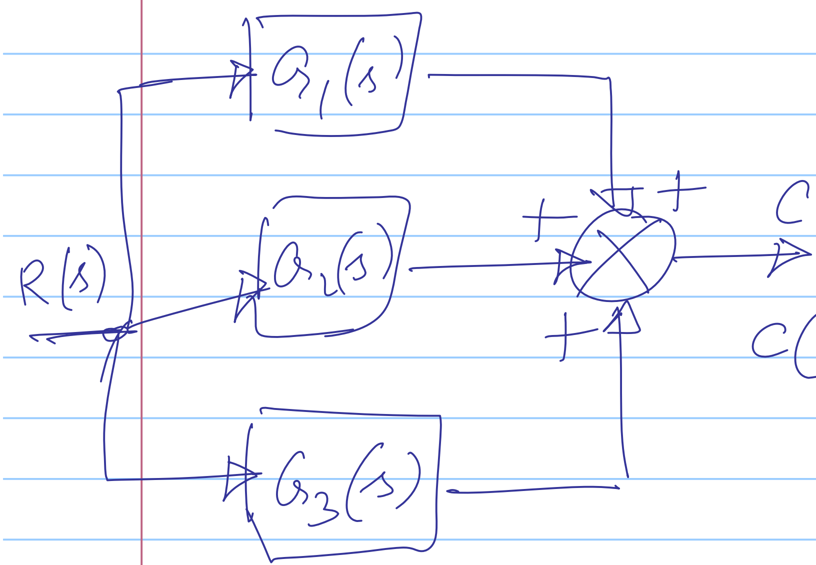
Note Title 11-06-2008

## Block Diagram Reductions



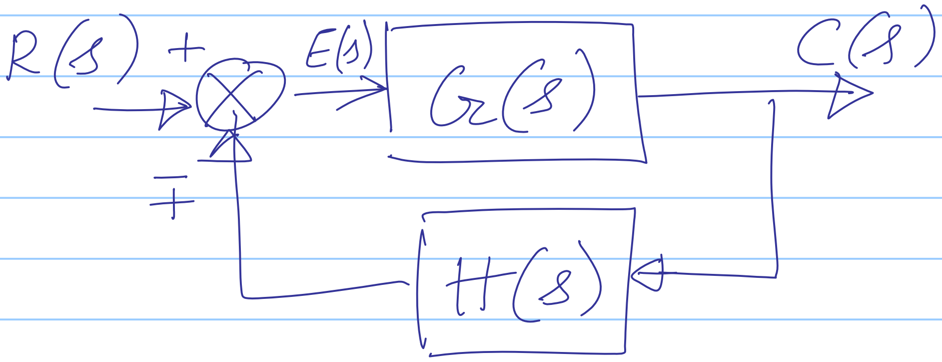

$$C(s) = \underbrace{G_3(s)\, G_2(s)\, G_1(s)}_{G_e(s)}\, R(s)$$




$$C(s) = \left[\pm G_1(s) \pm G_2(s) \pm G_3(s)\right] R(s)$$

## Feedback:



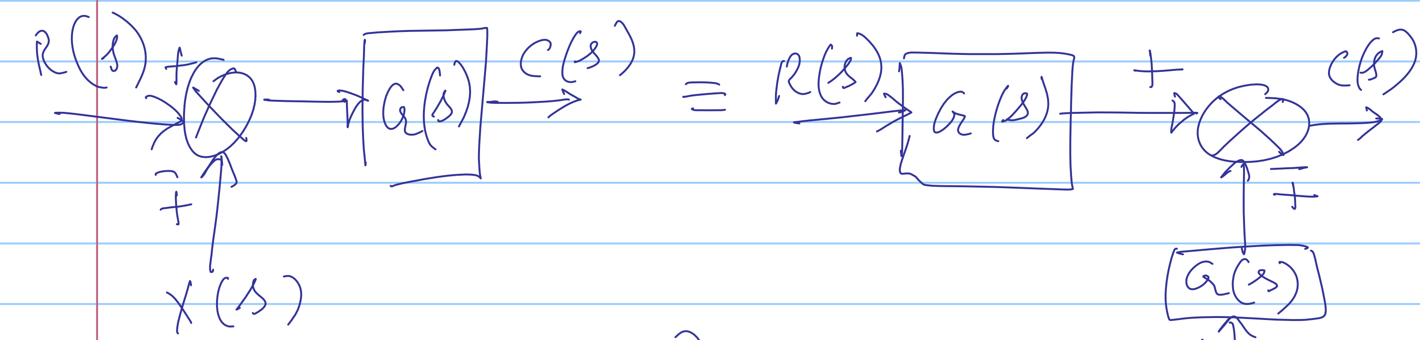

$$E(s) = R(s) \mp C(s)H(s)$$

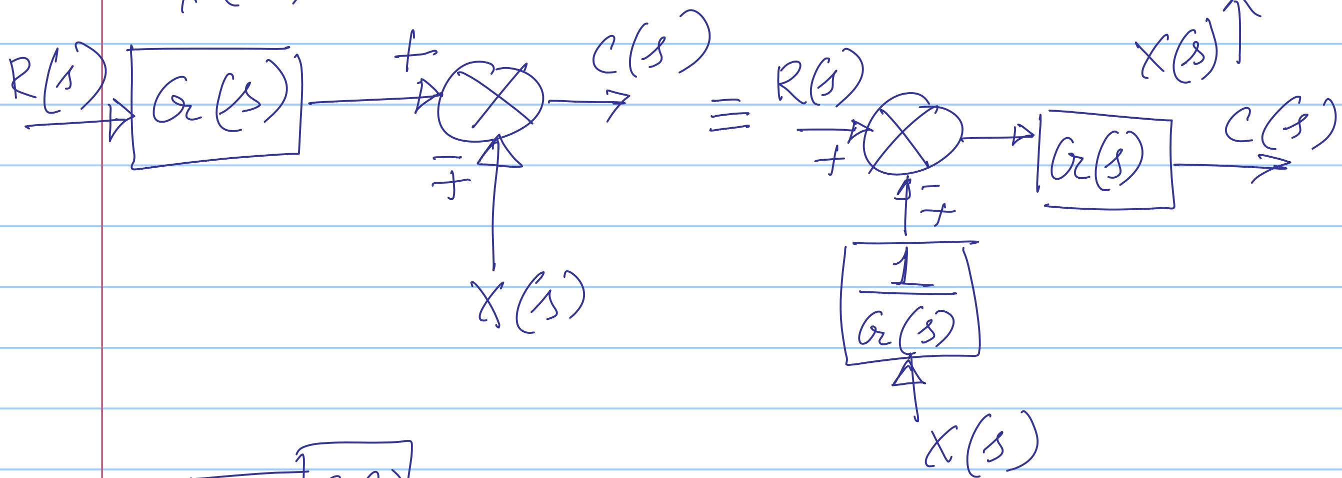
$$C(s) = E(s)\, G(s)$$

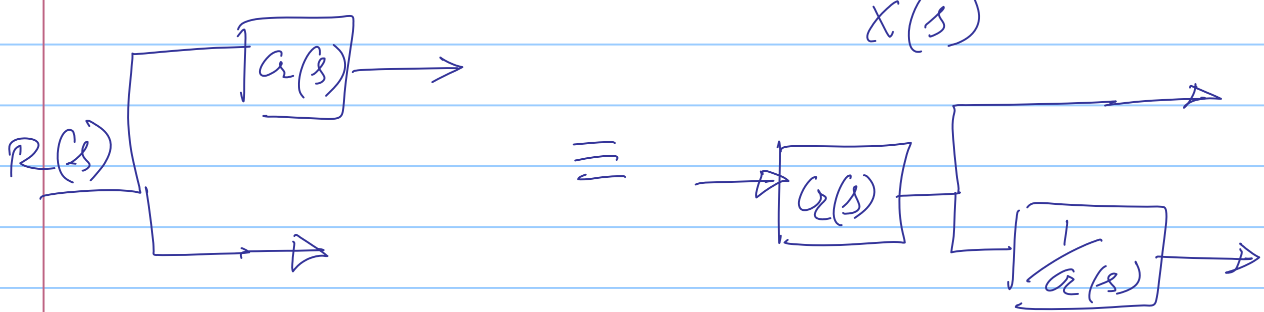
$$\frac{C(s)}{G(s)} = R(s) \mp C(s)H(s)$$

as $$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

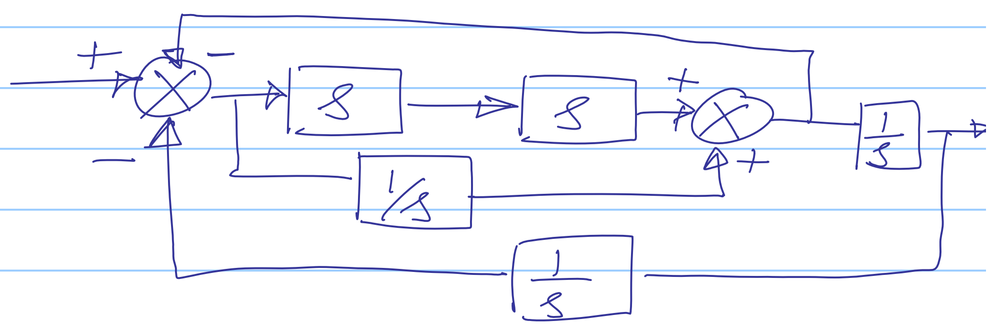
## Moving Blocks to simplify

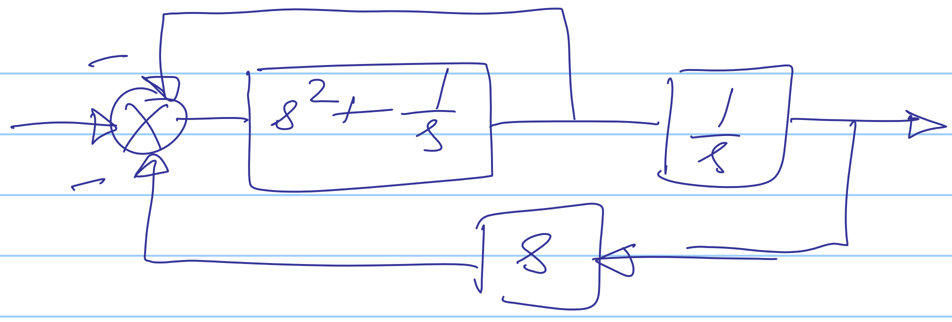


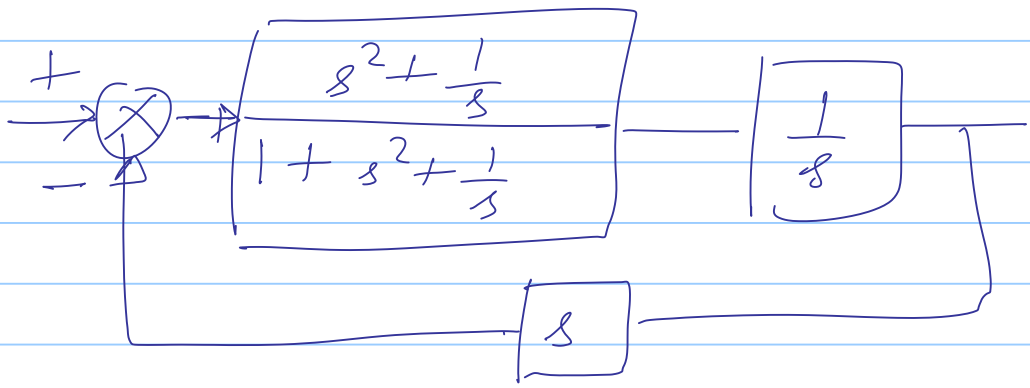




Example:






$$G(s) = \frac{\dfrac{s^2+\frac{1}{s}}{1+s^2+\frac{1}{s}} \cdot \dfrac{1}{s}}{1 + \dfrac{s^2+\frac{1}{s}}{1+s^2+\frac{1}{s}} \cdot \dfrac{1}{s} \cdot s}$$

$$= \frac{\dfrac{s^3+1}{s\left(s^3+s+1\right)}}{1 + \dfrac{s^3+1}{\left(s^3+s+1\right)}}$$

$$= \frac{\left(s^3+1\right)}{\left(2s^3+s+2\right)s} = \frac{s^3+1}{2s^4+s^2+2s}$$

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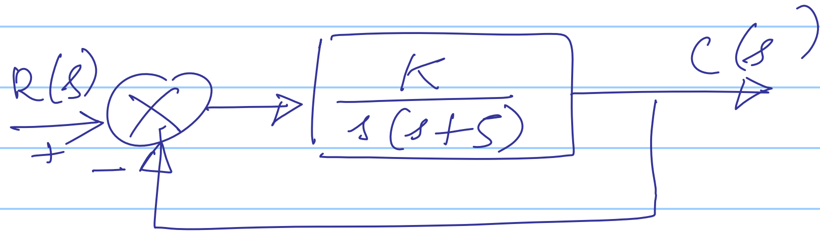
Example:




Design $K$ so that CL system will have

10% overshoot.

$$\frac{C(s)}{R(s)} = \frac{K}{s^2+5s+K}$$

Note real part same

$2\zeta\omega_n = 5$ $\qquad \omega_n = \sqrt{K} \Rightarrow \zeta = \frac{5}{2\sqrt{K}}$

$$\zeta \text{ req. for } 10\%\ OS = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}}$$

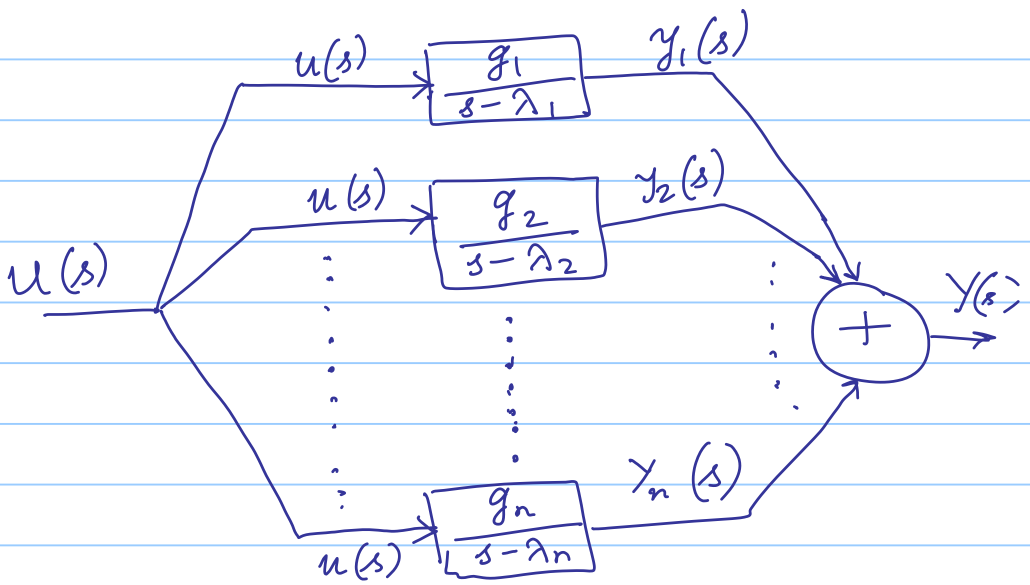
$$= 0.591$$

$\Rightarrow K = 17.9$

## Simplification of S.S. Realizations

### The Diagonal Realization

$$\frac{Y(s)}{U(s)} = \frac{g_1}{s-\lambda_1} + \frac{g_2}{s-\lambda_2} + \cdots + \frac{g_n}{s-\lambda_n}$$

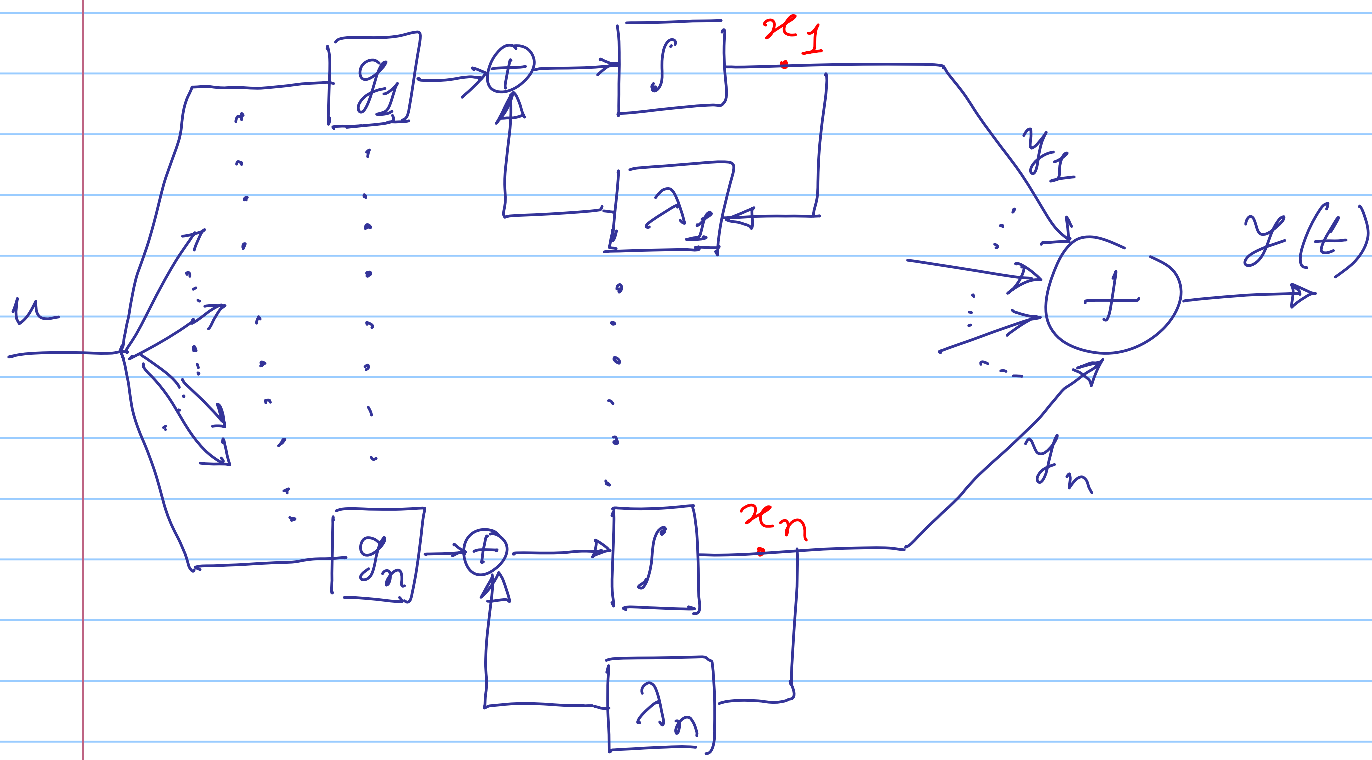


Consider the transfer function from

$$\frac{Y_1(s)}{U(s)} = \frac{g_1}{s-\lambda_1}$$

$$\Rightarrow s\,Y_1(s) - \lambda_1 Y_1(s) = g_1 U(s)$$

$$\Rightarrow \dot{y}_1(t) = \lambda_1 y_1(t) + g_1 u(t)$$




State Equations :

$$\dot{x}_1 = \lambda_1 x_1 + g_1 u$$
$$\dot{x}_2 = \lambda_2 x_2 + g_2 u$$
$$\vdots \qquad\qquad \vdots$$
$$\dot{x}_n = \lambda_n x_n + g_n u$$

In matrix form:

$$
\left\{
\begin{aligned}
\dot{x} &= \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\end{aligned}
\right.
\quad C
$$

Q. Can we use similarity transformation to create realizations, which are simpler than the original?

E.g. We can ask:

Q. Can we diagonalize (Recall the parallel realization) any realization by similarity transformation?

A. NO in general. Let us investigate further.

This reduces to the following question:

Q Is there a matrix $T$ such that $T^{-1}AT$ is diagonal?

<u>FACT 1</u> (from Linear Algebra): A $n \times n$ matrix $A$ is diagonalizable if and only if it has $n$ linearly independent eigenvectors.

Example: $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda - 1 & 2 \\ -1 & \lambda + 1 \end{pmatrix} = \lambda^2 - 1 + 2 = \lambda^2 + 1$$

$\therefore \lambda = \pm i \Rightarrow$ Eigenvalues are distinct

Let $t_1$ and $t_2$ be the eigenvectors.

$$A t_1 = (i) t_1 \iff \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} t_{11} \\ t_{12} \end{bmatrix} = \begin{bmatrix} i t_{11} \\ i t_{12} \end{bmatrix}$$

$$\left.\begin{aligned} t_{11} - 2t_{12} &= i t_{11} \\ t_{11} - t_{12} &= i t_{12} \end{aligned}\right\} \quad t_1 = \begin{bmatrix} 0.8165 \\ 0.4082 - i\,0.4082 \end{bmatrix}$$

Similarly $t_2 = \begin{bmatrix} 0.8165 \\ 0.4082 + i\,0.4082 \end{bmatrix}$

Form $T = [t_1 \;\; t_2] = \begin{bmatrix} 0.8165 & 0.8165 \\ (0.4082 - i\,0.4082) & (0.4082 + i\,0.4082) \end{bmatrix}$

$$\therefore \Lambda = T^{-1} A T = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

We see that though this is diagonal, this cannot be treated as a realization (This cannot be implemented!).

The Diagonalization should be over the real numbers.

FACT 2: A $n \times n$ REAL matrix $A$ is diagonalizable OVER THE REALS iff it has $n$ linearly independent REAL eigenvectors.

An easier to check condition: (Though only sufficient)

FACT 3: A $n \times n$ real matrix $A$ is diagonalizable over the reals if it has $n$ real distinct eigenvalues.

Building the matrix $T$

Assume $A$ has $n$ linearly independent eigenvectors $t_1, t_2 \cdots t_n$

$$At_i = \lambda_i t_i \qquad i = 1, 2, \cdots, n$$

where $\lambda_i$'s are the corresponding eigenvalues. Build a matrix $T_{(n \times n)}$ with $t_1, t_2, \cdots, t_n$ as its columns

$$T = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}_{n \times n}$$

Since $t_1, t_2, \cdots, t_n$ are linearly ind., $T$ is invertible.

$$AT = A\begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}$$

$$= \begin{bmatrix} At_1 & At_2 & \cdots & At_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 t_1 & \lambda_2 t_2 & \cdots & \lambda_n t_n \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}}_{T} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$= T \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

So $\bar{A} = T^{-1}AT = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

<u>Example</u> : 1) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ Eigenvectors are linearly dependent.

So cannot be diagonalized.

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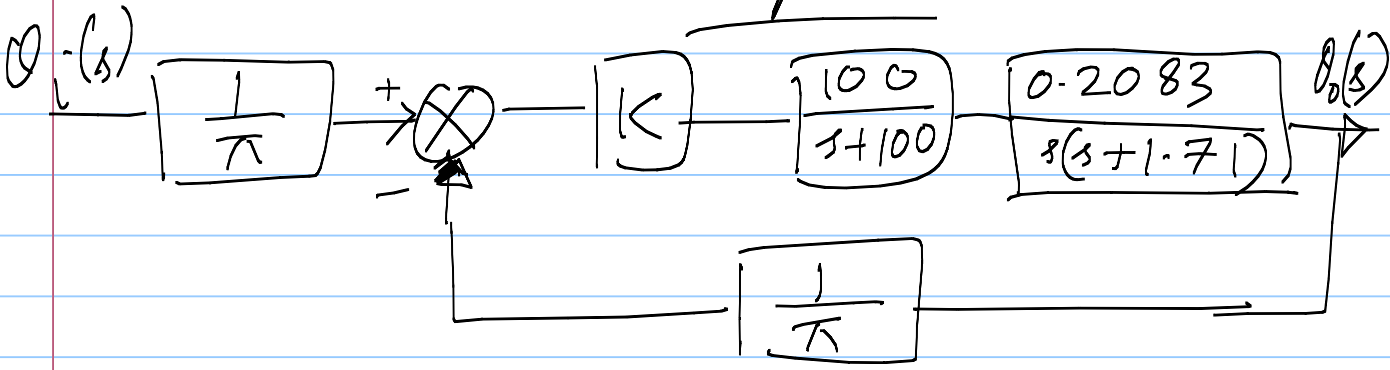
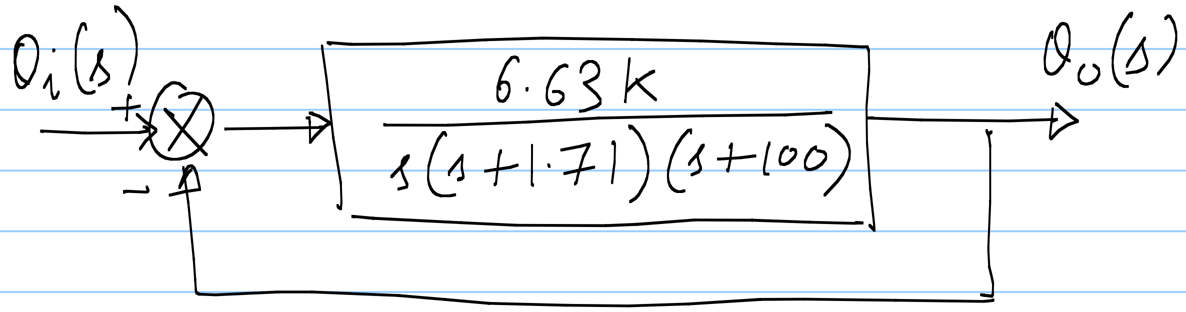
<u>Antenna Control</u> : <u>Designing the C.L. Response</u>



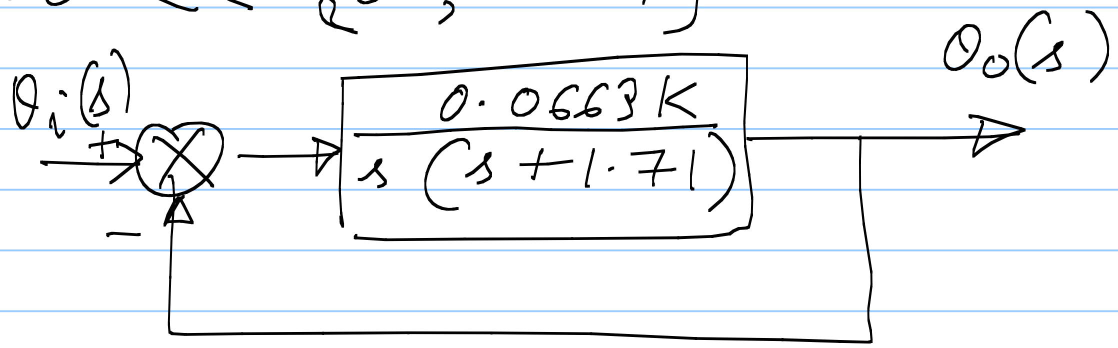



Problem: Choose $K$ to get a $10\%$ OS for the C.L. system.

Step 1 :: Dominant pole approx:

$$100 << \{0, -1.71\}$$

So




$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{0.0663K}{s^2 + 1.71s + 0.0663K}$$

For $10\%$ OS $\longrightarrow \zeta = 0.591$

$$\omega_n = \sqrt{0.0663K} \quad \& \quad 2\zeta\omega_n = 1.71$$

$$\Rightarrow K = 31.6$$

$$6.63K\left[\frac{\frac{1}{1.71}}{s} + \frac{-\frac{1}{1.71}}{(s+1.71)}\right] + \frac{\frac{1}{100\times 101.71}}{(s+100)}$$

$$\frac{\frac{1}{100}\times 6.63K}{s(s+1.71)}$$