Lecture 8: Root Locus - Analysis

Note Title 11-06-2008

(Compensator design is covered in next lecture)

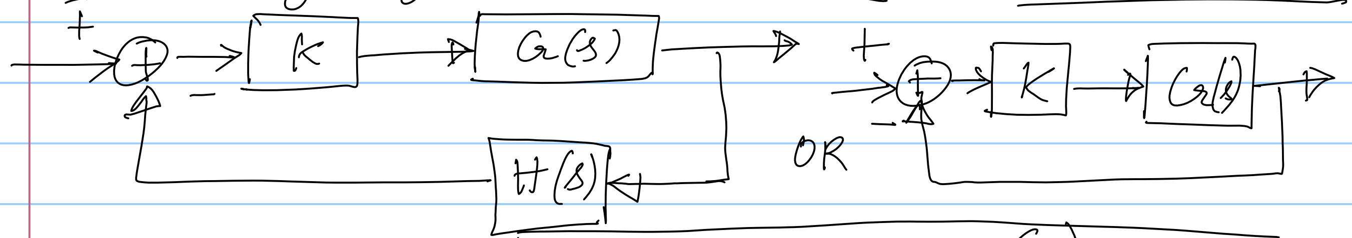
We know how to analyse/compute:

1) Transient behaviour of $\leq 2^{nd}$ order sys.
2) Stability analysis of any order sys. (R-H cri)

Root locus let us analyse:

1) Transient behaviour of any order sys
2) Stability of any order sys.-

* Underlying Assumption: Feedback Config



C.L Tr. fn: $\boxed{T(s) = \dfrac{K\,G(s)}{1 + K\,G(s)\,H(s)}}$

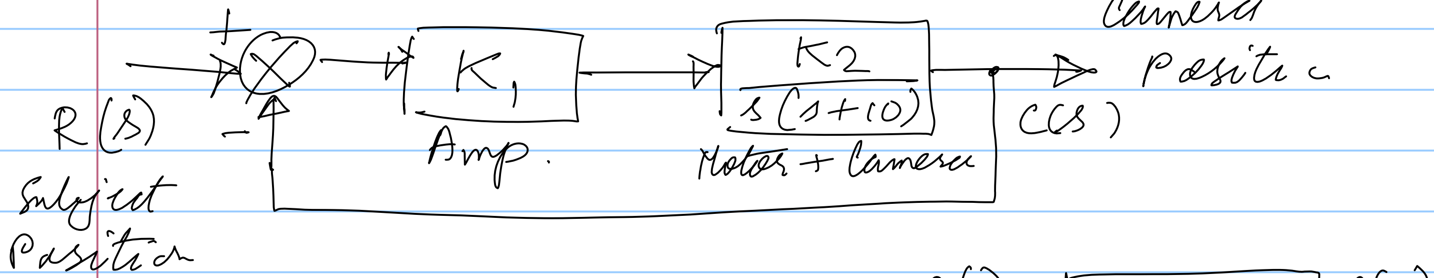
* C.L. pole location of $T(s)$ are dependent on the value of K.

Q) How does the closed loop pole locations change (in the above config) as K is varied?

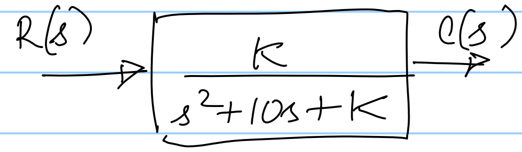
Answer: Plot the C.L poles as K is varied from 0 to ∞.

→ This plot is called Root locus

Example 1: Tracking Camera: It follows a subject wearing infrared sensors on the body.

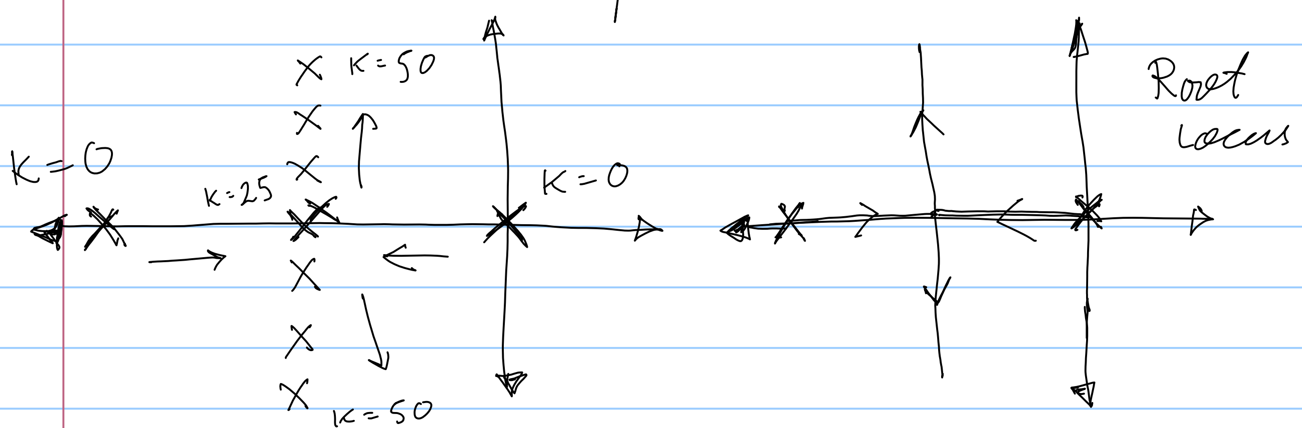


Let $K = K_1 . K_2$.



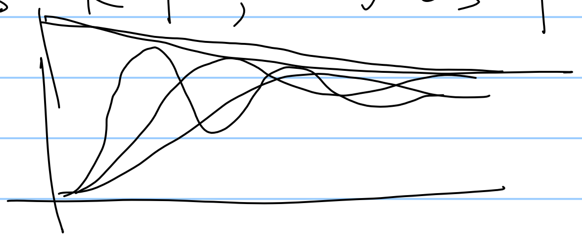
damping decreases ↓

K	Pole 1	Pole 2	
0	-10	0	Over damped
5	-9.47	-0.53	
10	-8.87	-1.13	
⋮	⋮	⋮	
25	-5	-5	Critically damped
30	$-5+j2.24$	$-5-j2.24$	Under damped
⋮	⋮	⋮	
50	$-5+j5$	$-5-j5$	
	⋮	⋮	



* For $K > 25$, T_s remains the same

* As $K \uparrow$, $\%OS \uparrow$, $T_p \downarrow$



Representation of T.F. in complex plane

$$F(s) = \frac{\prod_{i=1}^{m} (s+z_i)}{\prod_{k=1}^{n} (s+p_k)}$$

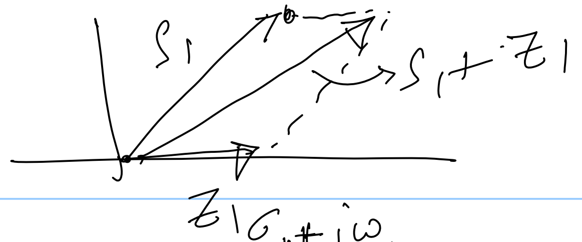
$$F(s)\Big|_{s=\sigma_1+j\omega_1} = \frac{\prod_{i=1}^{m} (\sigma_1+j\omega_1+z_i)}{\prod_{k=1}^{n} (\sigma_1+j\omega_1+p_k)}$$

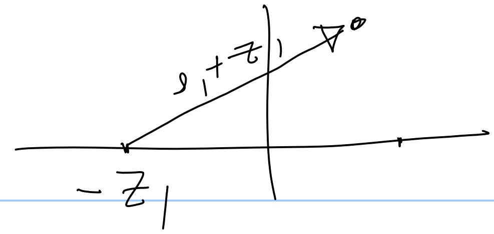
So $$|F(\sigma_1+j\omega_1)| = \frac{\prod_{i=1}^{m} |\sigma_1+j\omega_1+z_i|}{\prod_{k=1}^{n} |\sigma_1+j\omega_1+p_k|}$$

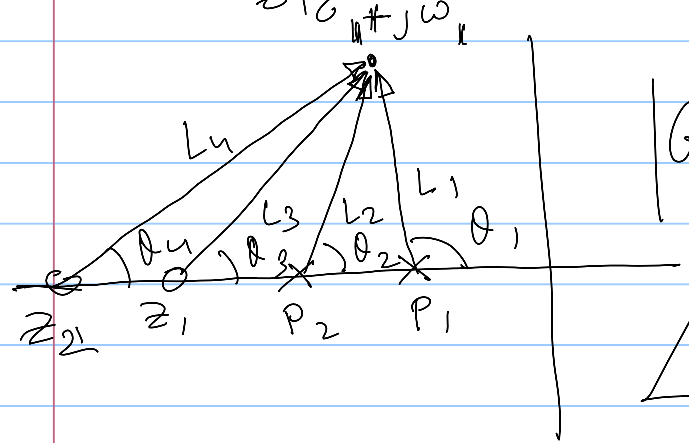
$$\angle F(\sigma_1+j\omega_1) = \sum \text{zero angles} - \sum \text{pole angles}$$

$$= \sum_{i=1}^{m} \angle(\sigma_1+j\omega_1+z_i) - \sum_{k=1}^{n} \angle(\sigma_1+j\omega_1+p_k)$$

Example: $$G(s) = \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}$$



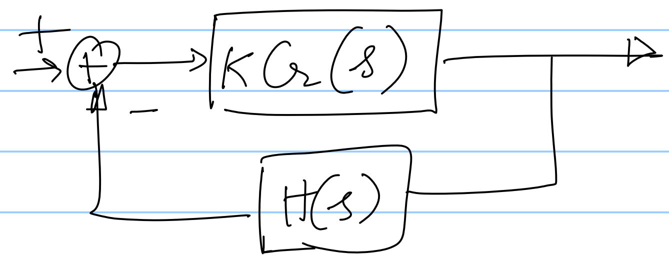




$$\left|G\left(\sigma_1+j\omega_1\right)\right| = \frac{L_3 L_4}{L_1 L_2}$$

$$\angle G\left(\sigma_1+j\omega_1\right) = \left(\theta_3+\theta_4 - \theta_1 - \theta_2\right)$$

Q. Predict whether any complex number $s_1 = (\sigma_1 + j\omega_1)$ is <u>on</u> the Root locus of



OR Alternatively

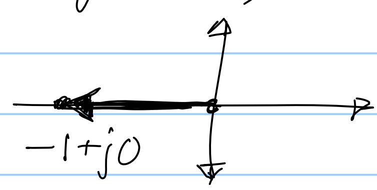
Q) Is $s_1 = (\sigma_1 + j\omega_1)$ the C.L. pole of the T.F. $\dfrac{KG(s)}{1+KG(s)H(s)}$ for some value of K?

$$T(s) = \frac{KG(s)}{1+KG(s)H(s)}$$

For $s = s_1$, to be the C.L. pole, for some K,

$$1 + KG(s_1)H(s_1) = 0$$

or $\quad KG(s_1)\,H(s_1) = -1$



i.e. for some K

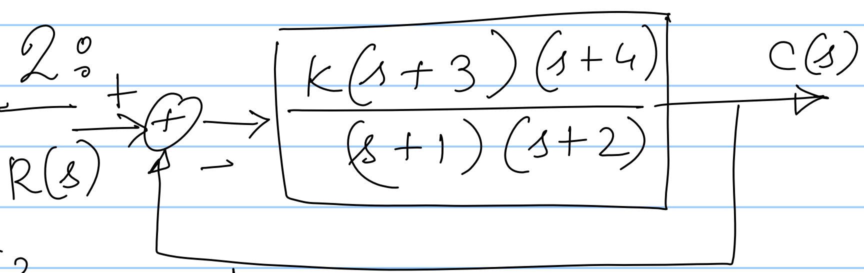
$$\left| K G(s_1) H(s_1) \right| = 1$$

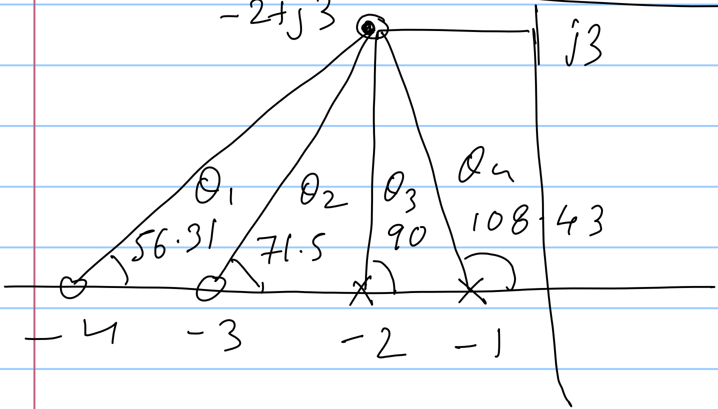
$$\angle K G(s_1) H(s_1) = (2j+1)180^\circ \quad \{ j = 0, \pm 1, \pm 2, \ldots$$

OR

$$\boxed{K = \frac{1}{|G(s_1)||H(s_1)|} \quad \text{and} \quad \angle G(s_1) H(s_1) = (2j+1)180 \; \{ j = 0, \pm 1, \ldots}$$

Example 2:





Q1) Can $-2+j3$ be a C.L. pole for some value of K?

Check: $\theta_1 + \theta_2 - \theta_3 - \theta_4$

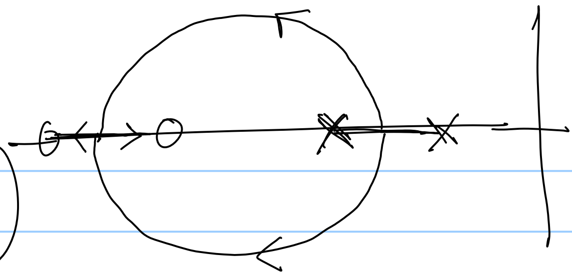
$$= 56.31^\circ + 71.57^\circ - 90^\circ - 108.43^\circ$$

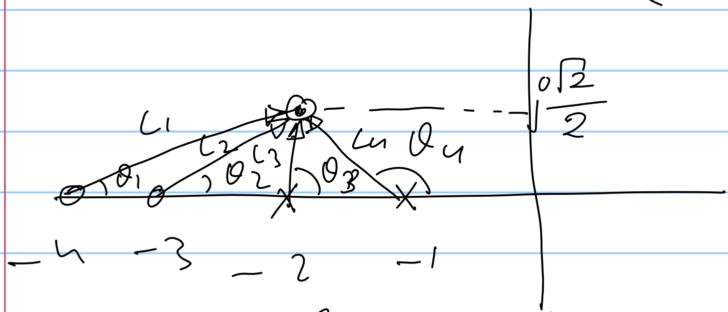
$$= -70.55^\circ$$

So $-2+j3$ cannot be a C.L. pole for any value of K.

$\Rightarrow$ $-2+j3$ is <u>not</u> on the root locus.

Q2) Can $-2 + j\left(\frac{\sqrt{2}}{2}\right)$





Check:
$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = -180^\circ$$

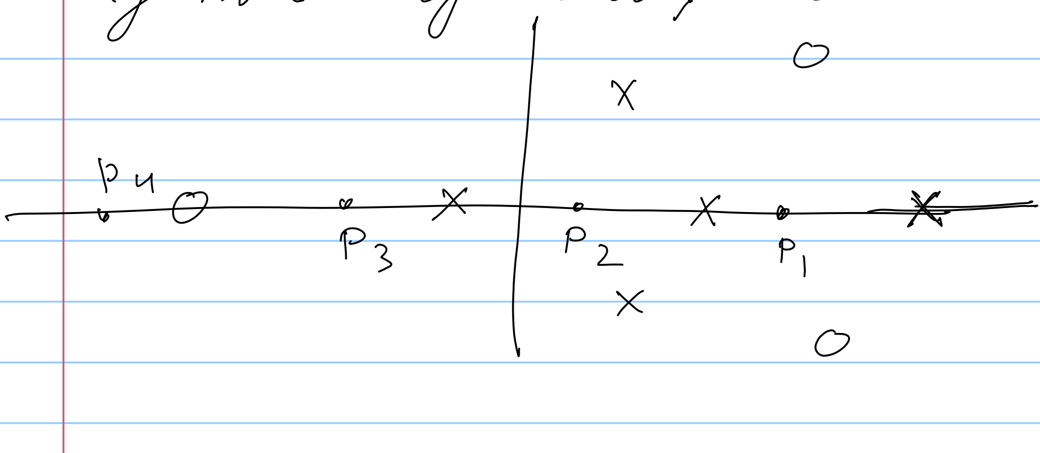
So, $\left(-2 + j\frac{\sqrt{2}}{2}\right)$ is a point on the R.L. for some value of K.

$$K = \frac{1}{|G(s_1)H(s_1)|} = \frac{1}{\left(\frac{L_1 L_2}{L_3 L_4}\right)} = \frac{L_3 L_4}{L_1 L_2}$$

$$= \frac{\frac{\sqrt{2}}{2} \cdot (1.22)}{(2.12)(1.22)} = 0.33.$$

Sketching the Root locus - Rules

1) No of branches = no of closed l. poles
2) The R.L. is symmetrical about real axis
3) On the real axis for $K > 0$, the R.L. exists to the left of an odd no. of real finite O.L poles and/or real finite open loop zeros.



4) Starting & Ending Point: $K = 0 \; \vdots \; K = \infty$ $\downarrow$

Let $G(s) = \frac{N_G(s)}{D_G(s)}$

$H(s) = \frac{N_H(s)}{D_H(s)}$

$$T(s) = \frac{KG(s)}{1+KG(s)H(s)} = \frac{KN_G(s)D_H(s)}{\underbrace{D_G(s)D_H(s) + KN_G(s)N_H(s)}_{D_{CL}(s)}}$$

As $K \to 0$, $D_{CL}(s) \to D_G(s)D_H(s)$

i.e. the C.L. poles $\to$ O.L. poles of $G(s)H(s)$

As $K \to \infty$, $T(s) \to \frac{N_G(s)D_H(s)}{N_G(s)N_H(s)}$

i.e. C.L poles $\to$ O.L. zeros of $G(s)H(s)$.

Q.) What if no of O.L. zeros < no of O.L poles

Infinite poles & Zeros: 1) If a function approaches zero as $s \to \infty$ then there is a zero at infinity

2) If a function approaches ∞ as $s \to \infty$ then there is a pole at ∞.

Every function has equal no of poles & zeros if we include the zeros/poles at infinity.

Rule 4: The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

5) Where are the infinite zeros?
(For real systems there are no infinite poles)

FACT: 1) The root locus approaches st. lines as asymptotes as the locus approaches infinity.
2) The eqn of the asymptotes is given by the real axis intercept σ_a, and angle θ_a:

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$$

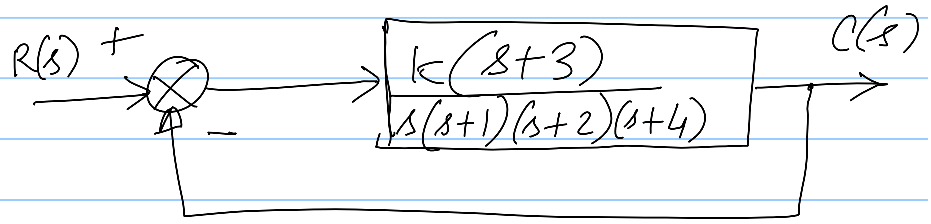
$$\theta_a = \frac{(2k+1)\pi}{\#\text{ finite poles} - \#\text{ finite zeros}}$$

where $K = 0, \pm 1, \pm 2, \pm 3$ and the angle is given in radians with respect to the +ve real axis.

The distinct θ_a's generated for different k's yield the different asymptotes.

\# Number of lines = # finite poles − # finite zeros
= no of infinite zeros.

Example 3:



Calculation of Asymptotes:

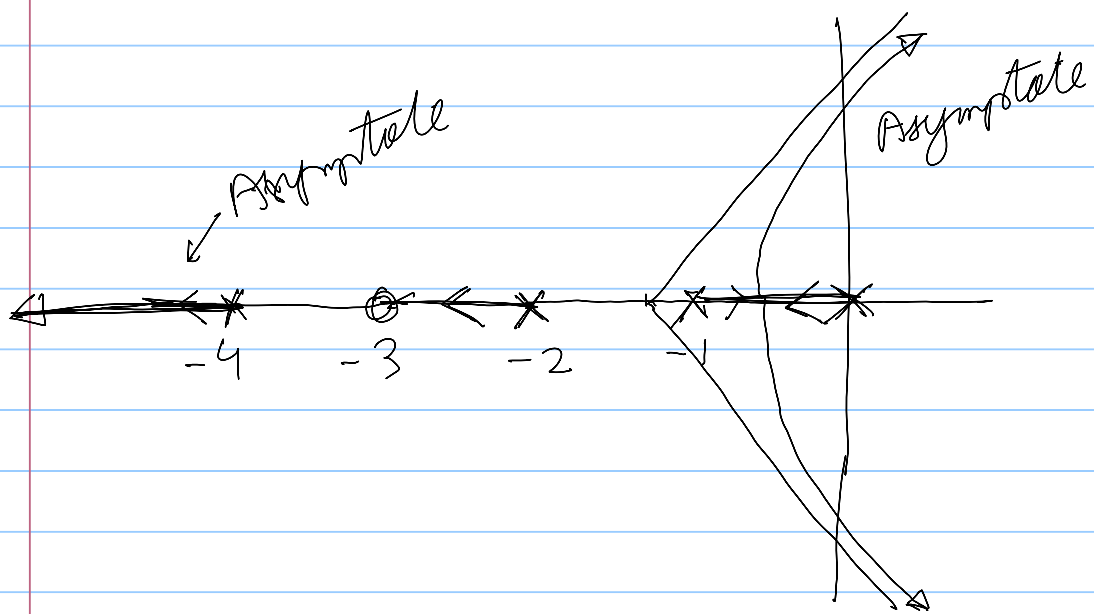
$$\sigma_a = \frac{(0-1-2-4)-(-3)}{(4-1)} = -\frac{4}{3}$$

$$\theta_a = \frac{(2k+1)\pi}{(4-1)} = \frac{\pi}{3} \qquad k=0$$

$$= \pi \qquad k=1$$

$$= \frac{5\pi}{3} \qquad k=2$$

For larger k's the angles repeat.



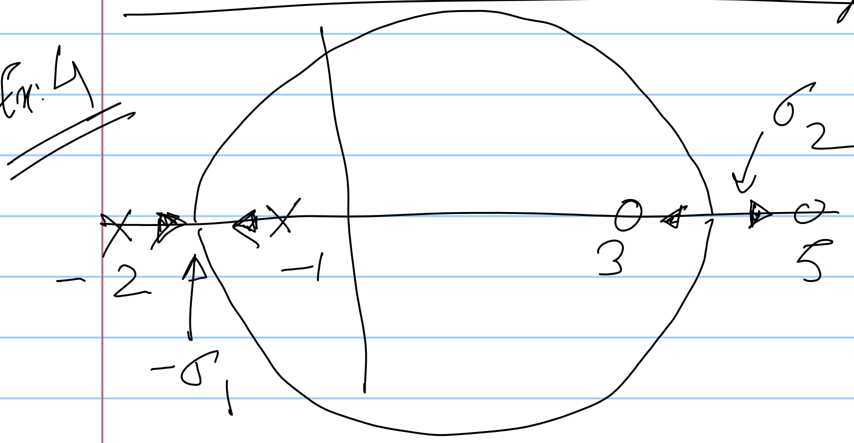
Additional Information:

1) Real axis breakaway/break in points
2) The $j\omega$-axis crossings

3) Angles of departure / arrival

Real axis breakaway / break in

Ex: 4



$-\sigma_1 \rightarrow$ breakaway pt

$\sigma_2 \rightarrow$ breakin pt.

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$

The gain attains a maxima along the real axis at the breakaway pt.

The gain attains a minima along the real axis at the breakin pt

$$K = -\frac{1}{G(s)H(s)}$$

On the real axis, $s = \sigma$

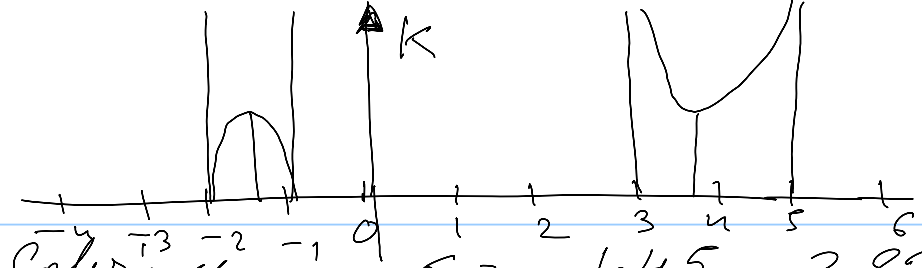
$$K = -\frac{1}{G(\sigma)H(\sigma)}$$

Find minima / maxima:

$$\frac{dK}{d\sigma} = \frac{d}{d\sigma}\left\{\frac{-1}{G(\sigma)H(\sigma)}\right\} = 0$$

For the example: $K = -\dfrac{(\sigma+1)(\sigma+2)}{(\sigma-3)(\sigma-5)}$

$$\frac{dK}{d\sigma} = \frac{11\sigma^2 - 26\sigma - 61}{(\sigma^2 - 8\sigma + 15)^2} = 0$$

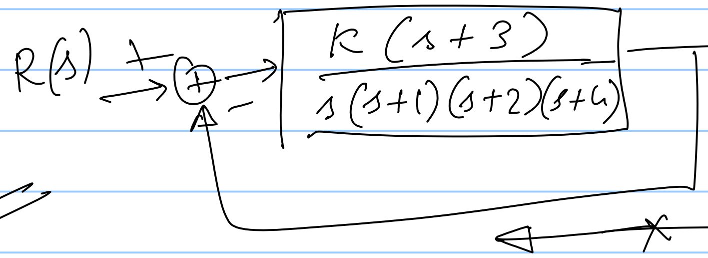


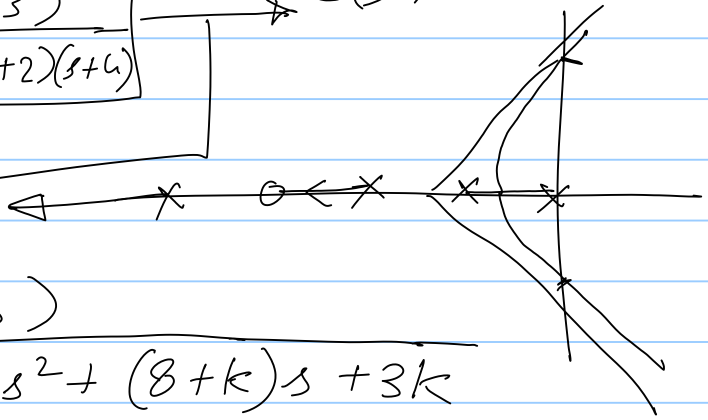
Solving, $\sigma = -1.45$, 3.82

σ_1 σ_2

$j\omega$-axis Crossings : Use R-H criterion

Ex 3





$$T(s) = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K}$$

s^4	1	14	$3K$
s^3	7	$8+K$	
s^2	$90-K$	$21K$	
s^1	$\dfrac{-K^2-65K+720}{90-K}$		
s^0	$21K$		

We want a row full of zeros:

$$-K^2 - 65K + 720 = 0$$

solving : $K = 9.65$ $(K > 0)$

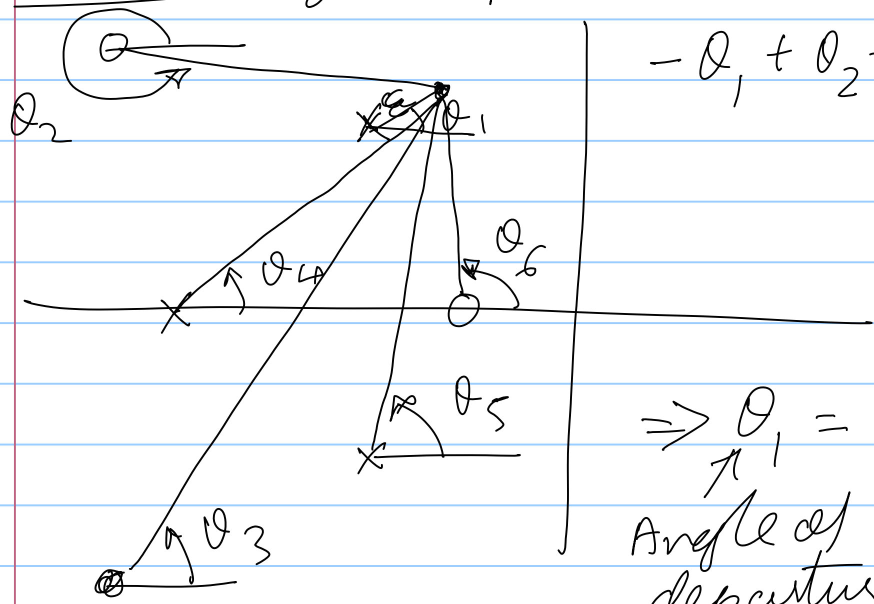
Form the even poly:

$$(90-K)s^2 + 21K = 80.35s^2 + 202.7 = 0$$

$$s = \pm j\,1.59$$

(Additionally: system is stable for $0 \leq K < 9.65$)

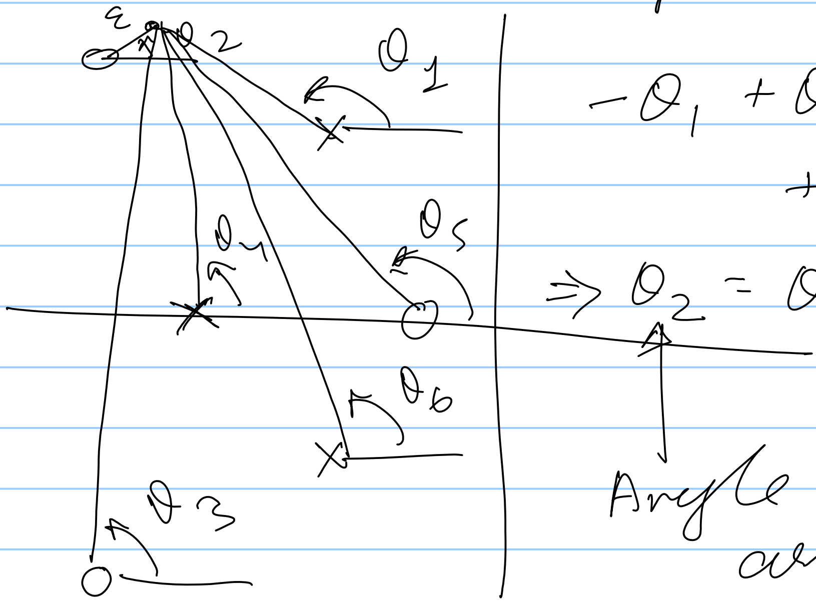
Angles of Departure / Arrival



$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+1)\,180^\circ$$

$$\Rightarrow \theta_1 = \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 - (2k+1)\,180^\circ$$

Angle of departure

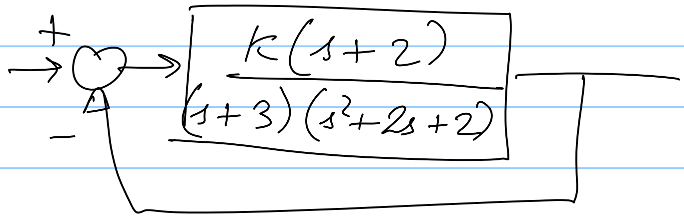


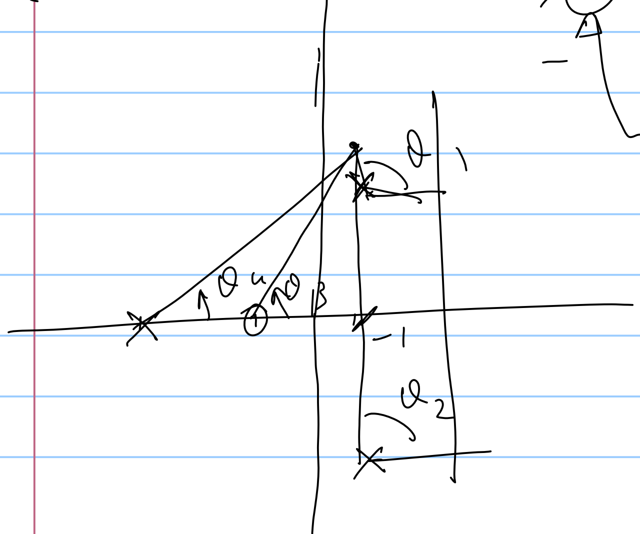
$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+1)\,180^\circ$$

$$\Rightarrow \theta_2 = \theta_1 - \theta_3 + \theta_4 + \theta_5 - \theta_6 + (2k+1)\,180^\circ$$

Angle of arrival

Example 5:





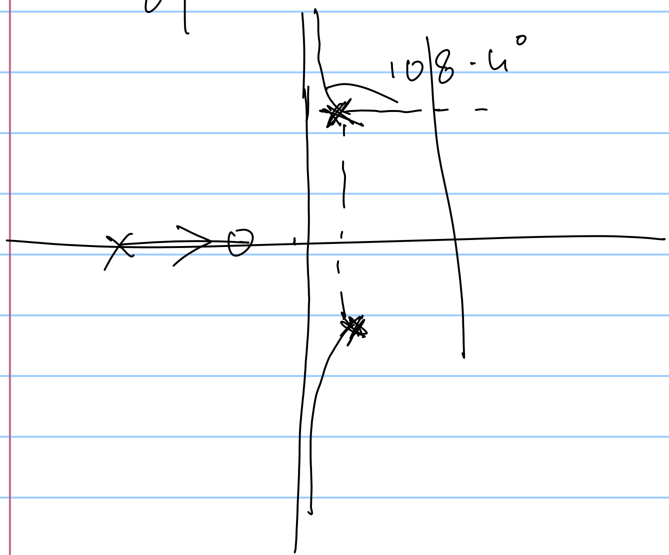
$$\sigma_a = \frac{(-3-2)+2}{3-1} = -\frac{3}{2}$$

$$\theta_a = \frac{(2k+1)\pi}{3-1} = \pi/2,\; 3\pi/2$$

$$-\theta_1 - \theta_2 + \theta_3 - \theta_4$$

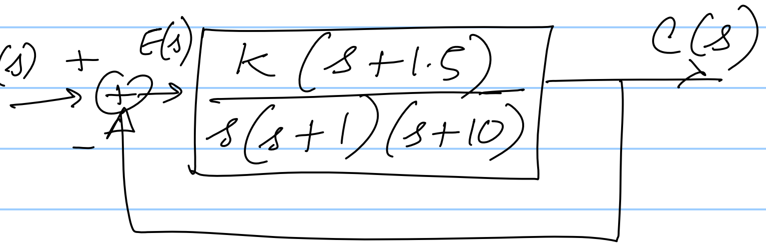
$$= -\theta_1 - 90^\circ - \tan^{-1}\left(\frac{1}{1}\right) - \tan^{-1}\left(\frac{1}{2}\right) = 180^\circ$$

$$\theta_1 = -251.6^\circ = 108.4^\circ$$

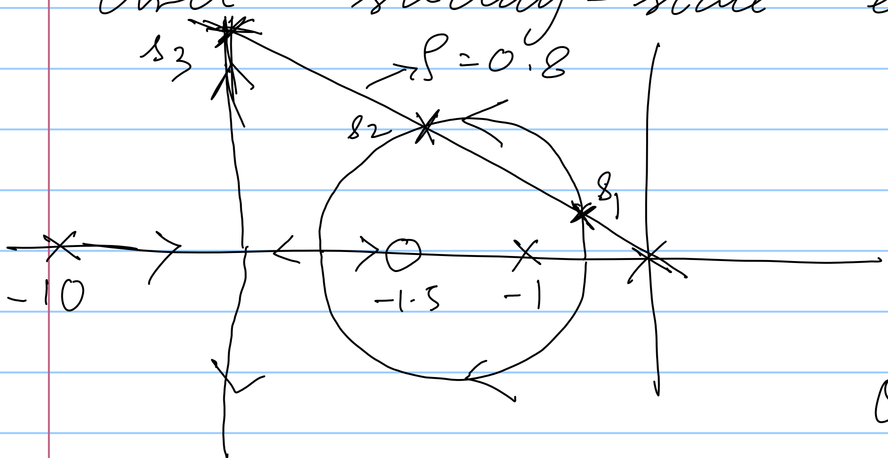


Third Order System Gain Design using Root locus

Example 6:



Design K to yield 1.52% OS. Also estimate the settling time peak time and steady-state errors.



Asymptotes:

$$\sigma_a = \frac{(0-1-10)+1.5}{2} = \frac{-9.5}{2}$$

$$\theta_a = \pi/2, \frac{3\pi}{2}$$

Breakaway pts: -0.62 for $K = 2.511$
$[-10, -1.5]$ or $[-1, 0]$ $\quad -4.4$ for $K = 28.89$

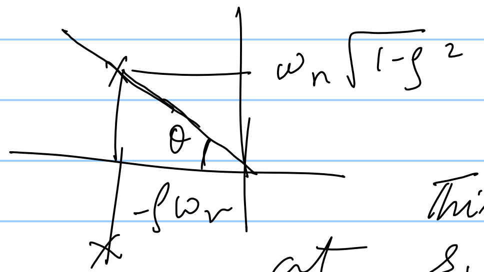
Breakin pt: -2.8 for $K = 27.91$

$[-10, -1.5]$

1) Assume that system can be approximated by a 2^{nd} order system with no zeros

$1.52\%\ OS \longrightarrow \zeta = 0.8$

2) Draw the $\zeta = 0.8$ line on R-L.



$\cos\theta = \zeta$

This line intersects R.-L at s_1, s_2, s_3

$s_1 = -0.87 \pm j\,0.66 \rightarrow K = 7.36$
$s_2 = -1.99 \pm j\,0.90 \rightarrow K = 12.79$
$s_3 = -4.6 \pm j\,3.45 \rightarrow K = 39.64$

3) Check T_s & T_p for each of these pole locations $T_s = \frac{4}{\zeta\omega_n}$, $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

4) Check the location of the third pole (to validate our 2nd order assumption) for each K.

	Third pole	T_s	T_p	K_v
$s_1 \longrightarrow$	-9.25	4.6	4.37	1.1
$s_2 \longrightarrow$	-8.61	3.36	3.49	1.9
$s_3 \longrightarrow$	-1.80	0.87	0.91	5.9

5) Check S.S. Error using K_v
(Type 1 system)

5) Check 2nd order approx for
$K = 39.64$
Poles at $-4.6 \pm j3.45$, -1.80

6) Check transient response and %OS

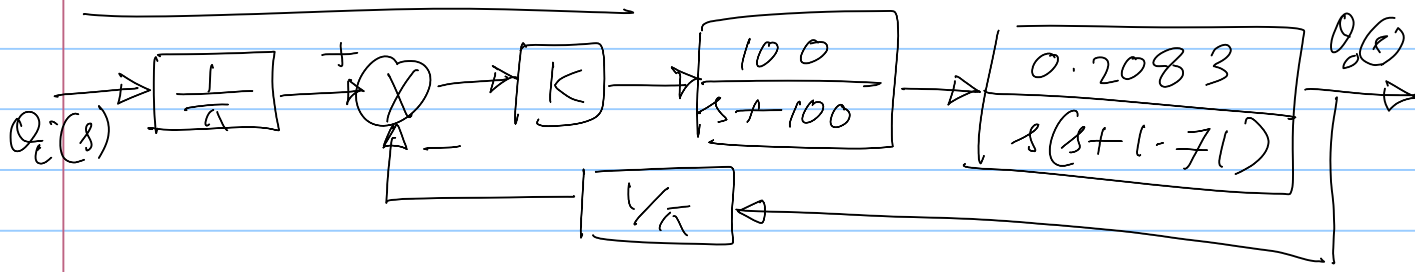
** USE SISOTOOL **

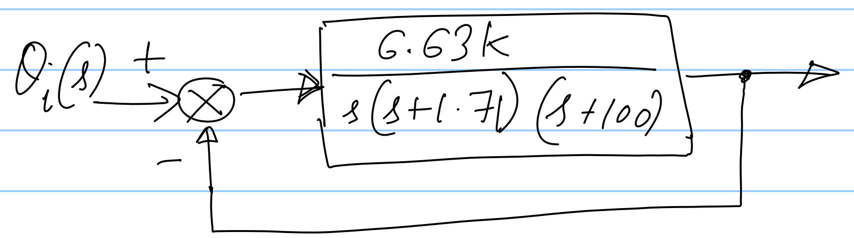
[Observation: The design does not work because the 2nd order approx is not good enough.

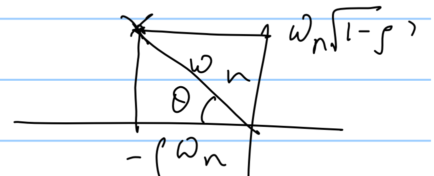
Root Locus for $K<0$ OR for +ve feedback systems

→ READ from TEXTBOOK

Antenna Control: (Exam 7)

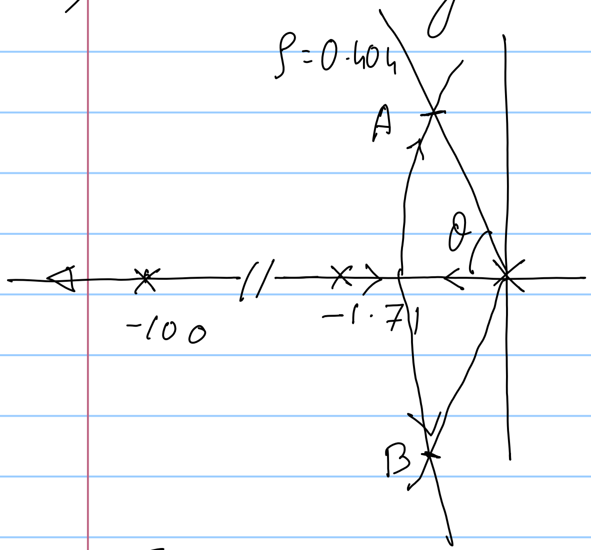






$\cos\theta = \zeta$

Q) Find K for 25% OS.



$\sigma_a = -33.9$

$\theta_a = 60°, 180°, -60°$

25% OS $\Rightarrow \zeta = 0.404$

$\theta = \cos^{-1}\zeta = 66.17°$

$A, B = -0.833 \pm j1.888$

The gain value $= 6.63K = \dfrac{1}{G_2(s)\big|_{s=A}} = 425.7$

$K = 64.21$.

\# Check 2nd order approx is valid as $Re\{3^{rd}\ \text{pole}\} < -100$.

\# Check with computer simulation: