Lecture 6 : Stability

Note Title 11-06-2008

In Chapter 1 we described 3 types of requirements on control systems:

1) Transient Response (Done)
2) Stability ⟸ This chap
3) Steady State Errors

Impulse Response : $r(t)=\delta(t) \leftrightarrow \mathcal{L}\{\delta(t)\}=1$

$$H(s) = G(s)R(s) = G(s)\cdot 1$$
$$h(t) = \mathcal{L}^{-1}\{G(s)\}$$

There are several concepts of stability. For our purpose: bounded input should produce bounded output

BIBO Stability:

DEFN: A system is BIBO stable/ Bounded input-bounded output stable (BIBO) stable if any bounded input $r(t) \leq M_1$ for $0 \leq t < \infty$ produces a bounded output, $c(t) \leq M_2$ for $0 \leq t < \infty$.

FACT: A system is BIBO stable if and only if the impulse response of the system $h(\cdot)$ is absolutely summable i.e.

$$\int_0^\infty |h(t)|\, dt < M < \infty$$

Proof: Let $r(t)$ be a bounded input

$$|r(t)| \leq M_1 \quad \text{for} \quad 0 \leq t < \infty$$

The corresponding output is

$$c(t) = \int_0^t h(\tau)\, r(t-\tau)\, d\tau$$

$$|c(t)| = \left| \int_0^t h(\tau)\, r(t-\tau)\, d\tau \right|$$

$$\Rightarrow |c(t)| \leq \int_0^t \left| h(\tau)\, r(t-\tau)\, d\tau \right|$$

$$\leq \int_0^t |h(\tau)|\, |r(t-\tau)|\, d\tau$$

$$\leq M_1 \int_0^t |h(\tau)|\, d\tau$$

$$\leq M_1 \int_0^\infty |h(\tau)|\, d\tau \quad \left[\text{since } |h(t)| \geq 0 \;\; \forall\, t \geq 0\right]$$

$$\Rightarrow \text{If} \quad \int_0^\infty |h(\tau)|\, d\tau < M < \infty$$

$$|c(t)| \leq M_1 M \quad (= M_2, \text{ say})$$

for all $0 \leq t < \infty$

Exercise: Prove necessity.

We can use the FACT to correlate BIBO stability with pole location

$$h(t) = \mathcal{L}^{-1}\{G(s)\}$$

Let $$G(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{s^N \prod_{k=1}^{Q}(s+\sigma_k)\prod_{m=1}^{R}(s^2+2\alpha_m s+\beta_m^2)}$$

* Assume poles are distinct. (for simplicity, NOT req.)
* For $h(t)$ to be absolutely summable, $N = 0$.

Example: $N = 1 \Rightarrow$

$$G(s) = \frac{K_1}{s} + \frac{K_2}{(\qquad)}$$

$$h(t) = K_1 u(t) + K_2(\cdots\cdots)$$

$$\int_0^\infty |u(t)|\,dt \not< \infty$$

<u>Exercise:</u> Which input makes the output unbounded? ($N=1$)

* For $N = 0$,

$$h(t) = \sum_{k=1}^{Q} A_k e^{-\sigma_k t} + \sum_{m=1}^{R} B_m e^{-\alpha_m t}\sin(\omega_m t + \theta_m)$$

For absolute summability,

$$\sigma_k > 0 \quad , \quad \alpha_m > 0$$

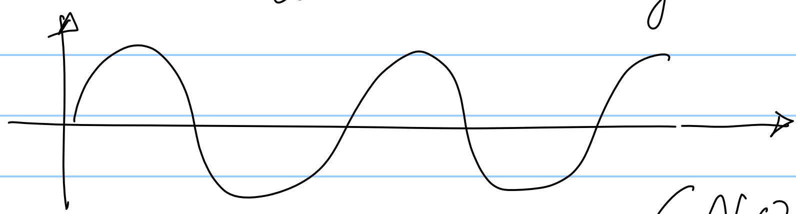
<u>FACT:</u> $G(s)$ is BIBO stable if

and only if all the poles have strictly -ve real part.

Exercise: Let $\alpha_m = 0$ for some m. Which input will make the output unbounded.

Marginal Stability: A transfer f^n is marginally stable if all the poles have non-positive real parts and all purely imaginary axis poles have unit multiplicity.

The impulse response is not absolutely summable but oscillates with fixed magnitude



$$G(s) = \frac{1}{s^2+1}$$

(NOT BIBO stable)

Q) Exercise: Calculate impulse response for

$$G(s) = \frac{1}{(s^2+1)^2}$$

Unstable Systems: At least one pole of RHP and/or poles of multiplicity greater than one on imaginary axis.

Routh – Hurwitz Stability Criterion

Q) Can we predict whether a system is BIBO stable from the denominator of $G(s)$?

Let $G(s) = \dfrac{n(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \; (= d(s))$

Let the poles be r_1, r_2, r_3, r_4. Then

$$d(s) = a_4 (s - r_1)(s - r_2)(s - r_3)(s - r_4)$$

$$\frac{d(s)}{a_4} = s^4 - (r_1 + r_2 + r_3 + r_4) s^3 + (r_1 r_2 + r_2 r_3 + \cdots) s^2 + (-1)^3 (r_1 r_2 r_3 + r_2 r_3 r_4 + \cdots) s + (r_1 r_2 r_3 r_4)$$

Sufficient condition for instability

If (i) all coefficients $a_0, \ldots, a_4$ are not of the same sign

(ii) any coeff is zero

then the system is unstable.

Q) How many poles are on RHP, LHP and $j\omega$-axis?

Ans: Routh Hurwitz Criterion

$$G(s) = \frac{N(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

Step 1: Generate Routh's Table

Initial Layout

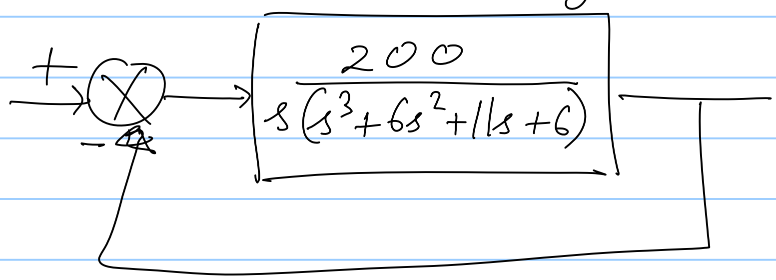
$$\begin{array}{llll}
s^4 & a_4 & a_2 & a_0 \\
s^3 & a_3 & a_1 & 0 \\
s^2 & & & \\
s^1 & & & \\
s^0 & & &
\end{array}$$

$$\begin{array}{llll}
s^4 & a_4 & a_2 & a_0 \\
s^3 & a_3 & a_1 & 0 \\
s^2 & -\frac{1}{a_3}\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix} = b_1 & -\frac{1}{a_3}\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix} = b_2 & -\frac{1}{a_3}\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix} = 0 \\
s^1 & -\frac{1}{b_1}\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix} = c_1 & -\frac{1}{b_1}\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix} = 0 & -\frac{1}{b_1}\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix} = 0 \\
s^0 & -\frac{1}{c_1}\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix} & -\frac{1}{c_1}\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix} = 0 & -\frac{1}{c_1}\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix} = 0
\end{array}$$

Step 2: Apply the following FACT

FACT: The number of roots of the polynomial $d(s)$ that are in the right half plane is equal to the number of sign changes in the first column

Example:



$$T(s) = \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200}$$

s^4	1	11	200	$+$
s^3	$\cancel{6}\ 1$	$\cancel{6}\ 1$	$\cancel{0}\ 0$	$+$
s^2	$\cancel{10}\ 1$	$\cancel{200}\ 20$	0	$+$
s^1	-19			$-$
s^0	20			$+$

Conclusion: 2 sign change

2 RHP poles
2 LHP poles

Numerically: $\left.\begin{array}{l} -4.27 \pm j2.54 \\ +1.27 \pm j2.54 \end{array}\right\}$

Q) How do we know that there are no poles on $j\omega$-axis from the ROUTH-table?

Example: $d(s) = s(s+2) = s^2 + 2s + 0$

s^2	1	0
s^1	2	0
s^0	0	0

Cannot predict instability.

Special Case 1: The first element of a row is zero but the entire row is not zero.

Problem → division by zero

Soln: Replace zero by ε & complete table.

$\uparrow$ small

Example: $G(s) = \dfrac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$

				Sign of 1st colm: ε +ve	ε -ve
s^5	1	3	5	+	+
s^4	2	6	3	+	+
s^3	$\cancel{0}$ ε	$7/2$	0	+	−
s^2	$\dfrac{6\varepsilon - 7}{\varepsilon}$	3	0	−	+
s^1	$\dfrac{42\varepsilon - 49 - 6\varepsilon^2}{12\varepsilon - 14}$	0	0	+	+
s^0	3	0	0	+	+

Conclusion: 2 sign changes → so two poles on RHP.

Special Case 2: An entire row is zero.

Example: $G(s) = \dfrac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$

s^5	1	6	8
s^4	$\cancel{7}\ 1$	$\cancel{42}\ 6$	$\cancel{56}\ 8$
s^3	$\cancel{0}\ \cancel{4}\ 1$	$\cancel{0}\ \cancel{12}\ 3$	$\cancel{0}\ \cancel{0}\ 0$
s^2	3	8	0
s^1	$1/3$	0	0
s^0	8	0	0

Create a polynomial with the previous row: start with the power of "s" in the index column and include alternate powers thereafter.

$$P(s) = s^4 + 6s^2 + 8$$

$$\frac{dP(s)}{ds} = 4s^3 + 12s + 0$$

Use these coefficients instead and complete the table.

Conclusion: No sign change. So no RHP poles.

Problem: Recall for BIBO stability purely imaginary poles are also not allowed.

But the above cases only gives information about RHP poles.

Example: Check numerically the poles of (last example)

$$G(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

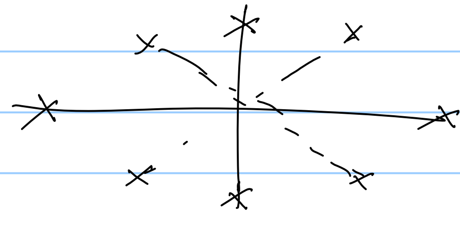
Poles: $-7, \pm 2j, \pm 1.414j$

Q) Is it possible to predict the no. of $j\omega$ axis poles also from Routh's table?

Ans: Yes

* An entire zero row can occur only if a polynomial with only even powers of s divides (is a factor of) the original polynomial.
e.g. $s^4 + 5s^2 + 8$ (Even poly)

* Such polynomials have roots only symmetrically placed about the origin



* The row previous to the zero row contain this Even polynomial.

* Everything from the row containing the even polynomial downward is a test only for the even poly.

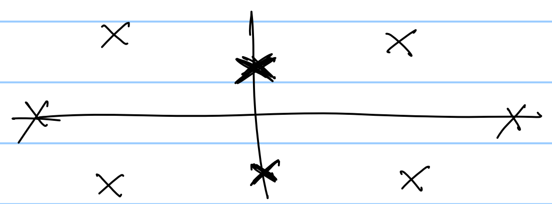
How to check no of jω-axis poles?

Check if there is a zero row?

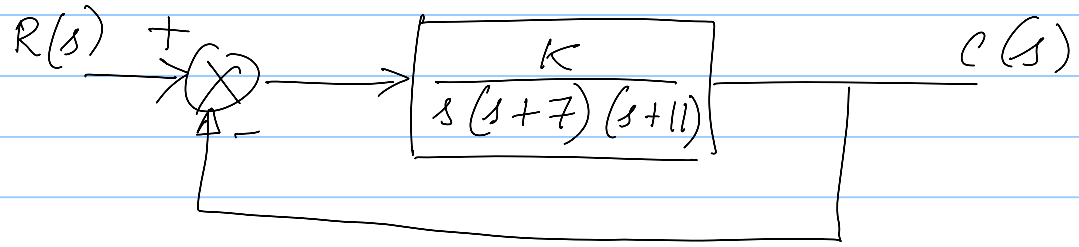
NO → there are no jω poles

YES →

No of jω poles = Order of even poly − 2 × (No of sign changes in Even poly part of table)



Stability Design via Routh-Hurwitz



Find the range of gain K for which the CL - system is BIBO stable. (poles are on jω-axis/unstable)
Assume K > 0.

$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

s^3	1	77	**BIBO stable** $0 < K < 1386$
s^2	18	K	
s^1	$\frac{1386-K}{18}$	0	**Unstable** $K > 1386$ (2 sign changes)
s^0	K	0	

If $K = 1386$, we have a zero row.

Form $P(s) = 18s^2 + 1386$

$$\frac{dP(s)}{ds} = 36s + 0$$

s^3	1	77	No sign changes
s^2	18	1386	So no RHP poles
s^1	$\cancel{0}$ 36	0	So 2 poles on
s^0	1386		$j\omega$-axis

STABILITY OF LTI SYSTEMS

Asymptotic Stability / Internal Stability

Internal / Asymptotic stability refers to the stability of a realization of a system:

$$(*) \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

(*) is asymptotically stable if the solution of

(**) — $\dot{x}(t) = Ax(t)$ $\quad x(0) = x_0$

satisfies $\|x(t)\|_2 \longrightarrow 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

Where:

$$\|x(t)\|_2 = \sqrt{x_1^2(t) + x_2^2(t) + \cdots + x_n^2(t)}$$

Recall: $x(t) = e^{At}x_0$ $\quad$ i.e $\|e^{At}x_0\| \to 0$

Let $\lambda_1, \ldots, \lambda_n$ be the n eigenvalues of A.

FACT: The realization (*) is asymptotically stable if and only if $Re(\lambda_i) < 0$ for all $i = 1, \ldots, n$.

i.e. if and only if all eigenvalues are inside the open left half of the complex plane.

Example

$$\dot{x} = \underbrace{\left[\begin{array}{cc|c} -2 & 0 & 0 \\ 1 & -2 & 0 \\ \hline 0 & 0 & -2 \end{array}\right]}_{A} x$$

with $x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Characteristic polynomial of A is

$$a(s) = \det(sI - A) = (s+2)^3$$

$$(sI - A)^{-1} = \frac{1}{(s+2)^3} \begin{bmatrix} (s+2)^2 & 0 & 0 \\ (s+2) & (s+2)^2 & 0 \\ 0 & 0 & (s+2)^2 \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}\{(sI - A^{-1})\} = \begin{bmatrix} e^{-2t} & 0 & 0 \\ te^{-2t} & e^{-2t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

For any arbitrary $x_0 = [x_{01} \; x_{02} \; x_{03}]^T \in \mathbb{R}^3$,

$$x(t) = \begin{bmatrix} e^{-2t} & 0 & 0 \\ te^{-2t} & e^{-2t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$$

$$\lim_{t \to \infty} x_1(t) = \lim_{t \to \infty} x_2(t) = \lim_{t \to \infty} x_3(t) = 0.$$

Hence A.S.

~~A slightly different (weaker) concept of stability is:~~

~~DEFINITION: The system is STABLE if $\|x(t)\|$ is bounded for all $t \geq 0$, for all initial conditions.~~

NOT REQ

FACT: The system is stable if and only if $\text{Re}\,\lambda_i \leq 0$ for all $i = 1, \cdots, n$ AND all eigenvalues with zero real part NOT have multiplicity of 1 in the minimal polynomial of A.

* Asymptotic stability implies BIBO stability. $h(t) = Ce^{At}B$

* **NOTE:** In general the converse is not true. i.e.

$$\text{BIBO stability} \not\Rightarrow \text{Asymptotic stability}$$

Example: Consider a realization:

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Eigenvalues of $A \equiv (1, -1)$.
Hence it is NOT asymptotically stable.

Now consider $h(t) = Ce^{At}B$

$$e^{At} = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} \quad \Bigg| \quad = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e^{t} \\ e^{-t} \end{bmatrix} = e^{-t}.$$

$$\int_0^\infty |h(t)|\, dt = \int_0^\infty |e^{-t}|\, dt = 1 < \infty$$

Hence the system is BIBO stable.

REF. MATERIAL (NOT REQ IN THIS COURSE)

FACT: Let $x(t)$ be the solution to

$$\dot{x}(t) = Ax(t) \qquad x(0) = x_0.$$

Then each coordinate $x_j(t)$ $(j = 1, \dots, n)$ of $x(t)$ is a linear combination of the following n functions:

a) the function $t^l e^{t\lambda}$ where λ runs through the distinct real eigenvalues of A, and l is an integer with $0 \leq l <$ multiplicity of λ in the minimal poly

AND

b) the functions

$$t^l e^{at} \cos bt \quad \text{and} \quad t^l e^{at} \sin bt$$

where $a + ib$ runs through the complex eigenvalues of A having $b > 0$ and l is an integer in the range $0 \leq l <$ multiplicity of $a + ib$ in the minimal poly.

Antenna Control : C. loop Tr. function:

$$T(s) = \frac{6.63K}{s^3 + 101.71s^2 + 171\,8\,16.63K}$$

Find range of K to keep the system stable.

s^3	1	171
s^2	101.71	6.63K
s^1	17392.41 − 6.63K	0
s^0	6.63K	

Hence for stability $0 < K < 2623$