Lecture 7: Steady State Errors

Control Specs: 1) Transient Response 2) Stability 3) Steady State Errors

Definition: Steady state error is the difference between input and output (for a particular test input) as $t \to \infty$

[Note: Stability is a system property (not defined in terms of any particular input)]

S.S.E. is defined for a particular I/P

Test inputs: 1) Step ($u(t)$) → position tracking
2) Ramp $(t \cdot u(t))$ → velocity
3) Parabola $(\frac{t^2}{2} \cdot u(t))$ → acceleration

Assumption (Chap. 7): All systems are STABLE

$$e_1(t) = r(t) - C_1(t) = 0$$
$$e_2(t) = r(t) - C_2(t)$$

S.S.E. for step input
SS.E. for general T.F. : T(s)

\[ R(s) \xrightarrow{T(s)} C(s) \]
\[ E(s) = R(s) - C(s) \]

SS.E. = \( \lim_{t \to \infty} e(t) = \lim_{s \to 0} \left[ R(s) - C(s) \right] \]

SS.E. for Unity Feedback Config

For this config: \( E(s) = R(s) - C(s) \)

If C.L. T.F. is \( T(s) \) the \( C(s) = T(s) R(s) \)

Then \( E(s) = R(s) \left[ 1 - T(s) \right] \)

SS.E. = \( \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \)

\[ \text{or} \]
\[ \text{SS.E.} = \lim_{s \to 0} s R(s) \left[ 1 - T(s) \right] \]
Similarly, \[ E(s) = R(s) - C(s) \]
and \[ C(s) = G(s)E(s) \]
\[ E(s) = \frac{R(s)}{1 + G(z)} \]
\[ E(z) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(z)}{1 + G(z)} \]

Using this, we can calculate \( e(z) \) for any test signal for the unity feedback configuration.

**Step 1: Input** \( R(s) = \frac{1}{s} \)

\[ e(s) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s}{1 + G(z)} \]

\[ = \frac{1}{1 + \lim\limits_{s \to 0} G(z)} = \frac{1}{1 + K_p} \]

\( K_p := \lim\limits_{s \to 0} G(z) \) \( \equiv \) Position Error Constant

To make \( e(s) = 0 \), \( K_p = 0 \) is required
i.e. \( \lim\limits_{s \to 0} G(z) = 0 \) is required.
Let \( G(s) = \frac{(s+z_1)(s+z_2)}{s^n(s+p_1)(s+p_2)} \)

For \( n = 0 \), \( \lim_{s \to 0} G(s) = \frac{z_1 z_2}{p_1 p_2} \)

Hence, \( e(\infty) \neq 0 \)

For \( n > 1 \), \( \lim_{s \to 0} G(s) = 0 \Rightarrow e(\infty) = 0 \)

**FACT**: The S.S.E. is zero for a step input only when there are \( \geq 1 \) integrators in the forward path (feed unity feedback system)

**Exercise**: 1) Explain this intuitively.
2) What is the S.S.E. for a pure gain in the forward path?

**Ramp Input**: \( R(s) = \frac{1}{s^2} \)

\( e(\infty) = \lim_{s \to 0} s \cdot \frac{1}{s^2} = \lim_{s \to 0} s e(s) \)

\( = \frac{1}{K_v} \)

\( K_v = \lim_{s \to 0} s e(s) = \text{velocity error constant} \)

For \( e(\infty) = 0 \) we require \( K_v = \lim_{s \to 0} s e(s) = 0 \)

\( e(\infty) = \frac{(s+z_1)(s+z_2)}{s^n(s+p_1)(s+p_2)} \)
Strictly speaking this calculation is not valid. (F. V. theorem is not applicable). Hence when this is conventionally done since it gives correct answer:

For \( n = 0 \), \( K_v \lim s G(s) = 0 \Rightarrow e(\infty) = 0 \) (Diverging ramps)

For \( n = 1 \), \( \lim s G(s) = \frac{Z_1}{P_1 P_2 \cdots} \Rightarrow 0 < |e(\infty)| < \delta \) (Finite error)

For \( n > 2 \), \( K_v \lim s G(s) = 0 \Rightarrow e(\infty) = 0 \)

**FACT**: The S.S.E. for unit ramp is zero only when there are at least 2 integrators in the feed path.

Parabolic Input: \( R(s) = \frac{1}{s^3} \)

\[
e(\infty) = \lim_{s \to 0} \frac{\frac{1}{s^3}}{1 + a(s)} = \frac{1}{\lim_{s \to 0} s^2 G(s)} = \frac{1}{K_a}
\]

\( K_a := \lim_{s \to 0} s^2 G(s) = : \text{Acceleration error constant} \)

Clearly \( n > 3 \Rightarrow e(\infty) = 0 \)

\( n = 2 \Rightarrow 0 < |e(\infty)| < \delta \)

\( n = 0, 1 \Rightarrow |e(\infty)| = \infty \)

This calculation is also not strictly valid (F. V. theorem is not applicable)
System Type: Since S.S.E. for various test inputs depend on the no. of integrations in feedback path, systems are classified into "Types" accordingly.

0 integrator \( \Rightarrow (n = 0) = \text{Type '0'} \)

1 integrator \( \Rightarrow (n = 1) = \text{Type '1'} \)

2 integrators \( \Rightarrow (n = 2) = \text{Type '2'} \)

Exercise: Read Table 7.2 \( [\text{IMP}] \)

Example: Find the static error constants and S.S.E. for step, ramp & parabolic input.

First verify stability: (Use MATLAB)

\[ K_p = \lim_{s \to 0} G(s) = \infty \]

\[ K_v = \lim_{s \to 0} sG(s) = \infty \]

\[ K_a = \lim_{s \to 0} s^2 G(s) = \frac{500 \times 2 \times 4 \times 5 \times 6 \times 7}{8 \times 10 \times 12} = 875 \]

For \( r(t) = u(t), \ e(s) = \frac{1}{1 + K_p} = 0 \)
\[ r(t) = tu(t), \quad e(\infty) = \frac{1}{kv} = 0 \]
\[ r(t) = \frac{1}{2} t^2 u(t), \quad e(\infty) = \frac{1}{ka} = 1.14 \times 10^{-3} \]

Example:

\[
\begin{array}{c}
\text{Find } K \text{ s.t. S.S.E. is 10.} \\
\text{Type 1 system \( \rightarrow \) S.S.E. spec must correspond to ramp input. (Ramp only yields finite error in)} \\
\text{\( e(\infty) = 0.1 = \lim_{s \to 0} \frac{K(s+5)}{s(6+6)(s+7)(s+8)} \) \quad \text{Type 1}}
\end{array}
\]

\[ \lim_{s \to 0} s \cdot e(s) = 10 \]
\[ \Rightarrow \frac{5K}{6 \times 7 \times 8} = 10 \Rightarrow K = 672 \]

Exercise: 1) Is the system stable at this gain? 2) Can we meet transient response specs?

**Steady State Error for Disturbances (Unity Feedback)**

Assumption: Disturbance is injected between controller & plant.
\[
\begin{align*}
R(s) + E(s) \xrightarrow{\text{Controller}} G_1(s) & \quad + \quad D(s) \quad \xrightarrow{\text{Plant}} C(s) \\
\text{Unity Feedback} & \\
C(s) & = E(s) G_1(s) G_2(s) + D(s) G_2(s) \\
C(s) & = R(s) - E(s) \\
\text{Eliminatory} \ C(s) \text{ yields:} & \\
E(s) & = \frac{R(s)}{1 + G_1(s) G_2(s)} - \frac{G_2(s)}{1 + G_1(s) G_2(s)} D(s) \\
C(s) & = \lim_{s \to \infty} \frac{s R(s)}{1 + G_1(s) G_2(s)} - \lim_{s \to \infty} \frac{s G_2(s) D(s)}{1 + G_1(s) G_2(s)} \\
& = e_R(s) + e_D(s) \\
& \text{Same as before} \\
& \text{new term due to disturbance} \\
\text{Step Disturbance:} \quad D(s) = \frac{1}{s} \\
e_D(s) & = \lim_{s \to 0} \frac{-s G_2(s)}{1 + G_1(s) G_2(s)} \cdot \frac{1}{s} \\
& = \frac{-1}{\lim_{s \to 0} G_1(s) G_2(s)} + \left[ \lim_{s \to 0} G_1(s) \right] \\
& \Rightarrow e_D(s) \downarrow \text{as} \lim_{s \to 0} G_2(s) \uparrow \lim_{s \to 0} G_1(s) \downarrow
\end{align*}
\]
S.S.E. for Non-unity feedback systems

\[
R(s) \xrightarrow{[G_1(s)]} \times \xrightarrow{E_a(s)} \xrightarrow{G_m(s)} \xrightarrow{H(s)} \xrightarrow{G_n(s)} C(s)
\]

**NOTE:** \(E_a(s)\) (Actuating signal) is NOT the S.S.E.

We need to calculate, \(E(s) = R(s) - C(s)\)

Then use the usual formulae.
Example: \( R(s) \rightarrow \left( \frac{100}{s(s+10)} \right) \rightarrow \left( \frac{1}{s+5} \right) \rightarrow C(s) \rightarrow G_c(s) \rightarrow C(s) \rightarrow H(s) \)

1) What 'type' is the system?
2) Determine appropriate error constant
3) Determine S.S.E. for unit step

- First check stability

Using the derivation done above,

\[ G_e(s) = \frac{G_c(s)}{1 + G_c(s)H(s) - \delta} = \frac{100(s+5)}{s^3 + 15s^2 - 50s - 400} \]

System is Type '0'; \( K_p = \lim_{s \to 0} G_e(s) = \frac{5}{4} \)

\[ e(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 - (5/4)} = -4 \]

\( 0/p \) larger than \( 1/p \)

Steady-State Error for St. Sp. System
\[ i = A n + B r \]
\[ y = e x \]
\[ E(s) = R(s) - Y(s) \]
\[ = R(s) - \frac{1}{s} R(s) \]
\[ = R(s)[1 - C(sI - A)^{-1}B] \]

\[ E(s) = \lim_{s \to 0} s E(s) \]
\[ = \lim_{s \to 0} s R(s)[1 - C(sI - A)^{-1}B] \]

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**Antenna Control: Steady-State Error Design**

\[ \theta_i(s) \]

\[ \frac{1}{\tau} \quad + \quad \times \quad - \quad \frac{1}{\tau} \]

\[ \theta_o(s) \]

\[ 100 \quad \frac{1}{s+100} \quad 0.2083 \quad \frac{1}{s(s+1.71)} \]

\[ \theta_i(s) \]

\[ 6.63K \]

\[ \frac{1}{s(s+1.71)(s+100)} \]

\[ \theta_o(s) \]

\[ \epsilon(s) = \lim_{s \to 0} [\theta_i(t) - \theta_o(t)] = e(\infty) \]

\[ e(\infty) = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \theta_i(s) \]

**Type 1:**

\[ s \theta_e(s) = \lim_{s \to 0} \theta_e(s) = \infty \]

\[ \Rightarrow \text{Step Input} \to e(\infty) = 0 \]

**Ramp Input:**

\[ e(\infty) = \frac{1}{K_v} = \frac{25.79}{K} \]
Parabolic $T/P \rightarrow e(\theta) = \frac{1}{Ko} = \infty$

Suppose we want a 10\% S.S. error for ramp $T/P$.

$e(\theta) = 0.1 = \frac{1}{Ku} = \frac{25.79}{K} \rightarrow K = 257.9$

Using ROUTH-HURWITZ criterion, we found the range of stability for $K: 0 < K < 2623$

Hence $K = 257.9$ makes system stable

**NOTE:** We need to check STABILITY explicitly before calculating $TSE$.

**REASON:** Our errors calculation use F. Value that is valid only if $sE(s)$ has poles on open CHP.

Requirement for Stability

\[ T(s) = \frac{G(s)}{1+G(s)} = \frac{1}{s(s-2)} \]

\[ T(s) = \frac{1}{s^2 - 2s + 1} \]

\[ = \frac{1}{(s-1)^2} \]

\[ = \frac{s(s-2)}{s^2 - 2s + 1} \]

\[ = \frac{1}{s^2 - 2s + 1} \]
So for any input $R(s) = \frac{1}{s}, \frac{1}{s^2}, \frac{1}{s^3}$ output diverges. No question of S.S.E.

$$T(s) = \frac{1}{s^2 + 2}$$

For unit step, $C(s) = \frac{1}{s^2 + 2} \cdot \frac{1}{s}$

$$= \frac{1}{2} \left[ \frac{1}{s} - \frac{8}{s^2 + 2} \right]$$

$$= \left[ \frac{1}{2} - \frac{1}{2} \cos \sqrt{2} t \right] u(t)$$

So $\lim_{t \to \infty} e(t)$ does not exist.

Exercise: Try to give an example where the system is unstable/marginally stable but still has finite S.S.E. for step/ramp/parabola.