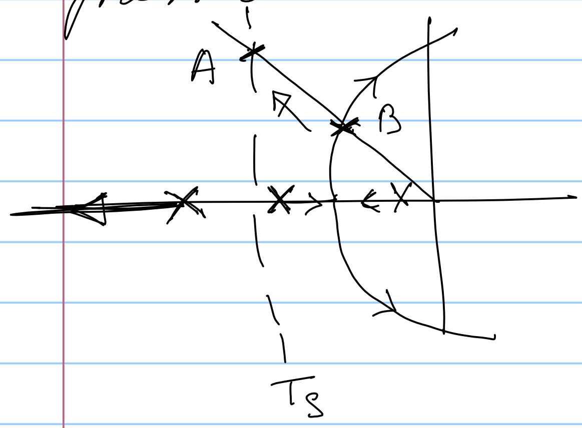
Note Title 14-03-2010

Lecture 9: Compensator Design using Root-Locus

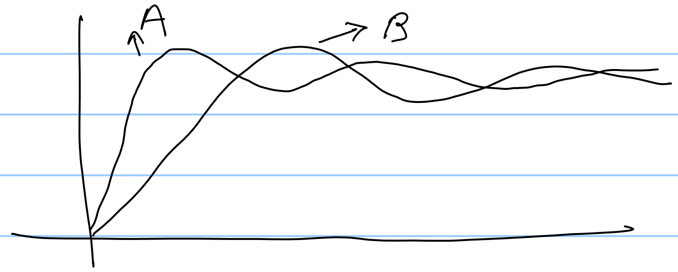
Limitations of Gain Design →

→ points on the given R-L only are feasible.

1) %OS + Settling time may not be possible:



A meets %OS + T_s specs
→ But not possible with this R-L.



2) Steady State Error vs Transient Response

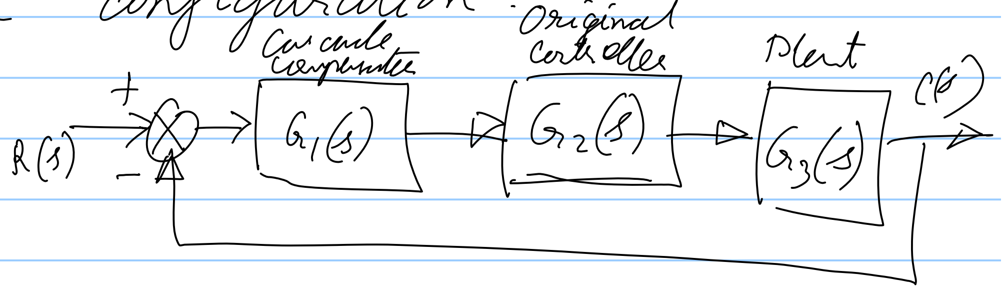
K ↑ %OS ↑ S.S.E. ↓

Compensators are needed →

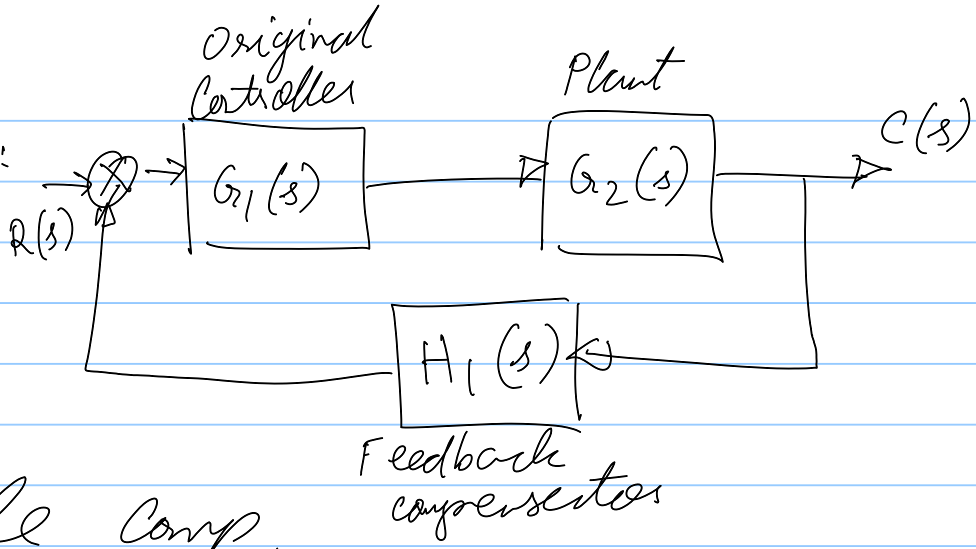
Active elements	Passive Element based Approx.
S.S.E → 1a) PI controllers	1b) Lag compensator
Tr. specs 2a) PD controllers	2b) Lead compensator
Both 3a) PID controllers	3b) Lead-Lag compensator

Two basic configuration:

Cascade:

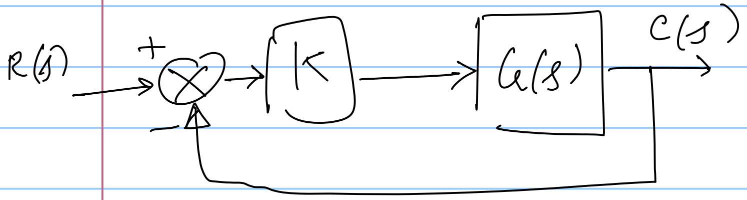


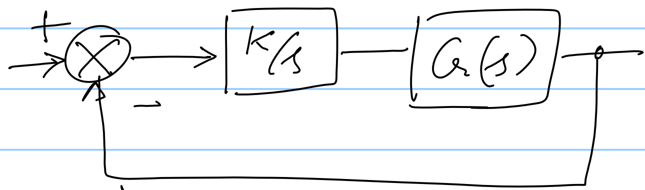
Feedback:

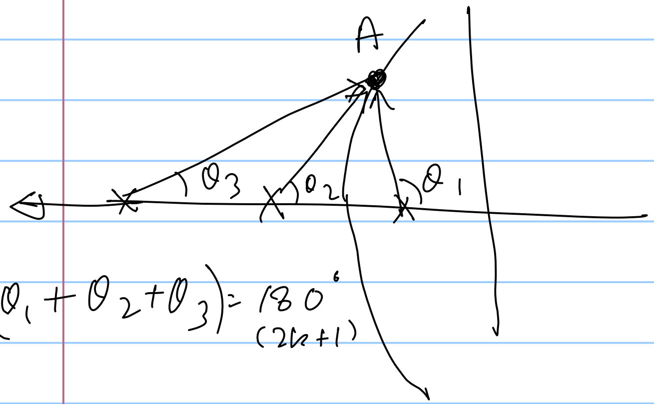


Cascade Comp

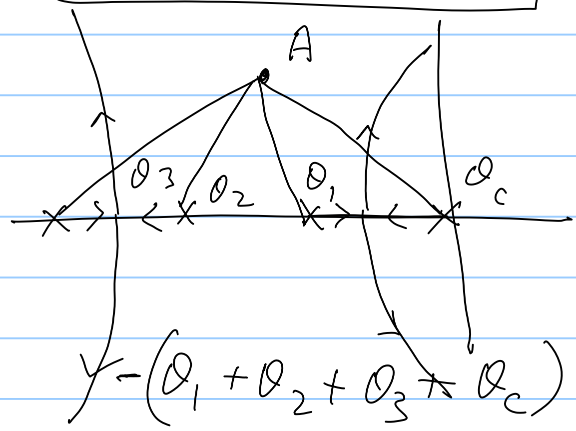
S.S.E. improvement via PI controller





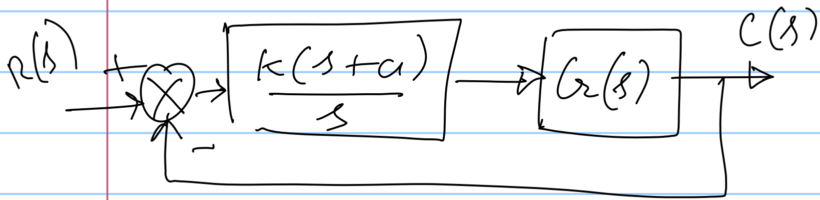


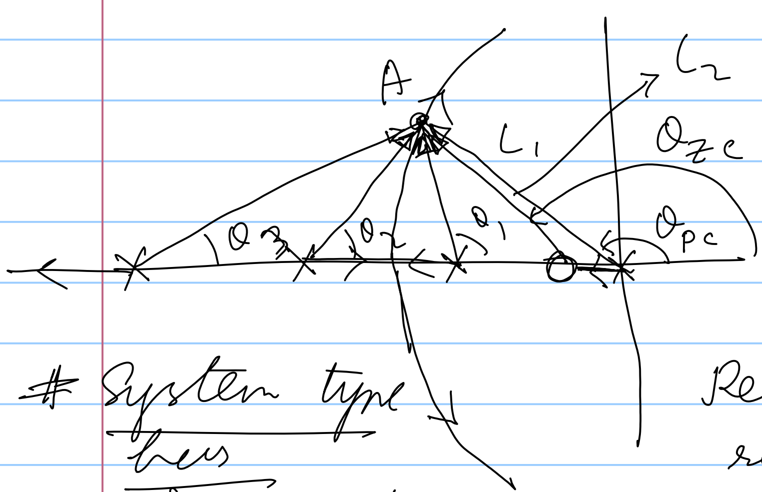
$-(\theta_1+\theta_2+\theta_3) = 180^\circ (2k+1)$



$-(\theta_1+\theta_2+\theta_3+\theta_c) \neq (2k+1)180^\circ$

A is not on R-L.





$\theta_{pc} \simeq \theta_{zc}$

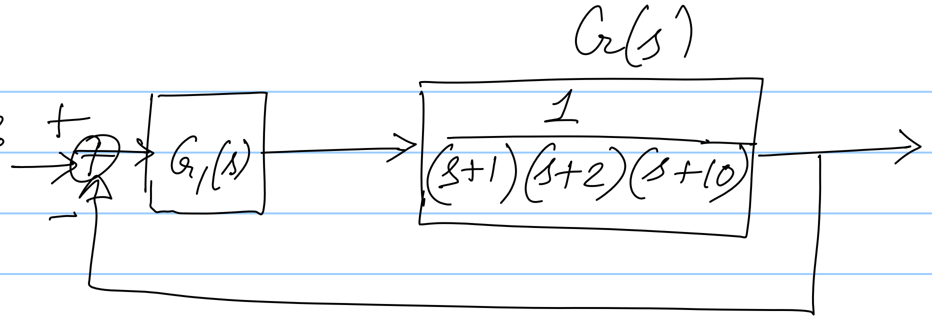
Hence

$-(\theta_1+\theta_2+\theta_3+\theta_{pc})+\theta_{zc} \simeq (2k+1)180^\circ$

System type has increased.

Req gain for A also remains unchanged as $L_1 \simeq L_2$

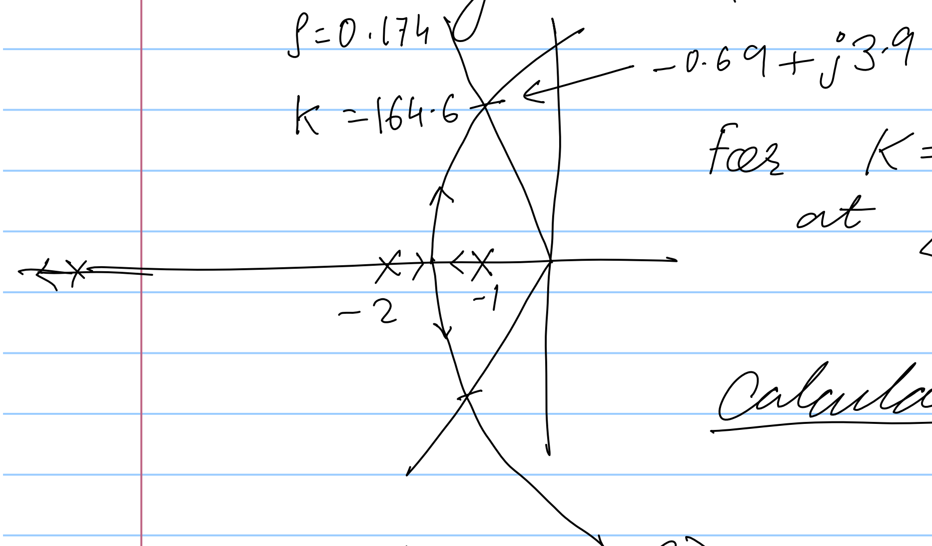
Example :



1) Let $G_1(s) = K$. Design K for $\zeta = 0.174$. Calculate corresponding S.S.E.

2) Let $G_1(s) =$ PI controller $= \frac{K(s+a)}{s}$
Design K and a s.t. $\zeta = 0.174$ and SSE = 0. (for position com)

Ans 1 : R.-L. for $G_1(s) = K$.



for $K = 164.6$ roots are at $\begin{cases} -0.69 \pm j3.9 \\ -11.61 \end{cases}$

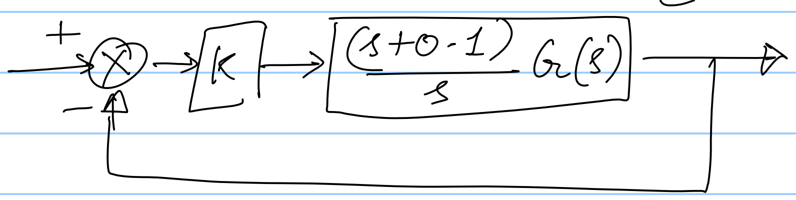
Calculation of SSE:
(Type 0)

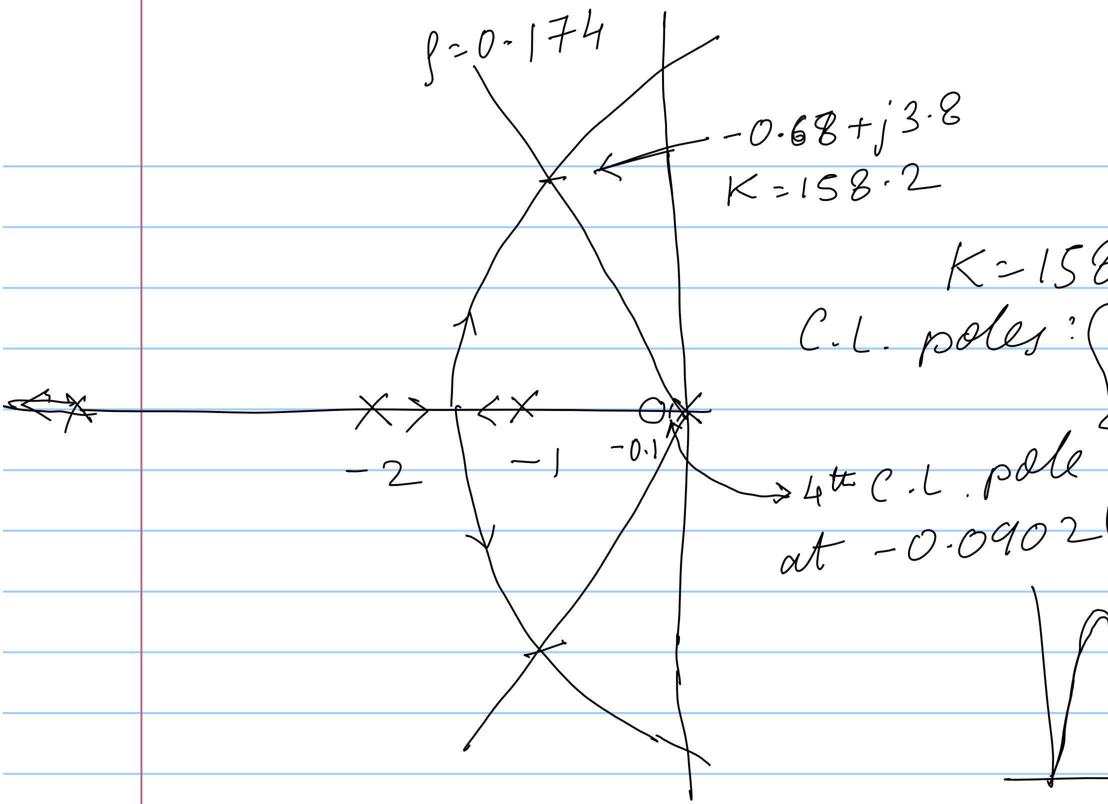
$$K_p = \lim_{s\to 0} G(s)G_1(s) = \lim_{s \to 0} \frac{164.6 \cdot 1}{1 \times 2 \times 10}$$
$$= 8.23$$

$$e(\infty) = \frac{1}{1+K_p} = \frac{1}{9.23} = 0.108$$

Ans 2 : $G_1(s) = \frac{K(s+0.1)}{s}$ $\begin{cases} \text{Pole at zero} \\ \text{Zero at } -0.1 \end{cases}$

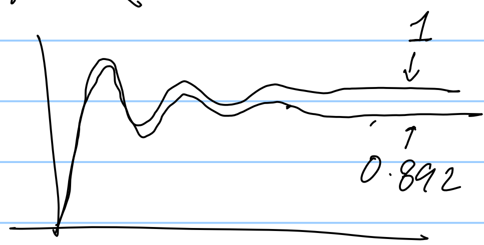
R − L for →





K = 158.2

C.L. poles: $-0.68 \pm j3.8$, -11.55, -0.0902

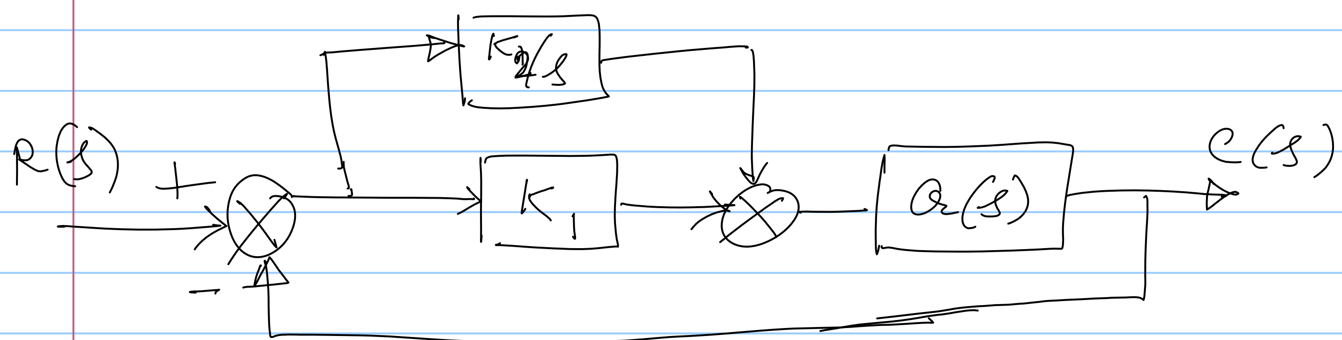


Transient Response (1) Zero (-0.1) and pole (-0.09) are close. (2) 3rd pole is < -10

⇒ 2nd order approx is valid so ρ = 0.174 is achieved.

SSE : This is now a 'Type 1' system so S.S.E to step is Zero.

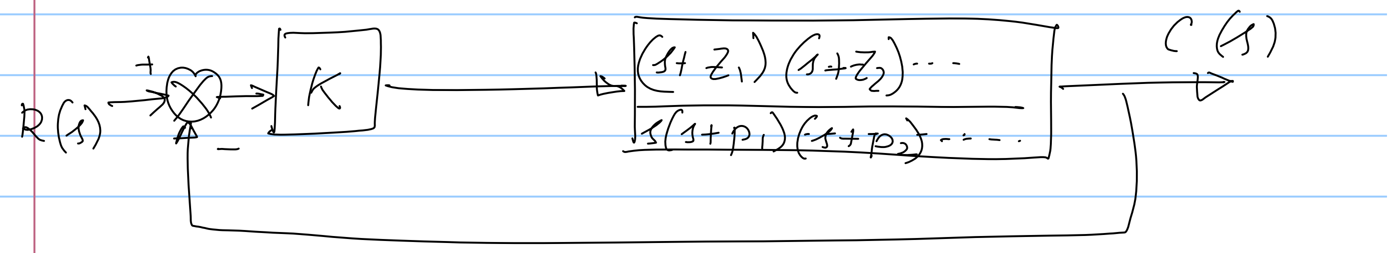
Implementation:

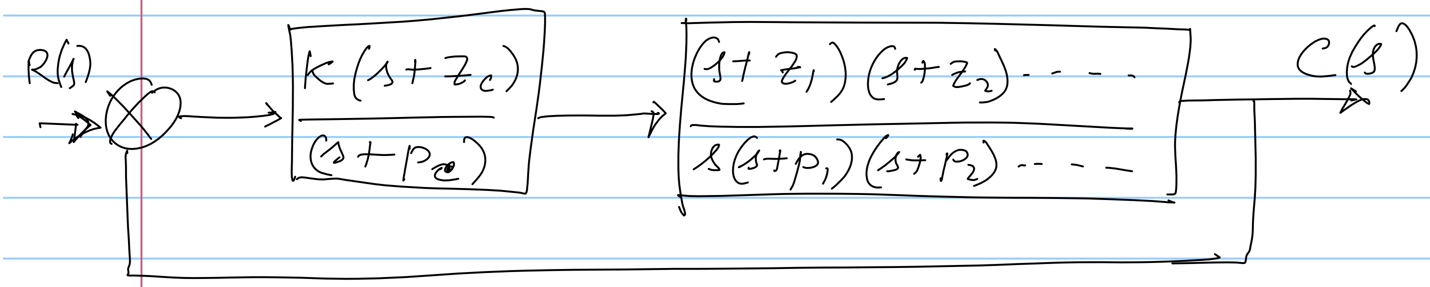


NOTE: For implementing K2/s we need an OP-AMP.

LAG COMPESATION: (Passive approx of PI)

Assume "Type 1" $G(s)$. (Equally true for any type)





For uncompensated:

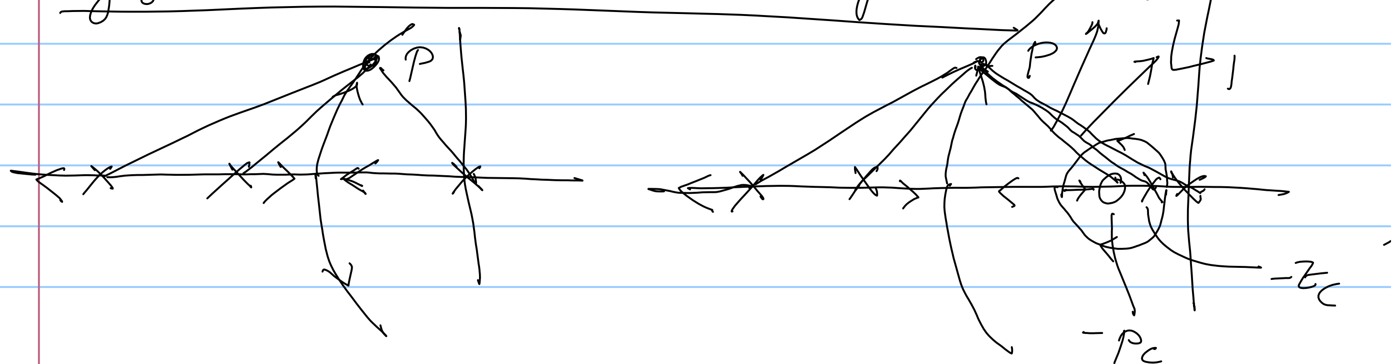
$$e_0(\infty) = \frac{1}{K_{v_0}} \; ; \; K_{v_0} = \lim_{s\to 0} sG(s) = \frac{(K z_1 z_2 \cdots)}{(p_1 p_2 \cdots)}$$

For compensated system:

$$e_N(\infty) = \frac{1}{K_{v_N}} \; ; \; K_{v_N} = \frac{(K z_1 z_2 \cdots)\, z_c}{(p_1 p_2 \cdots)\, p_c}$$

$$K_{v_N} = K_{v_0} \frac{z_c}{p_c} \implies \left[\frac{z_c}{p_c} \uparrow \;\approx\; e_N(\infty) \downarrow \right]$$

Effect on Transient Response



To preserve the transient response
→ R-L should be almost same.
→ 2nd order approx should remain valid.

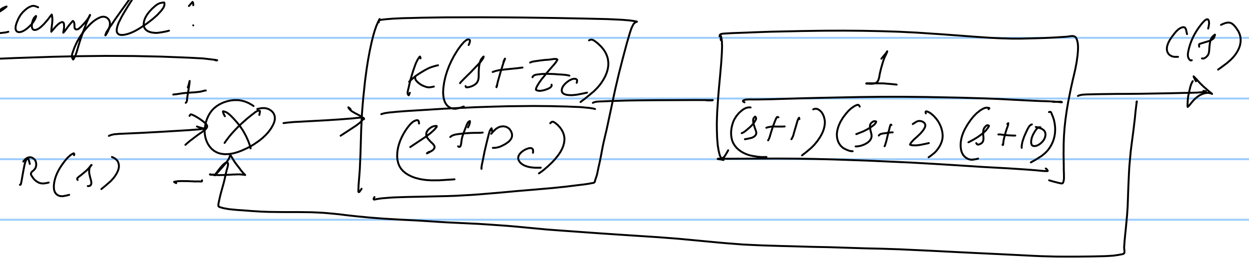
⇒ Z_c and P_c should be close to each other.

Conflicting requirements? (with SSE req)
→ Both can be met if Z_c & P_c are close to origin
(e.g. $Z_c = -0.01$ & $P_c = -0.001$)
$\frac{Z_c}{P_c} = 10$ (SSE improves 10 times)

But $|Z_c - P_c| = 0.009$

* The K req. to achieve P remains almost the same as $\frac{L_2}{L_1} \approx 1$.

Example:



Design spec: (1) $\zeta = 0.174$
(2) Improve SSE by a factor of 10

For the uncompensated system:

$e(\infty) = 0.108$, $K_{p_0} = 8.23$
10 fold improvement: $e(\infty) = 0.0108$

$$\frac{1}{1+K_{p_N}} = 0.0108 \Rightarrow K_{p_N} = 91.59$$

Now: $K_{p_N} = \lim_{s\to 0} G(s) K \frac{(s+z_c)}{(s+p_c)}$

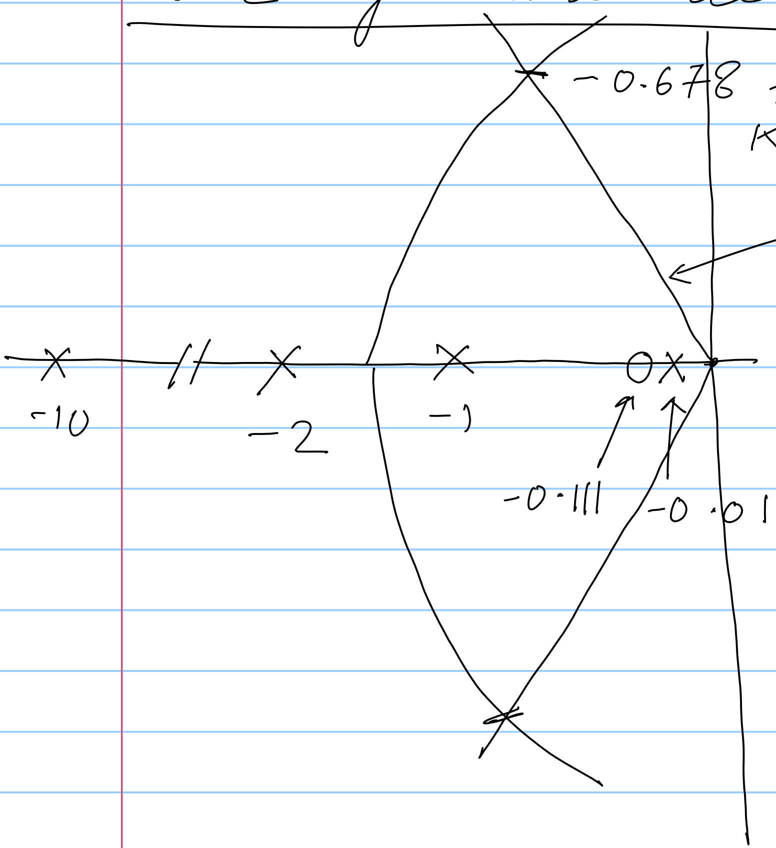
$$K_{p_N} = K_{p_0} \frac{z_c}{p_c}$$ $\quad K_{p_N} = 91.59$, $K_{p_0} = 8.23$

$$\frac{z_c}{p_c} = \frac{91.59}{8.23} = 11.13$$

Choose (arbitrarily) $p_c = 0.01$

$\Rightarrow$ $z_c = 11.13 p_c \approx 0.111$

R-L for this design:



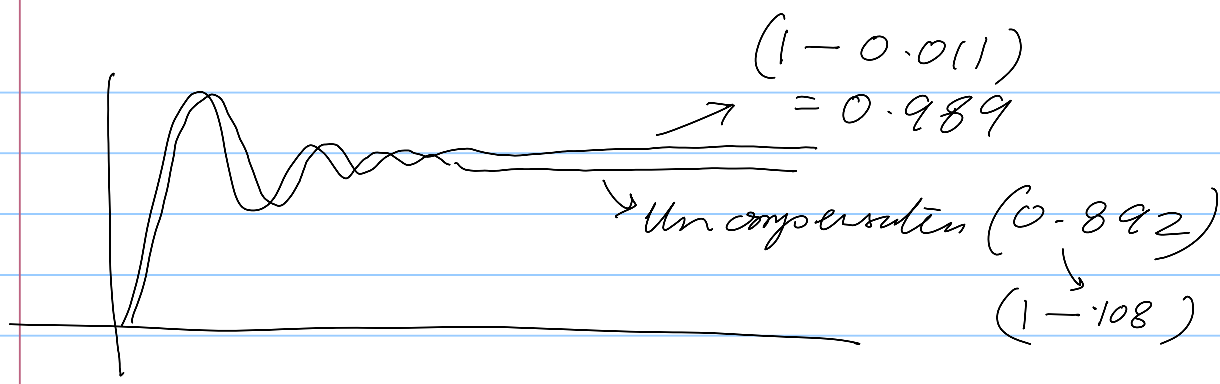
For $K = 158.1$
$z_c = 0.111$
$p_c = 0.01$

C.L. poles are at:

$$\begin{cases} -0.678 \pm j3.8 \\ -0.101 \\ -11.55 \end{cases}$$

Zero is at 0.111

Transient Response Comparison



Exercise: Compare this design with

$z_c = 0.0111$

$p_c = 0.001$

Transient Error Improvement

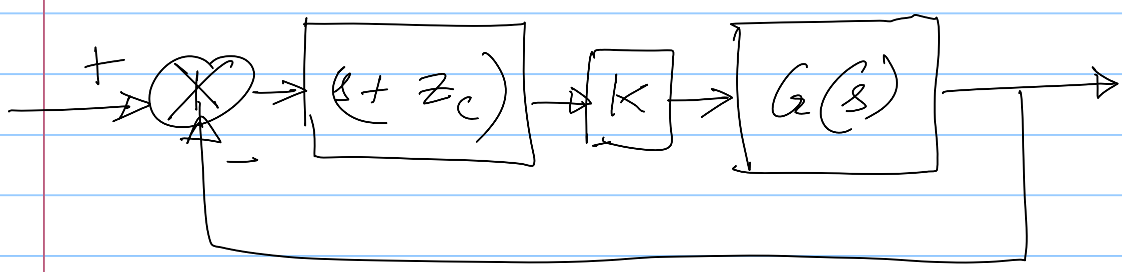
(Cascade Comp.)

Objective: Meet (i) %OS spec

(ii) T_s spec

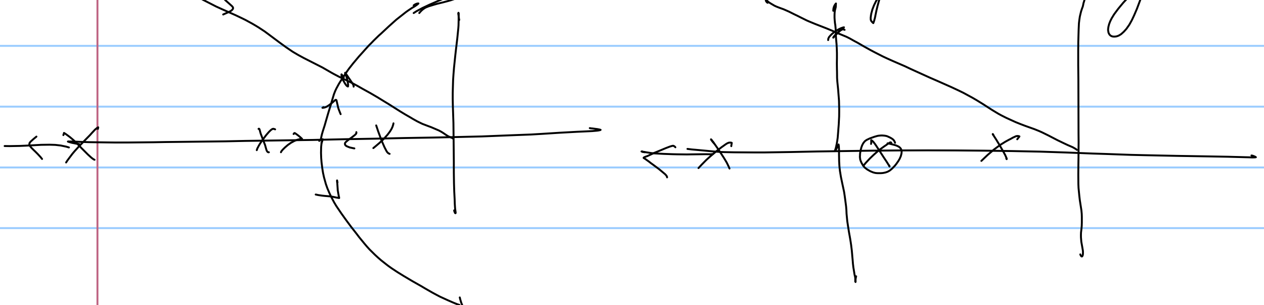
* Usually one of these can be met with just gain adjustment, but not both.

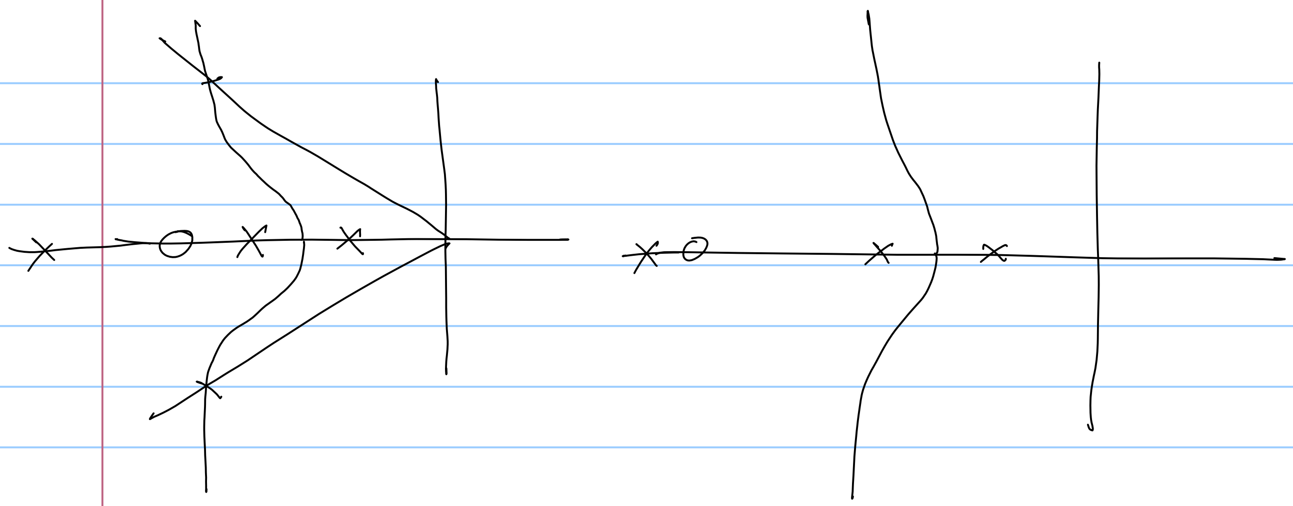
P-D compensator



Main idea: Add a real zero

→ makes response faster





Try moving the zero in SISOTOOL

- %OS remains almost same
- T_s improves over uncompensated case
- As the zero moves farther left, the response / R-L approaches uncompensated case.
- SSE may or may not improve in general.

P-D compensator design

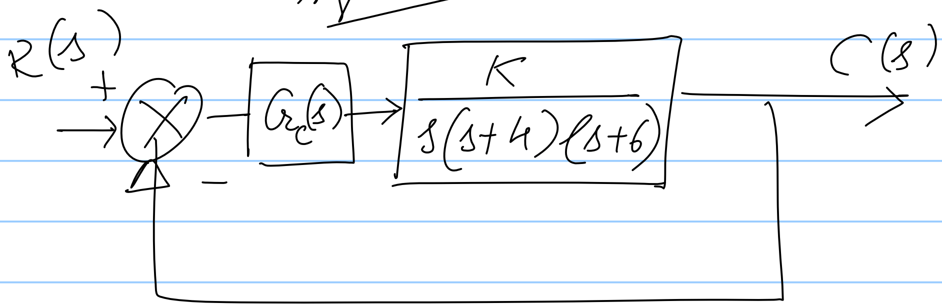
Step 1: Choose C.L. pole location according to specs. (say $\sigma + j\omega$)

Step 2: Calculate total angle contribution to $\sigma + j\omega$ from O.L. poles/zeros ($= \theta_{OL}$) say.

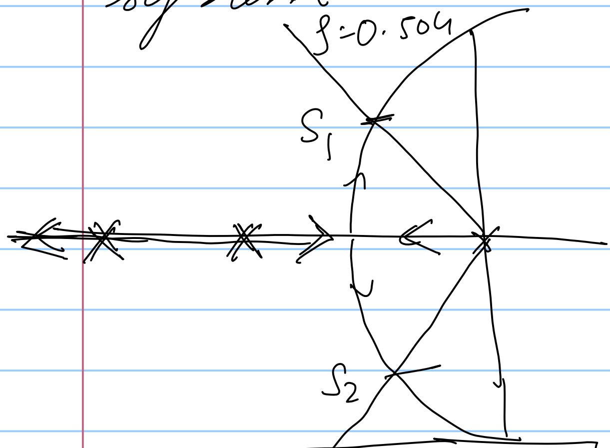
Step 3: Design the zero location to contribute an angle of $[(2k+1)180^\circ - \theta_{OL}]$.

smp 1

Example:



Design a P-D compensator to yield 16% OS, and a threefold reduction in settling time than the uncompensated system.



$16\%\ OS = \zeta = 0.504$

$S_{1,2} = -1.205 \pm j\,2.064$

$K = 43.35$ (third pole $= -7.59$)

2nd order approx OK.

Simulation values = $\boxed{\begin{array}{l} \%OS = 14.8 \\ T_s = 3.6 \end{array}}$

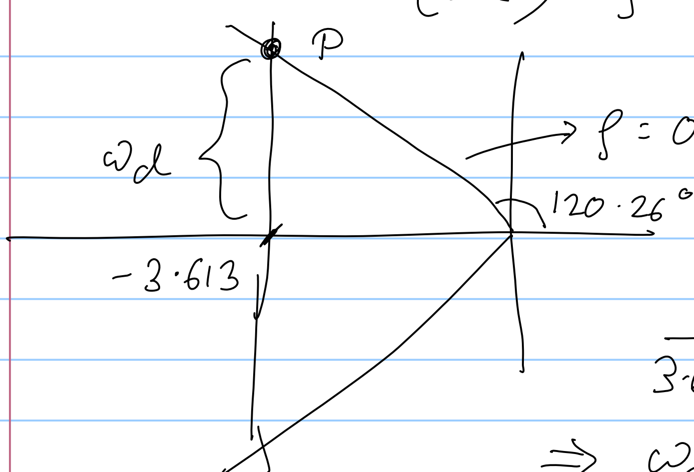
$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{1.204} = 3.320$$

$K_v = 1.806$

We want $\quad T_{s_N} = \dfrac{3.320}{3} = 1.107 \left(= \dfrac{4}{\sigma} \right)$

$\Rightarrow \sigma = 3.613$

But %OS (i.e) ζ must remain same:



P ← desired C.L. pole

$p = -3.613 \pm j\omega_d$

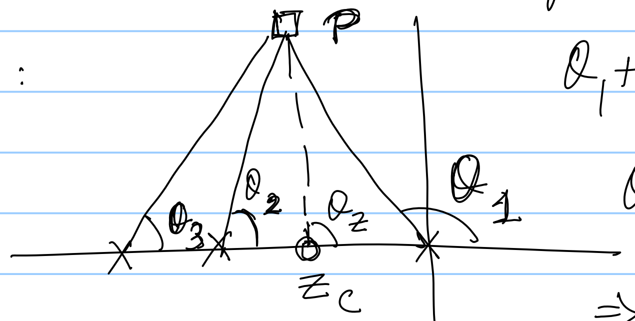
$$\frac{\omega_d}{3.613} = \tan(180° - 120.26°)$$

$\Rightarrow \omega_d = 6.193$

Desired C.L. (Dominant) poles $\overset{P}{=} -3.613 \pm j6.19$

Q) Where should we add a Real zero to make the R-L pass through P?

Step 2:



$\theta_1+\theta_2+\theta_3 = -275.6^\circ$

$\theta_z - (\theta_1+\theta_2+\theta_3) = -180^\circ$

$\Rightarrow \theta_z = 95.6^\circ$

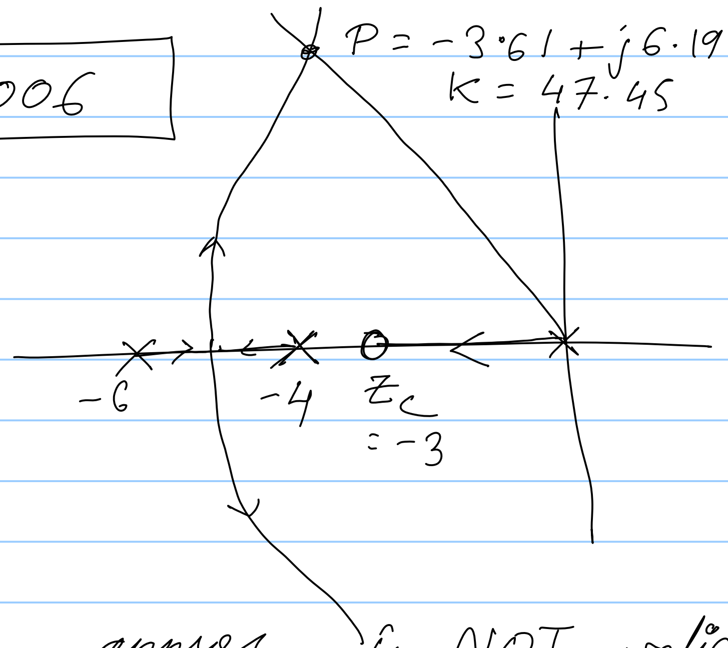
Why -180°?

From the geometry,

$$\frac{6.193}{(3.613 - z_c)} = \tan(180^\circ - 95.6^\circ)$$

$$\boxed{\Rightarrow z_c = 3.006}$$

Complete R-L



K = 47.45

poles: $\begin{cases} -3.61 \pm j6.19 \\ -2.775 \end{cases}$

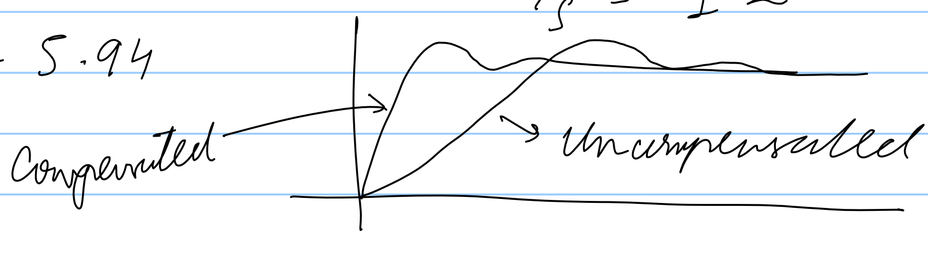
Zero : -3.006

So second order approx is NOT valid.

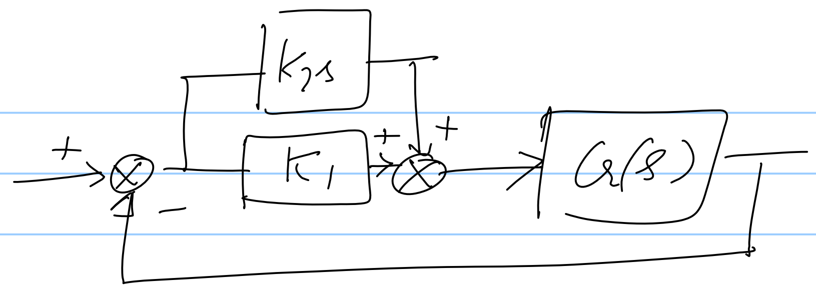
Simulation values: $\%OS = 11.8$

$T_s = 1.2$

$K_v = 5.94$



Implementation :

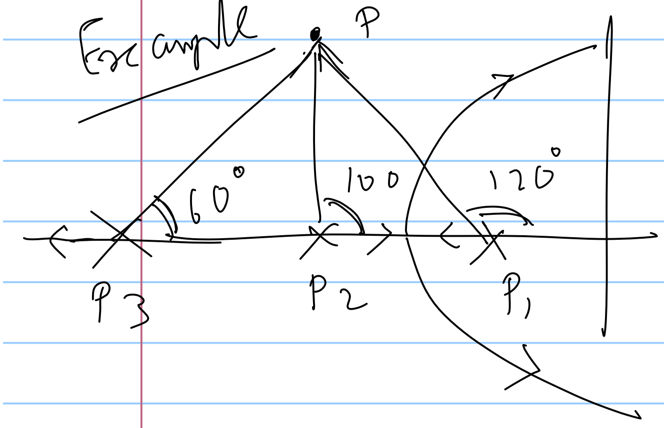


Disadvantages : 1) Active elements needed
2) Noise amplification

Additional advantage: Reduces (usually) the number of branches crossing the $j\omega$ axis to the RHP

Lead - Compensation (Passive approx. of P-D)

Example



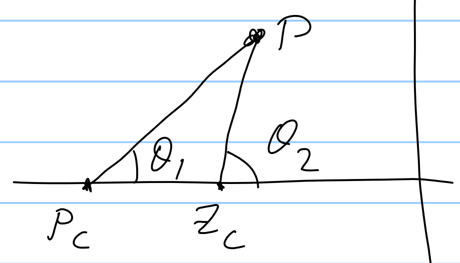
To place poles at P:

$$-280^\circ + \theta = (2k+1)180 \begin{cases} = -180^\circ \\ = 180^\circ \\ = -540^\circ \\ = +540^\circ \end{cases}$$

A +ve angle contribution is needed.

* This can be realized with a passive network

$$G_C(s) = K\left(\frac{s+Z_C}{s+P_C}\right)$$



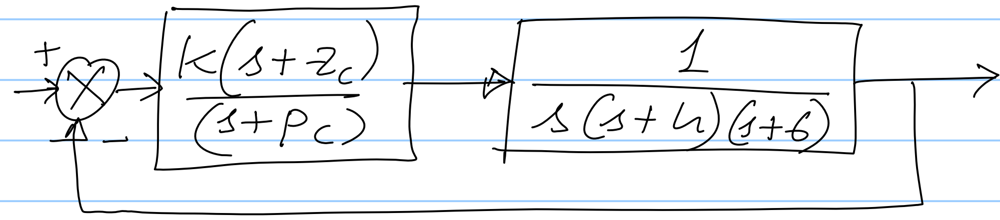
Total angle contribution at P $= (\theta_2 - \theta_1) > 0$

* **Note** that $|Z_C| > |P_C|$ in Lag Network
$|Z_C| < |P_C|$ in lead network

* There can be many choices of Z_C, P_C Chosen according to :

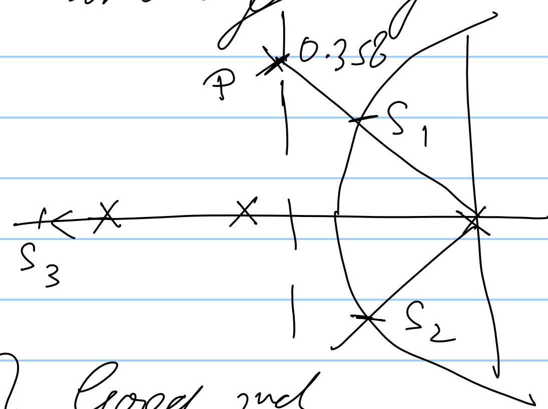
→ gain required
→ SSE achieved
→ 2^{nd} order approx validity.

Example:



Design lead compensation to achieve
(i) 30% OS
(ii) improve settling time by a factor of 2.

$30\%\ OS \Rightarrow \zeta = 0.358$



Uncompensated system:

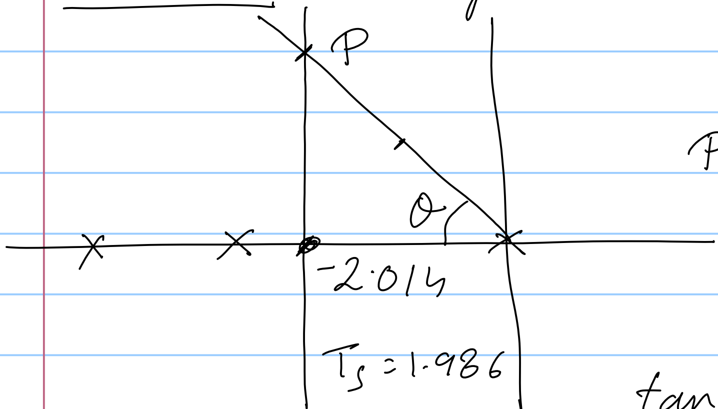
$K = 63.21$

CL Poles: $s_{1,2} = -1.007 \pm j\,2.627$
$s_3 = -7.98$
} Good 2nd order approx

We can use settling time formula:

$$T_s = \frac{4}{\text{Real part}} = \frac{4}{1.007} = 3.972$$

Problem: Keep $\%OS = 30$, make $T_s = \frac{3.972}{2} = 1.986$



$P := \sigma + j\omega$;

$$T_s = \frac{4}{\sigma} = 1.986$$

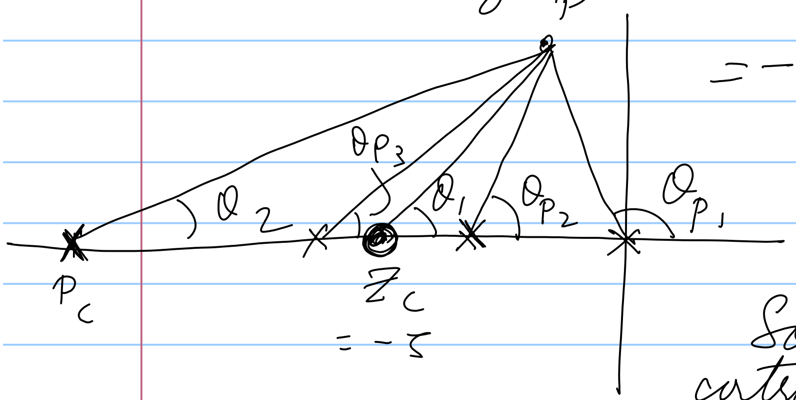
$\sigma = 2.014$

$\tan\theta = \frac{\omega}{2.014}$; $\cos\theta = 0.358$

$\Rightarrow \omega = 5.252$ and $P = -2.014 + j5.252$

Next we need to choose z_c & p_c to include P in the R-L.

Total angle contribution from O-L poles



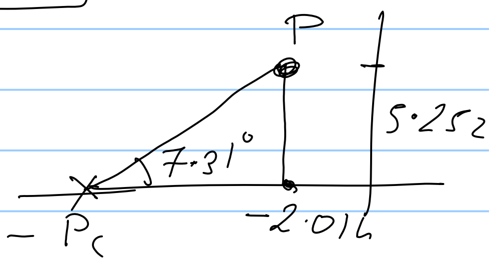
$= -(\theta_{p_1} + \theta_{p_2} + \theta_{p_3}) + \theta_1$

$= -172.69°$

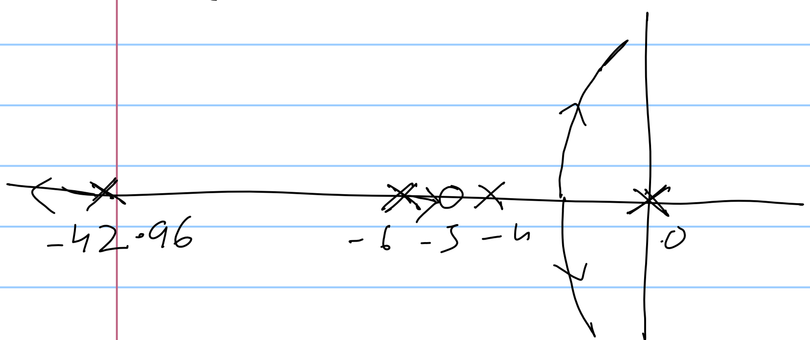
So a total angle contribution of $-7.31°$ is needed from the lead comp. POLE.

i.e. $\boxed{\theta_2 = 7.31°}$

$$\Rightarrow \frac{5.252}{p_c - 2.014} = \tan 7.31°$$



$\Rightarrow p_c = 42.96$



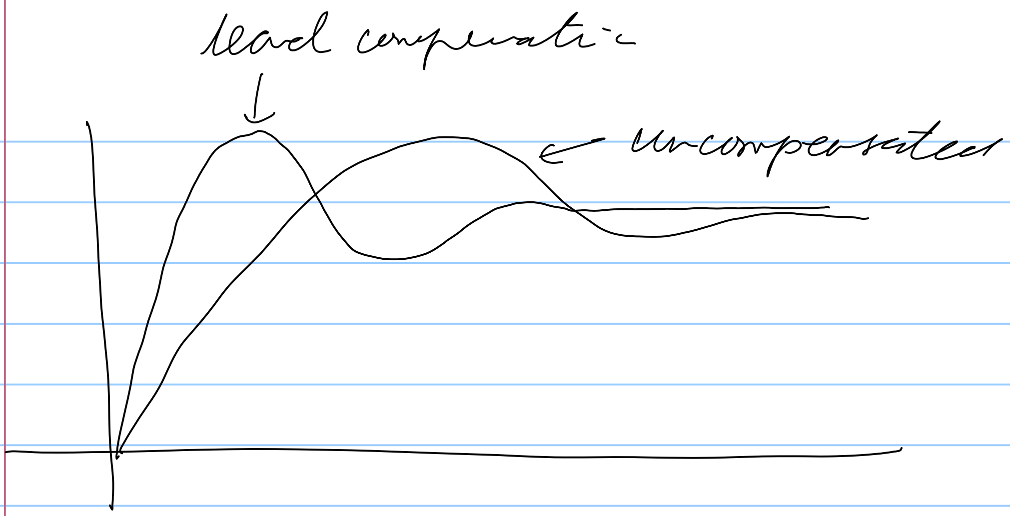
$K = 1423$

C.L poles $\begin{cases} -2.014 \pm j5.252 \\ -43.8 \\ -5.134 \end{cases}$

C.L. zero $\rightarrow -5$

$$e(\infty) = \frac{1}{K_v} = 0.145 \quad \text{(Type 1)}$$

2nd order approx OK since pole zero are close & other pole is at -42.96.



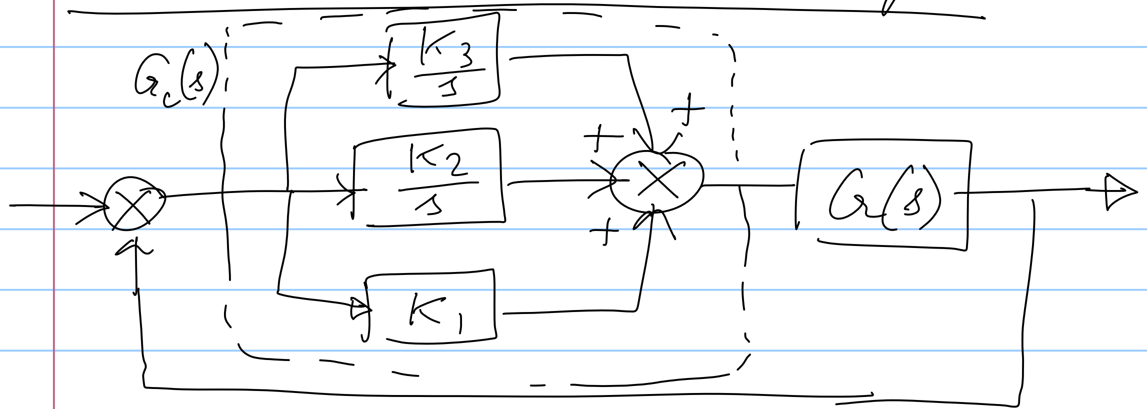
Exercise: Design two other lead compensators for the same system with $z_c = -4$ and $z_c = -2$.

Improving SSE + Transient Response

Method: i) 1st improve Tr. Resp. (using P-D)
ii) Then improve SSE (using PI)

* Reverse order is also possible but not to be covered here.

PID Controller Design



$$G_c(s) = K_1 + \frac{K_2}{s} + K_3 s$$

or $$G_c(s) = \frac{K_3 s^2 + K_1 s + K_2}{s}$$

* two zeroes
* one pole at origin

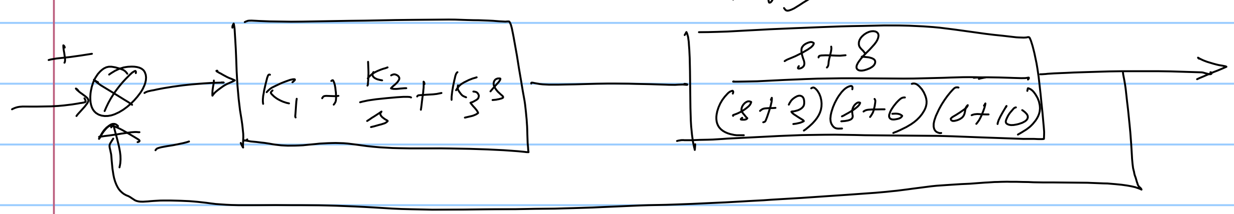
* One zero + 1 pole at origin $\equiv$ PI control
* Other zero $\equiv$ PD

Design Steps:

1) Evaluate performance of Uncomp. System
2) Design PD controller to meet transient specs — Verify with simulations
3) Design PI control to yield SSE
4) Determine gains K_1, K_2, K_3. $\rightarrow$ Verify with simulation

Example:

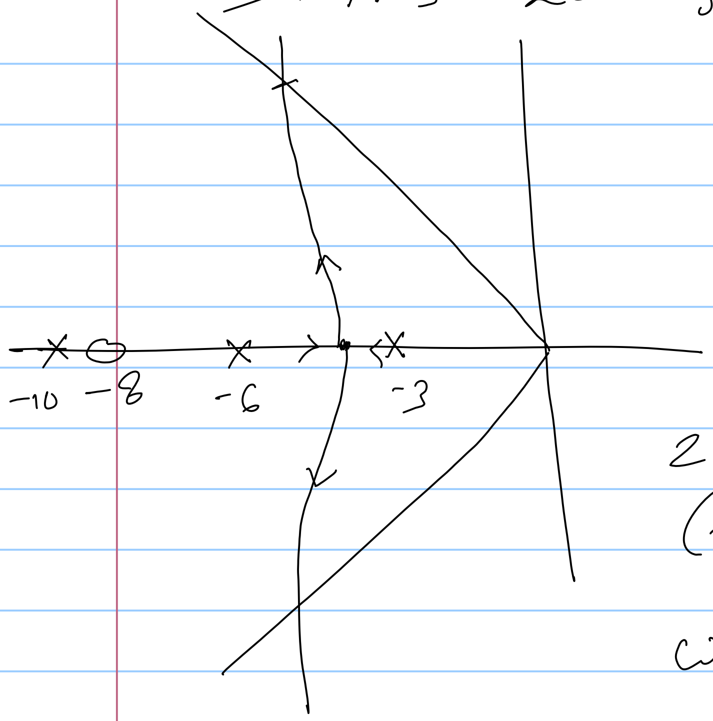
sys 2



Design PID controller so that (i) peak time reduces to $\frac{2}{3}$ of uncompensated system
(ii) $\%OS = 20$
(iii) $SSE = 0$ for step input.

Step 1: Uncompensated System (Gain design)

$\% OS = 20 \approx \zeta = 0.456$



$K = 121.5$

CL poles: $\begin{cases} -5.4 \pm j\,10.57 \\ -8.169 \end{cases}$

C.L. zero $= -8$

2nd order approx is valid. (should be verified with simulation)

$\omega_n = 11.88$; $\zeta = 0.456$

$\% OS = 20$; $T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.297$

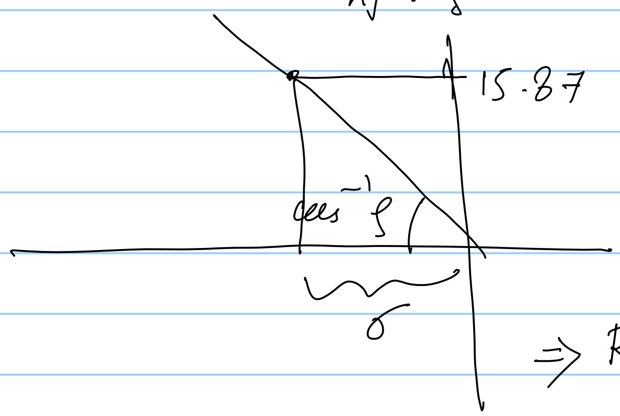
$$K_p = \frac{121.5 \times 8}{3 \times 6 \times 10} = 5.4 \; ; \; e(\infty) = 0.156$$

Step 2 : PD controller design

We want peak time

$$T_{p_N} = \frac{2}{3} T_{p_o} = \frac{2}{3} \times (0.297)$$

$$T_{p_N} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \Rightarrow \underbrace{\omega_n\sqrt{1-\zeta^2}}_{\text{imag part of desired c.l. dominant pole}} = \frac{\pi}{\frac{2}{3} \times (0.297)} = 15.87$$



$\sigma = -8.13$

$\Rightarrow$ Req. C.L poles $= \underbrace{-8.13 \pm j15.87}$

Total angle contribution of the O.L

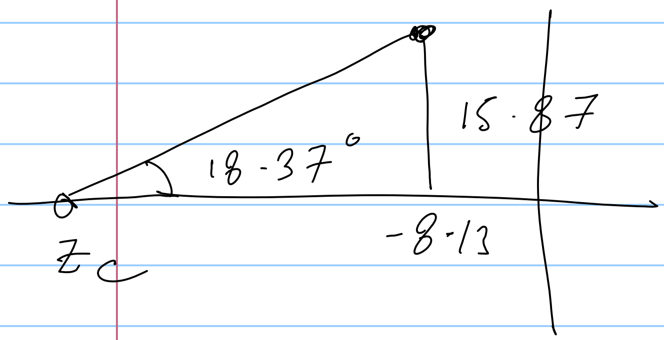
poles at $\underbrace{-3, -6, -10}$ & zero at $\underbrace{-8}$

at $(-8.13 + j15.87)$ is

-198.37°

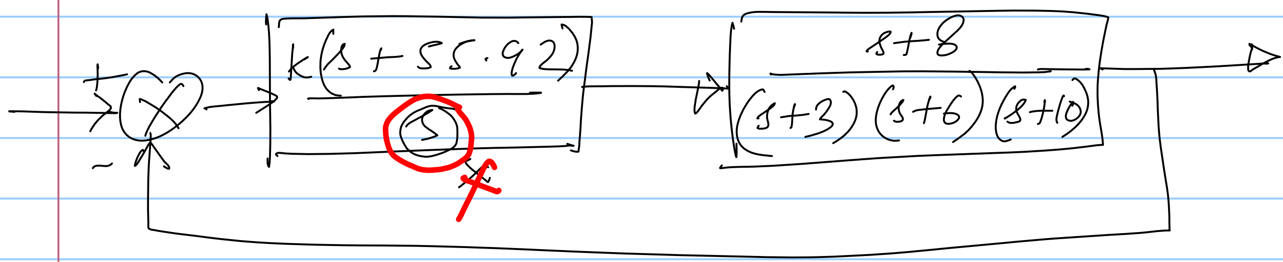
Angle contribution req. from zero

$= +18.37^\circ \Rightarrow \dfrac{15.87}{z_c - 8.13} = \tan 18.37$



$\Rightarrow z_c = 55.92$

$G_{PD}(s) = (s + 55.92)$



$K = 5.34$: C.L poles: $\begin{cases} -8.13 \pm j15.87 \\ -8.079 \end{cases}$

C.L. zeros $\begin{cases} -8 \\ -55.92 \end{cases}$

Second Order approx OK

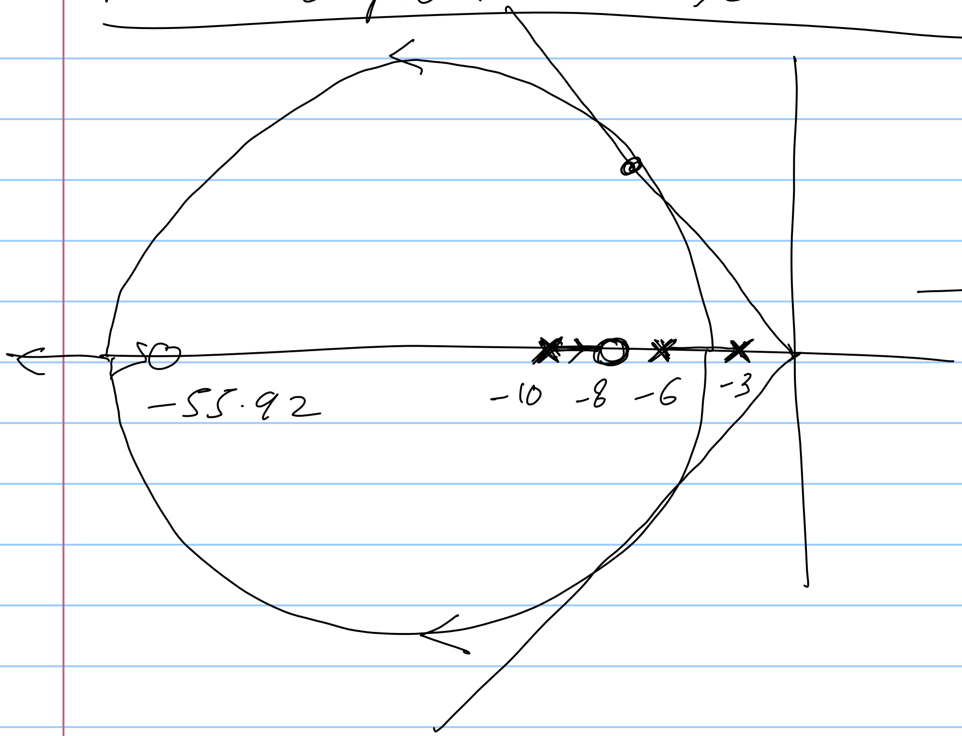
$T_p = 0.198$;

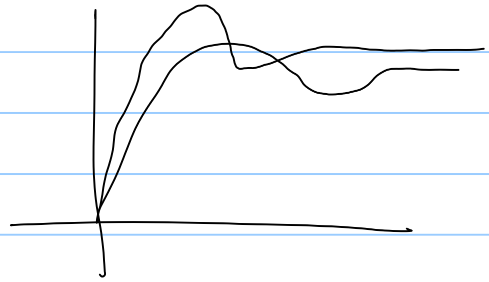
$K_p = 13.27 \qquad e(\infty) = 0.070$

Verify

Simulation results closely match

P-D compensated root-locus

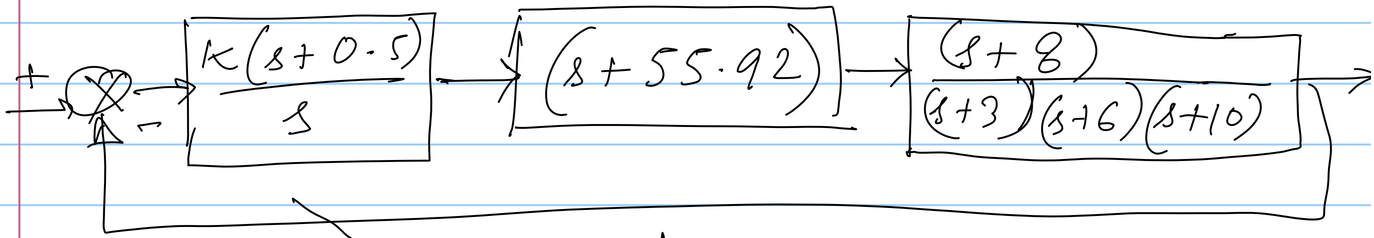


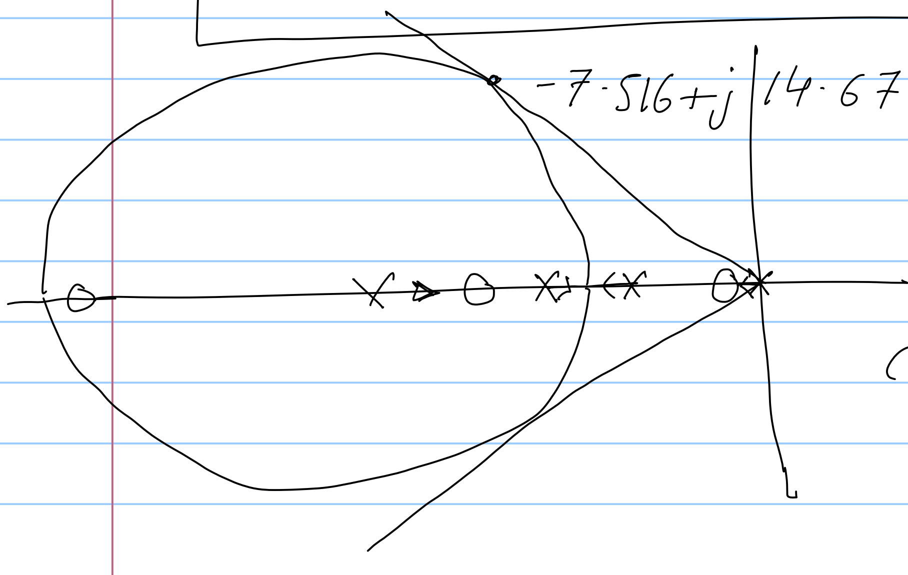


PI - Compensator Design

* Zero should be close to origin

$$G_{PI}(s) = K\left(\frac{s+0.5}{s}\right)$$





Draw R−L
and draw $\zeta = 0.456$
line.

$K = 4.6$

C.L poles: $\begin{cases} -7.516 \pm j\,14.67 \\ -8.099 \\ -0.468 \end{cases}$

C.L zeros: $-8, -55.92, -0.5$.

2nd order approx is OK.

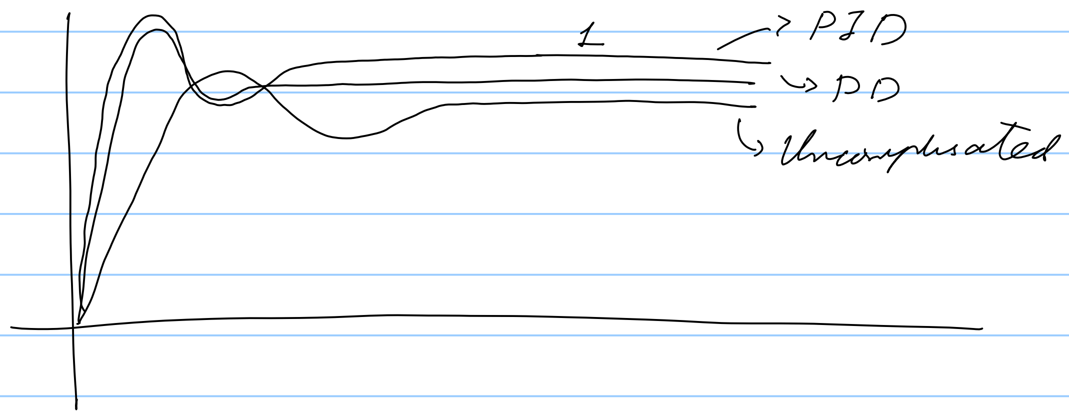
$e(\infty) = 0$ (Type 1 system)

Determine K_1, K_2, K_3

PID comp: $\dfrac{4.6(s+55.92)(s+0.5)}{s}$

$$= \frac{4.6(s^2 + 56.42s + 27.96)}{s}$$

$K_1 = 259.5$, $K_2 = 128.6$, $K_3 = 4.6$



Exercise: PID control slows down the system. What is the soln?

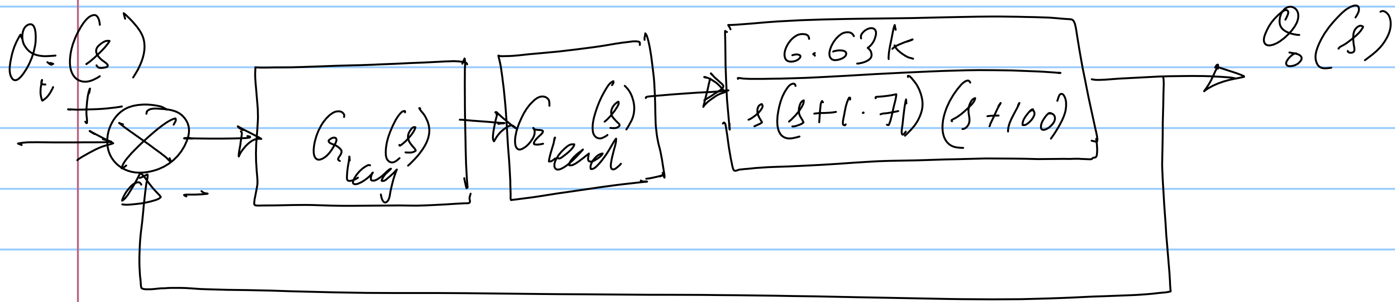
Lag - Lead Compensation

1) Evaluate performance of Uncomp. System
2) Design LEAD controller to meet transient specs — Verify with simulations

3) Design SSE to check how much improvement is still required.

4) Design LAG comp. to yield SSE — Verify with simulation

Example : Antenna Control :



Design cascade comp. to meet :
1) 25% OS 2) $T_s = 2$ sec (3) $K_v = 20$

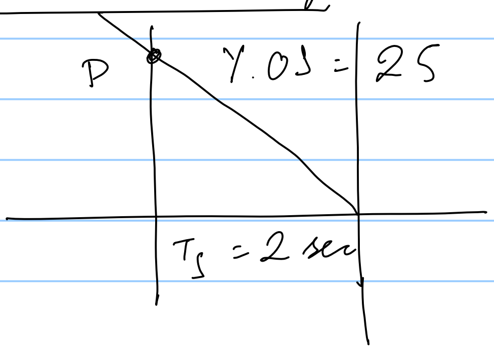
Gain design : (done in Chap 8) :

% OS = 25 was achieved with $K = 64.21$
C.L. poles : $-0.833 \pm j1.888$ } 2nd order
Third pole < -100 } Ap. OK

$$T_s = \frac{4}{0.833} = 4.8 \text{ seconds.}$$

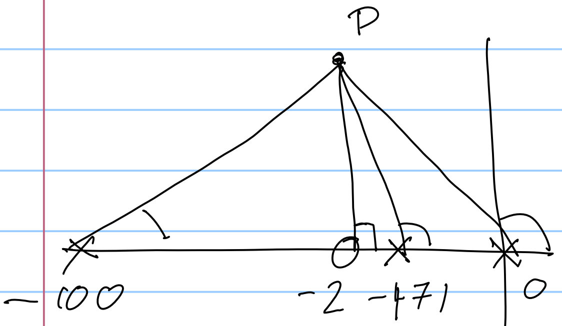
$$K_V = \frac{6.63 \times 64.21}{(1.71) \times 100} = 2.49$$

Lead Comp : improve T_s (= 2 sec)



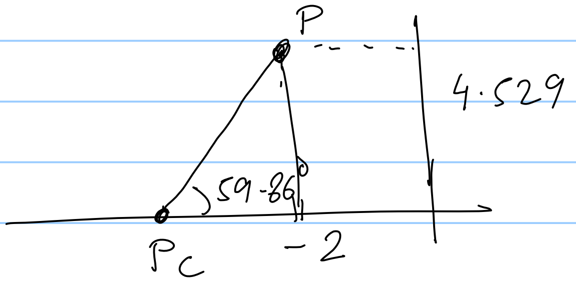
$P = -2 + j4.529$

Assume lead comp zero at -2 ($z_c = 2$)



Total angle contrib
$= -120.14^\circ$

Pole should contrib $= -59.86^\circ$



$$\frac{4.529}{P_c - 2} = \tan 59.86^\circ$$

$$P_c = 4.63$$

Lead comp system: $\dfrac{6.63\,K\,(s+2)}{s(s+1.71)(s+100)(s+4.63)}$

Find $K \doteq \left(\dfrac{2549}{6.63}\right)$

Check using simulation

Check $K_v = \dfrac{2549 \times 2}{(1.71)(100)(4.63)} = 6.44$

But we want $K_v = 20$. $\left(\text{Improvement} = \dfrac{20}{6.44} = 3.1\right)$

LAG Compensation: Choose pole $P_c = -0.01$

$$Z_c = 3.1 \times (-0.01)$$
$$= -0.031$$

Composite gain

$$\left(\frac{s+0.031}{s+0.01}\right)\left(\frac{s+2}{s+4.63}\right)\left(\frac{6.63K}{s(s+1.71)(s+100)}\right)$$

$6.63K = 2533 \Rightarrow K = 382.1$

Summary: $K = 382.1$

$$K_V = \frac{2533 \times 2 \times 0.031}{(0.01)(1.71)(4.63)(100)} = 19.84$$

Exercise: Check C.L. poles / zeros & time domain simulation.

Exercise: Challenge problem on Pg 493.

Reading Exercise: 1) Active ckt Realization
2) Passive ckt "