Lecture 7: Steady State Errors

Note Title 28-02-2010

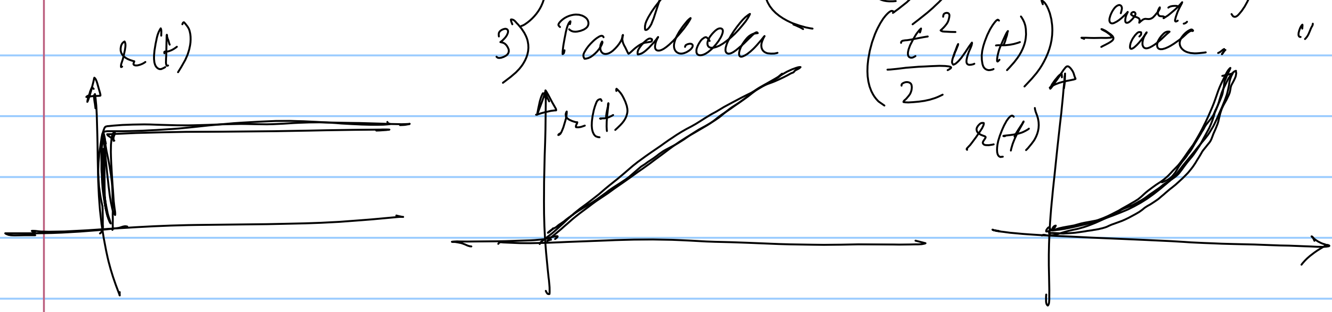
Control Specs: 1) Transient Response ✓
2) Stability ✓
3) Steady State Errors ←

Def^{n}: Steady state error is the difference between input and output (for a particular test input) as $t \to \infty$

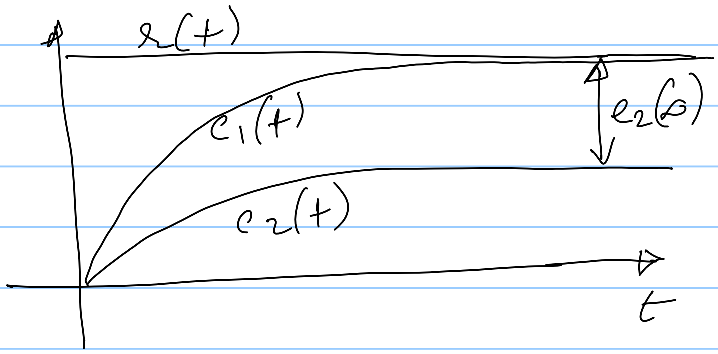
[NOTE: Stability is a system property (not defined in terms of any particular input)]

S.S.E. is defined for a particular I/P.

Test inputs: 1) Step $(u(t)) \to$ const. position tracking
2) Ramp $(t\,u(t)) \to$ const. velocity "
3) Parabola $\left(\frac{t^2}{2}u(t)\right) \to$ const. acc. "



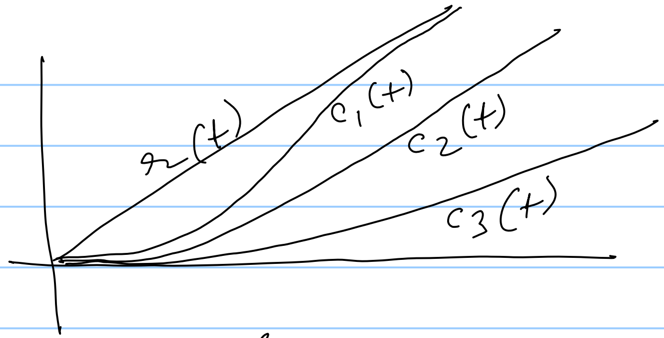
Assumption (Chap. wide): All systems are STABLE



$e_1(\infty) = r(\infty) - c_1(\infty) = 0$

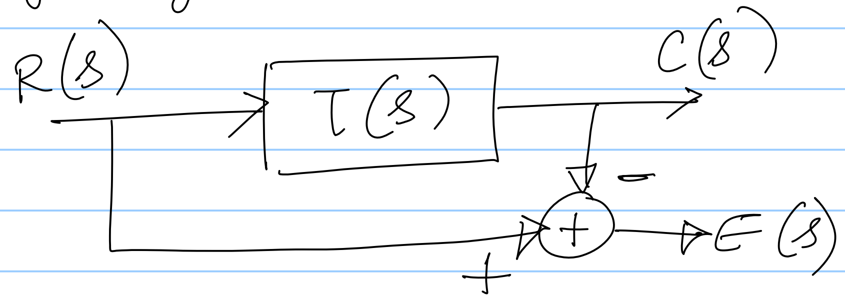
$e_2(\infty) = r(\infty) - c_2(\infty)$

S.S.E. for step input



S.S.E for Ramp Input

SS.E. for general T.F. : $T(s)$

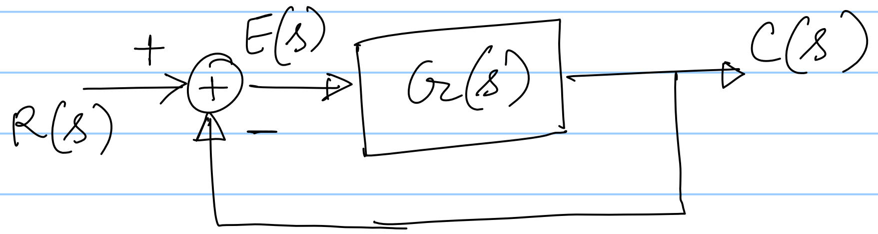


$$S.S.E = \lim_{t\to\infty} e(t) = \lim_{t\to\infty}\left[r(t) - c(t)\right]$$

$$= \lim_{s\to 0} s\left[R(s) - C(s)\right]$$

(by F.V. thm)

S.S.E. for Unity Feedback Config



For this config: $E(s) = R(s) - C(s)$

$= S.S.E$ (by defn)

If C.L. tf. is $T(s)$ then $C(s) = T(s)\,R(s)$

Then $E(s) = R(s)\left[1 - T(s)\right]$

$$S.S.E. = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

or $\boxed{S.S.E. = \lim_{s\to 0} sR(s)\left[1 - T(s)\right]}$

Similarly, $E(s) = R(s) - C(s)$

and $C(s) = G(s)E(s)$

$$E(s) = \frac{R(s)}{1+G(s)}$$

$$\boxed{e(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1+G(s)}} \quad \circledast$$

Using $\circledast$ we can calculate $e(\infty)$ for any test signal for the unity feedback configuration.

Step Input: $R(s) = \frac{1}{s}$

$$e(\infty) = \lim_{s \to 0} \frac{s \cdot \frac{1}{s}}{1+G(s)}$$

$$= \frac{1}{1 + \underbrace{\lim_{s \to 0} G(s)}_{K_p}} = \frac{1}{1+K_p}$$

$K_p := \lim_{s \to 0} G(s) =:$ Position Error Constant

* To make $e(\infty) = 0$, $K_p = \infty$ is required

i.e. $\lim_{s \to 0} G(s) = \infty$ is required.

Let $G(s) = \dfrac{(s+z_1)(s+z_2)\cdots\cdots}{s^n (s+p_1)(s+p_2)\cdots}$

* For $\underline{\underline{n=0}}$, $\lim\limits_{s \to 0} G(s) = \dfrac{z_1 z_2 \cdots}{p_1 p_2 \cdots}$

Hence, $e(\infty) \neq 0$

* For $n \geqslant 1$, $\lim\limits_{s \to 0} G(s) = \infty \Rightarrow e(\infty) = 0$

<u>FACT</u>: The S.S.E. is zero for step input only when there are $\geqslant 1$ integrators in the forward path (for unity feedback system)

<u>Exercise</u>: 1) Explain this intuitively.
2) What is the S.S.E for a pure gain in the forward path?

<u>Ramp Input</u>: $R(s) = \dfrac{1}{s^2}$

$$e(\infty) = \lim_{s \to 0} \frac{s \cdot \frac{1}{s^2}}{1 + G(s)} = \frac{1}{\underbrace{\lim\limits_{s \to 0} s\,G(s)}_{K_v}}$$

$$= \frac{1}{K_v}$$

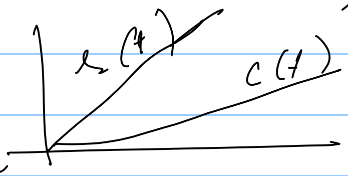
$K_v := \lim\limits_{s \to 0} s\,G(s) =:$ velocity error constant

For $e(\infty) = 0$ we require $K_v = \lim\limits_{s \to 0} s\,G(s) = \infty$

$G(s) = \dfrac{(s+z_1)(s+z_2)\cdots}{s^n (s+p_1)(s+p_2)\cdots}$

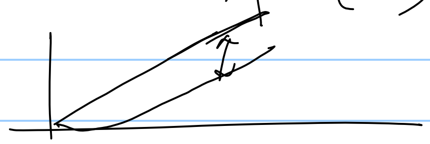
[Strictly speaking this calculation is not valid. (F.V. thm is not applicable). However this is conventionally done since it gives correct answer.]

* For $n = 0$, $K_v = \lim_{s \to 0} s\,G(s) = 0 \Rightarrow e(\infty) = \infty$ (Diverging ramps)



* For $n = 1$, $K_v = \lim_{s \to 0} s\,G(s) = \dfrac{z_1 z_2 \cdots}{P_1 P_2 \cdots}$

$\Rightarrow 0 < |e(\infty)| < \infty$ (Finite error)



* For $n \geq 2$, $K_v = \lim_{s \to 0} s\,G(s) = \infty \Rightarrow e(\infty) = 0$

<u>FACT</u>: The S.S.E. for unit ramp is zero only when there are at least 2 integrators in the fwd. path.

<u>Parabolic Input</u>: $R(s) = \dfrac{1}{s^3}$

$$e(\infty) = \lim_{s \to 0} \frac{\frac{1}{s^3} \cdot s}{1 + G(s)} = \frac{1}{\lim_{s \to 0} s^2 G(s)} = \frac{1}{K_a}$$

$K_a := \lim_{s \to 0} s^2 G(s) =:$ Acceleration error constant

Clearly
$n \geq 3 \Rightarrow e(\infty) = 0$
$n = 2 \Rightarrow 0 < |e(\infty)| < \infty$
$n = 0, 1 \Rightarrow |e(\infty)| = \infty$

This calculation is also not strictly valid. (F.V. thm is not applicable)

System Type: Since S.S.E. for various test inputs depend on the no. of integrations in fwd. path systems are classified into "Types" accordingly.

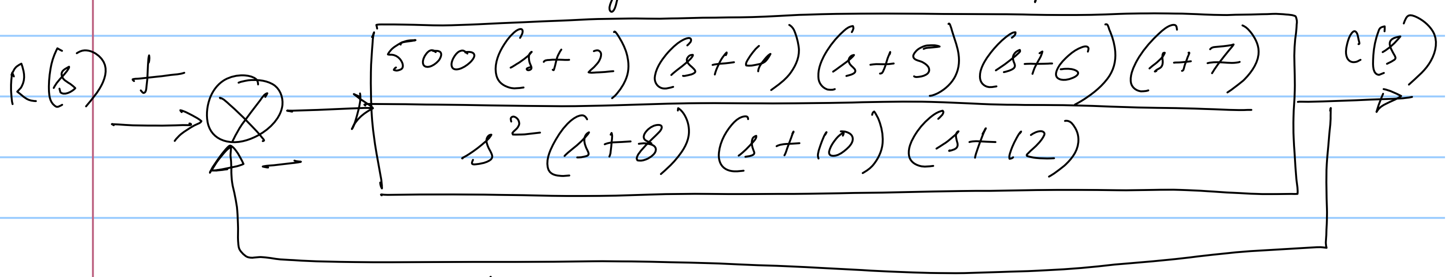
0 integrations $\Leftrightarrow (n=0)$ =: Type '0'

1 integrators $\Leftrightarrow (n=1)$ =: Type '1'

2 integrators $\Leftrightarrow (n=2)$ =: Type '2'

* **Exercise**: Read Table 7.2 [IMP]

Example: Find the static error constants and S.S.E. for step, ramp & parabolic I/P.



First verify stability: (Use MATLAB) ←

$$K_p = \lim_{s\to 0} G(s) = \infty$$

$$K_v = \lim_{s\to 0} sG(s) = \infty$$

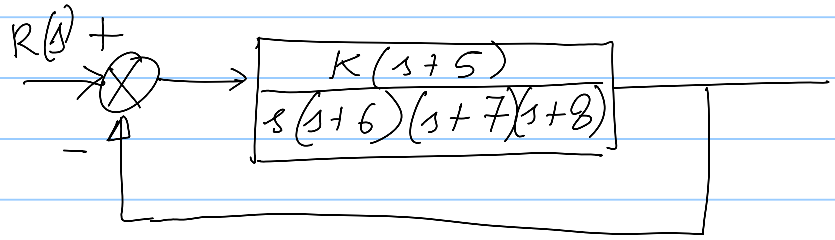
$$K_a = \lim_{s\to 0} s^2 G(s) = \frac{500\times 2\times 4\times 5\times 6\times 7}{8\times 10\times 12} = 875$$

For $r(t) = u(t)$, $e(\infty) = \frac{1}{1+K_p} = 0$

$r(t) = t\,u(t)$, $\quad e(\infty) = \dfrac{1}{K_v} = 0$

$r(t) = \frac{1}{2}t^2 u(t)$, $\quad e(\infty) = \dfrac{1}{K_a} = 1.14 \times 10^{-3}$

Example:



Find K s.t. S.S.E. is 10%.

Type 1 system → So S.S.E. spec must correspond to ramp input. (Ramp only yields finite error in Type 1)

$$e(\infty) = 0.1 = \frac{1}{\lim_{s\to 0} sG(s)}$$

i.e. $\lim_{s\to 0} sG(s) = 10$

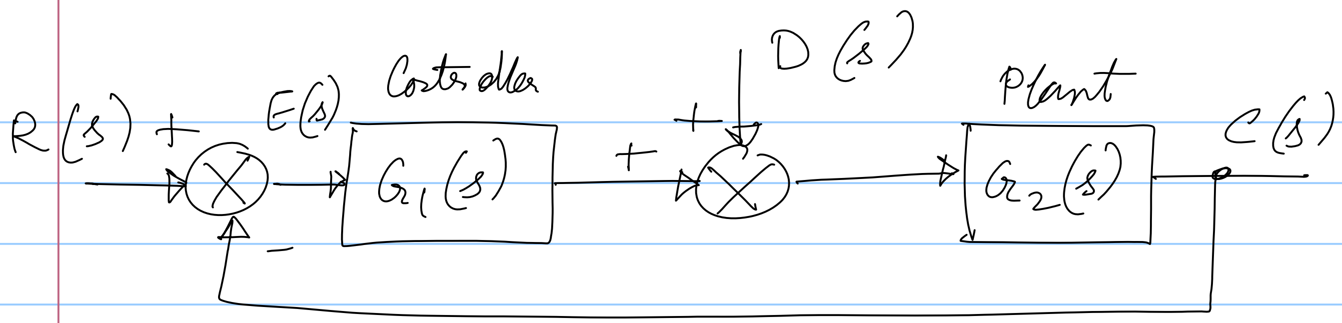
$$\Rightarrow \frac{5K}{6\times7\times8} = 10 \Rightarrow K = 672$$

Exercise: 1) Is the system stable at this gain?

2) Can we meet transient response specs?

Steady State Error for Disturbances (Unity Feedback)

* Assumption: Disturbance is injected between controller & plant.



Unity Feedback

$$\begin{cases} C(s) = E(s)\,G_1(s)\,G_2(s) + D(s)\,G_2(s) \\ C(s) = R(s) - E(s) \end{cases}$$

Eliminating $C(s)$ yields:

$$E(s) = \frac{R(s)}{1+G_1(s)G_2(s)} - \frac{G_2(s)}{1+G_1(s)G_2(s)} D(s)$$

$$e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G_1(s)G_2(s)} - \lim_{s\to 0} \frac{s\,G_2(s)D(s)}{1+G_1(s)G_2(s)}$$

$$= \underbrace{e_R(\infty)}_{\text{same as before}} + \underbrace{e_D(\infty)}_{\text{new term due to disturbance}}$$

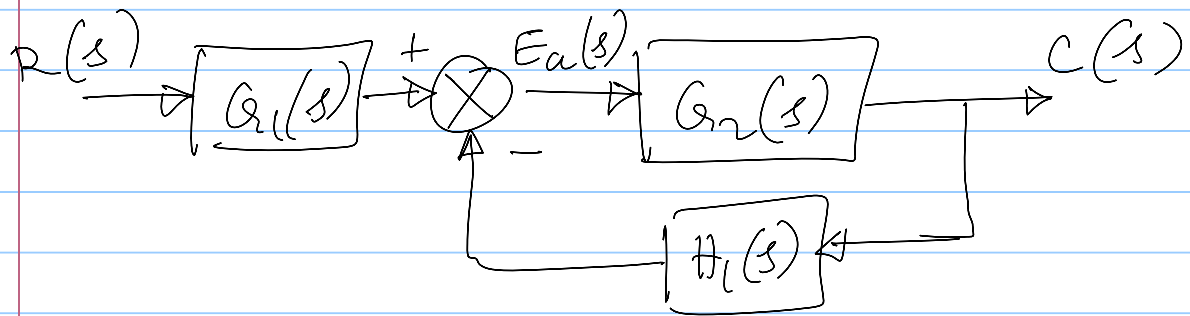
<u>Step Disturbance</u> : $D(s) = \frac{1}{s}$

$$e_D(\infty) = \lim_{s\to 0} \frac{-s\,G_2(s)\cdot\frac{1}{s}}{1+G_1(s)\,G_2(s)}$$

$$= \frac{-1}{\left[\lim_{s\to 0}\frac{1}{G_2(s)}\right] + \left[\lim_{s\to 0} G_1(s)\right]}$$

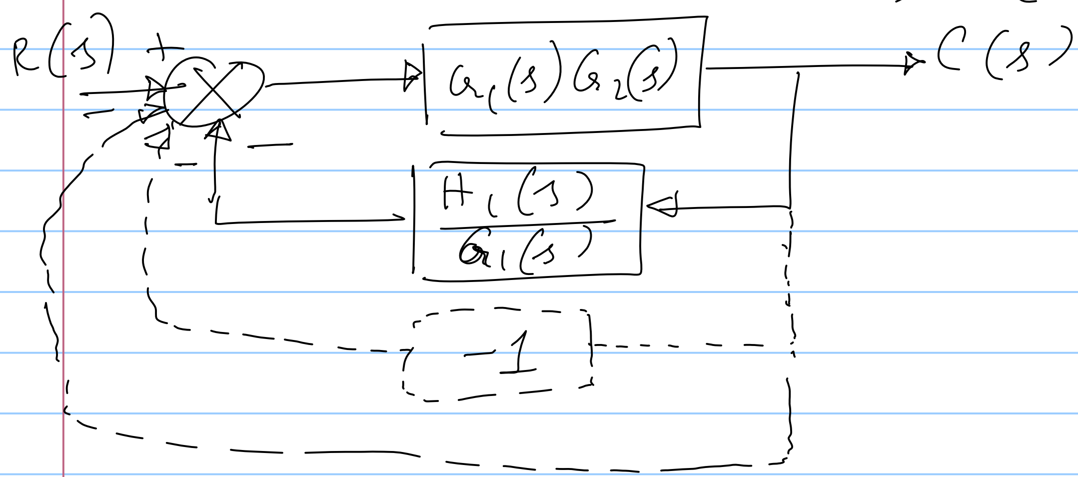
$\Longleftrightarrow$ $e_D(\infty)\downarrow$ as $\lim_{s\to} G_2(s)\downarrow$ $\lim_{s\to 0} G_1(s)\uparrow$

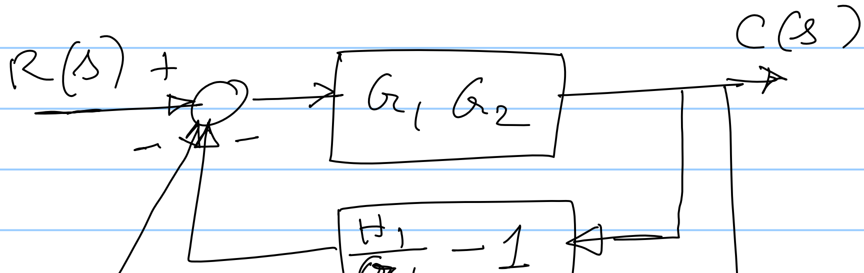
S.S.E. for Non-unity feedback systems

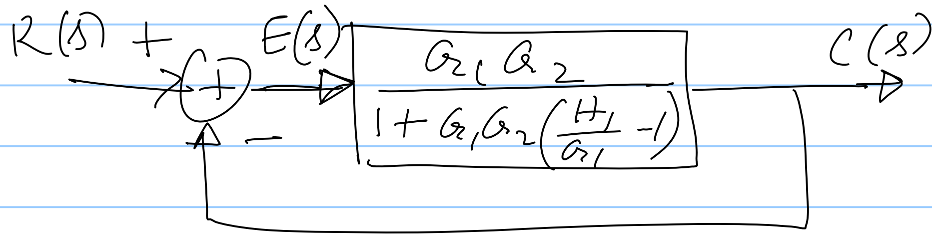


NOTE: $E_a(s)$ (=: Actuating signal) is NOT the S.S.E.

We need to calculate, $E(s) = R(s) - C(s)$

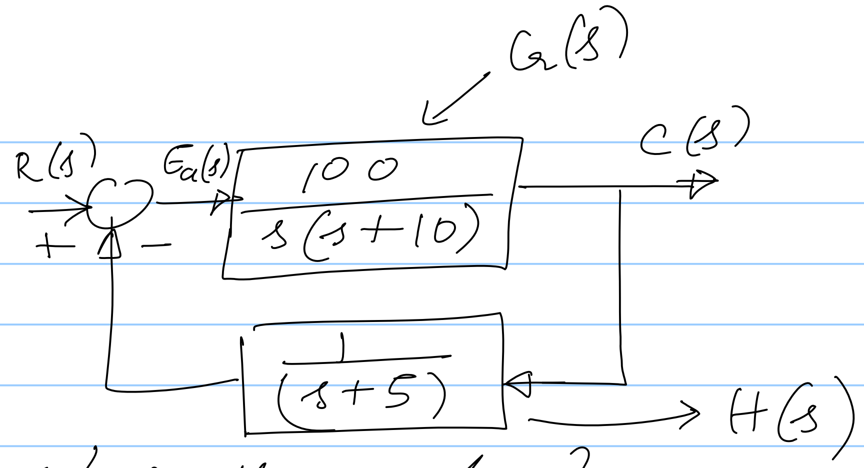






Then use the usual formulae.

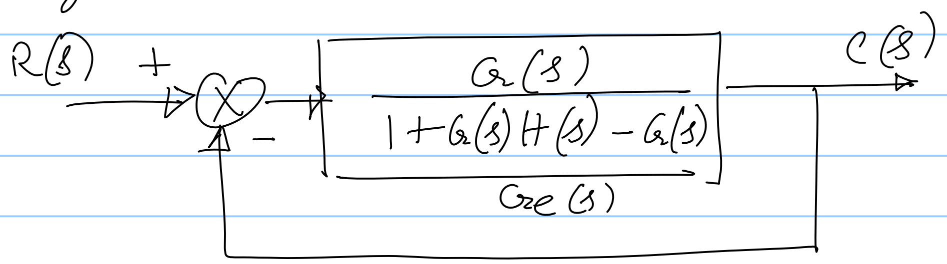
Example :



1) What 'type' is the system?
2) Determine appropriate error constants
3) Determine S.S.E. for unit step.

First check stability ✓

Using the derivation done above,



$$G_e(s) = \frac{G(s)}{1+G(s)[H(s)-1]} = \frac{100(s+5)}{s^3+15s^2-50s-400}$$

System is Type '0' ; $K_p = \lim_{s \to 0} G_e(s)$

$$= -\frac{5}{4}$$

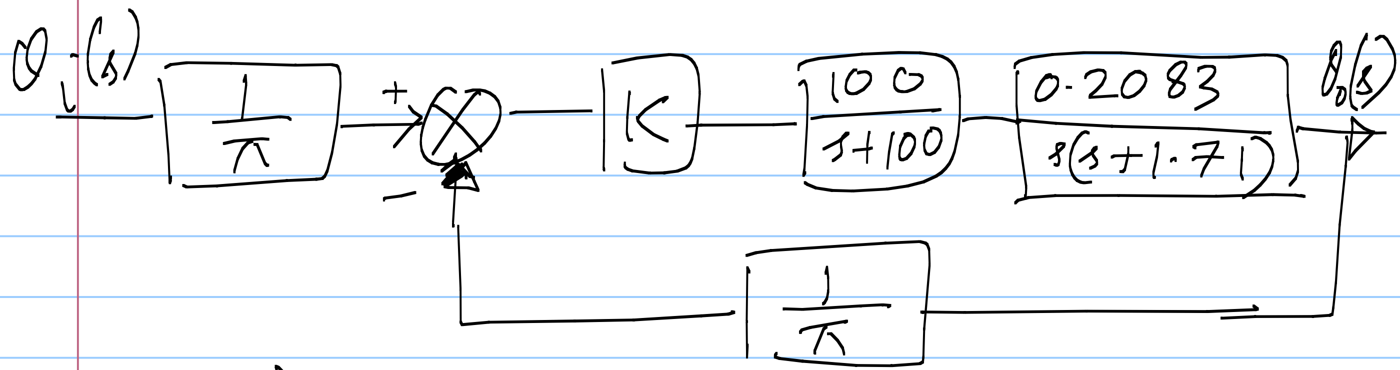
$$e(\infty) = \frac{1}{1+K_p} = \frac{1}{1-(5/4)} = -4$$ (O/P larger than I/P)

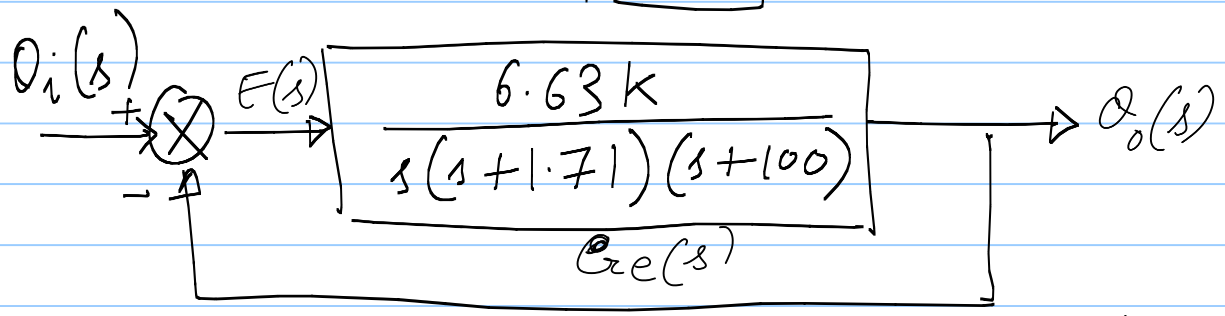
Steady-State Error for St. Sp. System

$$\dot{x} = Ax + Br \qquad E(s) = R(s) - Y(s)$$
$$y = Cx \qquad\qquad = R(s) - T(s)R(s)$$
$$= R(s)\left[1 - C(sI-A)^{-1}B\right]$$

$$e(\infty) = \lim_{s \to 0} sE(s)$$
$$= \lim_{s \to 0} sR(s)\left[1 - C(sI-A)^{-1}B\right]$$

Antenna Control: Steady-State Error Design





$$\text{S.S.E.} = \lim_{t \to \infty}\left[\theta_i(t) - \theta_o(t)\right] = e(\infty)$$

$$e(\infty) = \lim_{s \to 0} sE(s) = \lim \frac{s\,\theta_i(s)}{1 + G_e(s)}$$

Type 1: So $K_p = \lim_{s \to 0} G_e(s) = \infty$

$\Rightarrow$ Step Input $\to e(\infty) = 0$

Ramp Input $\to e(\infty) = \frac{1}{K_v} = \frac{25.79}{K}$

Parabolic I/P $\rightarrow e(\infty) = \frac{1}{K_a} = \infty$

Suppose we want a 10% S.S. error for ramp. I/P.

$$e(\infty) = 0.1 = \frac{1}{K_V} = \frac{25.79}{K} \Rightarrow K = 257.9$$

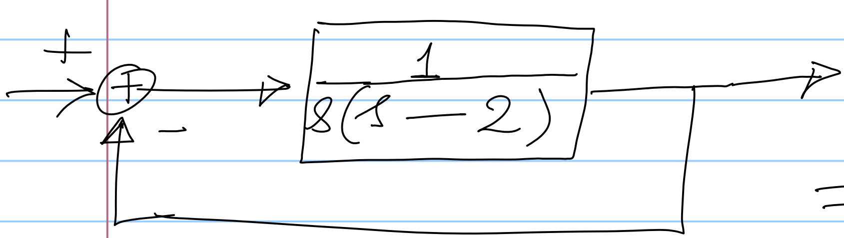
Using ROUTH-HURWITZ criterion, we found the range of stability for K : $0 < K < 2623$.

Hence $K = 257.9$ makes system stable.

$\boxed{\text{NOTE}}$: We need to check STABILITY explicitly before calculating SSE.

REASON : Our error calculations use F. Value thm which in turn is valid only if $sE(s)$ has poles on open LHP.

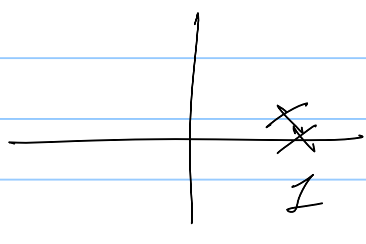
Requirement for Stability



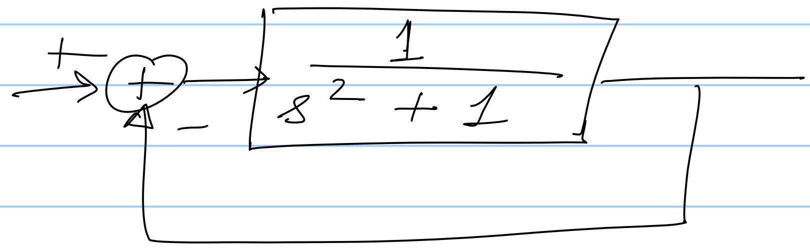
$$T(s) = \frac{G(s)}{1+G(s)} = \frac{\frac{1}{s(s-2)}}{1+\frac{1}{s(s-2)}}$$

$$= \frac{1}{s^2 - 2s + 1}$$

$$= \frac{1}{(s-1)^2}$$



So for any input $R(s) = \frac{1}{s}, \frac{1}{s^2}, \frac{1}{s^3}$ output diverges. No question of S.S.E.



$$T(s) = \frac{1}{s^2 + 2}$$

For unit step, $C(s) = \frac{1}{s^2+2} \cdot \frac{1}{s}$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+2}\right]$$

$$= \left[\frac{1}{2} - \frac{1}{2}\cos\sqrt{2}t\right]u(t)$$

So $\lim_{t \to \infty} e(t)$ does not exist.

Exercise: Try to give an example where the system is unstable / marginally stable but still has finite s.s.e. for step / ramp / parabola.

Ans: