Pontryagin's Minimum principle.

$$\min_{u(t)} J = h(x(t_f), t_f) + \int_0^{t_f} [g(x(t), u(t), t)]\, dt$$

s.t. $\dot{x} = \cancel{g}\, a(x(t), u(t), t)$

augment constraint $\dot{x} = a(x(t), u(t), t)$ using auxiliary ~~J_a~~ Variables $p^T(t)$

$$J_a = h(x(t_f), t_f) + \int_0^{t_f} [g(x,u,t) + p^T(a - \dot{x})]\, dt$$

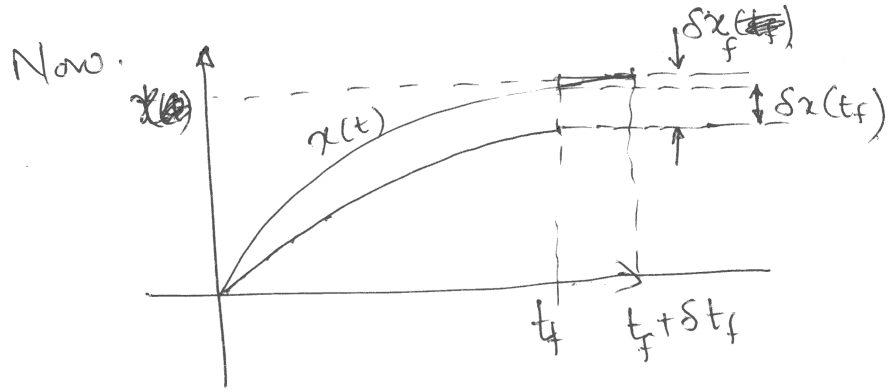
Variation in J_a.

$$\delta J_a = h_x \delta x_f + \frac{\partial h}{\partial t_f}\delta t_f + \int_0^{t_f} [g_x \delta x + g_u \delta u + (a - \dot{x})^T \delta p + p^T a_x \delta x + p^T a_u \delta u - p^T \delta \dot{x}]\, dt + [g(x,u,t_f) + p(t_f)^T (a - \dot{x}(t_f))]\,\delta t_f$$

$$= h_x \delta x_f + \frac{\partial h}{\partial t_f}\delta t_f + \int_0^{t_f} [(g_x + p^T a_x)\delta x + (g_u + p^T a_u)\delta u + (a - \dot{x})^T \delta p]\, dt + [g + p^T(a - \dot{x})](t_f)\,\delta t_f$$

$$- p^T(t_f)\delta x(t_f) + \int_0^{t_f} [\dot{p}^T \delta x]\, dt.$$

Now.



$$\delta x(t_f) = \delta x_f - \dot{x}^*(t_f)\delta t_f.$$

$$\delta J_a = \left(h_x - p^T(t_f)\right)\delta x_f + \left[h_{t_f} + H\right](t_f)\,\delta t_f$$

$$+ \int_{t_0}^{t_f} \left[\left(H_x + \dot{p}^T\right)\delta x + H_u \delta u + \left(H_p - \dot{x}^T\right)\delta p(t)\right] dt$$

Now, ~~ess~~ we will assume that we have $x(t)$, which satisfies all the diff. Eqn.

i.e. $\dot{x}^T = \dfrac{\partial H}{\partial p}$; $\dot{p}^T = -\dfrac{\partial H}{\partial x}$; $h_x(t_f) = p^T(t_f)$

$\left[h_{t_f} + H\right](t_f) = 0$ (Transversality condition)

Then all that remains is

$$\delta J_a = \int_{t_0}^{t_f} \frac{\partial H}{\partial u}\,\delta u(t)\,dt.$$

* for $u(t)$ to be minimizing $\delta J_a \geq 0$ for all admissible $\delta u(t)$
* Or $\delta H \geq 0$ for all time t and variations δu.
* if control constraints are inactive $\dfrac{\partial H}{\partial u} = 0$.
* Stronger condition (Pontryagin's min. principle)

$$u^*(t) = \arg \min_{u(t) \in U} H(x(t), u(t), t)$$

Minimum time problem

$$\dot{x} = F(t)x(t) + G(t)u.$$

$$x(0) = x_0.$$

Let Scalar Control Variable be $|u(t)| \leq 1$

Problem: find $u(t)$ Such that $x(t_f) = 0$ in achieved in minimum time possible.

i.e

$$J = \int_0^t 1 \, dt.$$

s.t.

$$\dot{x} = F(t)x + G(t)u.$$

$$x(t_0) = x_0$$

$$x(t_f) = 0$$

$$|u(t)| \leq 1$$

Let Hamiltonian be defined as.

$$H = \lambda^T(Fx + Gu) + 1$$

$$H = \lambda^T Fx + \lambda^T Gu + 1.$$

~~note~~ by PMP optimal Control u^* must be such that it minimizes H for all time t.

Thus

$$u^*(t) = \arg\min_{u(t)} H(x(t), u(t), t)$$

note: since Control u appears linearly in H one can not use $\frac{\partial H}{\partial u} = 0$.

But, $|u(t)| \leq 1$,

so, By ~~observation~~ observing H we see that control that minimizes H satisfyies

$$~~H=~~ \quad u = \begin{cases} 1 & \text{if } \lambda^T G < 0 \\ -1 & \text{if } \lambda^T G > 0. \end{cases}$$

$\lambda^T G$ is <u>switching function</u>.

moreover ~~by~~ we have

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x}. \qquad [\text{necessary condition}].$$

$$= -\lambda^T F.$$

Since this is free final time problem.

By Transversality condition.

$$\left[\lambda^T (Fx(t) + Gu(t)) + 1\right]_{t=t_f} = 0$$

This helps in finding final time t and ~~input~~ auxiliary variables $\lambda(t)$ and hence input $u(t)$.

moreover we observe that input $u(t)$ switches between $+1$ and -1. So the optimal control input is Bang-Bang in nature.

Now we state few theorems related to linear systems's time-optimal Control problem that we just described. We consider <u>Linear time-invariant systems</u>

(i) $\boxed{\text{Existence}}$:

<u>Theorem</u>: If all Eigenvalues of F are with <u>non-positive</u> real parts then there Exist an optimal Control which will transfer <u>any</u> initial State x_0 to the origin. (Assuming (F, G) Controllable).

However even if all E.values of F are of arbitrarily signed real parts, then there exist a region in state-space (called (R) reachable region); initial conditions $x_0 \in R$ can be time-optimally transferred to the origin

(ii) $\boxed{\text{Uniqueness}}$:

Theorem. If Extremal Control Exists then it is unique.

(iii) $\boxed{\text{Number of Switchings}}$:

If E.values of F are all real and a (unique) time optimal control Exists, then each Control Component can switch atmost (n-1) times. (where n is system order).

Minimum time Control of Double - integrator system:

$\dot{x}_1 = x_2$

$\dot{x}_2 = u$

$|u(t)| \leq 1$

$$\min_{u \in U} J = \int_0^{t_f} 1 \, dt.$$

Form Hamiltonian.

$$H = 1 + \lambda_1 x_2 + \lambda_2 u.$$

$\dot{\lambda}_1 = 0 \implies \lambda_1 = \text{Const.} = c_1$

$\dot{\lambda}_2 = -\lambda_1 \implies \lambda_2 = -c_1 t + c_2$

Switching function is $\lambda_2(t)$

So
$$u(t) = \begin{cases} 1 & \text{if } \lambda_2(t) \leq 0 \\ -1 & \text{if } \lambda_2(t) > 0 \end{cases}$$

Moreover by transversality condition

$$\left[1+\lambda_1 x_1 + \lambda_2 u\right]_{t=t_f} = 0$$

$$\Rightarrow \quad 1 + \lambda_2(t_f)\, u(t_f) = 0$$

and hence $\quad \lambda_2(t_f)\, u(t_f) = -1$

Situation at $t = t_f$. Either $\lambda_2(t_f) = 1$ & $u(t_f) = -1$.

or $\lambda_2(t_f) = -1$ and $u(t_f) = 1$

Since $\lambda_2(t) = -c_1 t + c_2$ is linear in t, input $u(t)$ can change sign ~~only~~ atmost once.

Going backwards in time from t_f and using $u = \pm 1$ we can form switching surface. (Curve in this case) in state-space (which can be used to generate feedback law.

For input $u = -1$

$\dot{x}_2 = -1 \Rightarrow x_2 = -1t + c$ but since at $t = t_f$ $x(t_f) = 0$

$$x_2 = -t + t_f$$

$$\dot{x}_1 = x_2 \Rightarrow x_1 = -\frac{(t_f - t)^2}{2}$$

This gives a switching curve. $x_1 = -\dfrac{x_2^2}{2}$

x_1 and x_2 characterized in this manner gives set of (x_1, x_2) which can reach $(0,0)$ without any switch by using $u = -1$

For input $u = 1$.

$\dot{x}_2 = 1 \Rightarrow x_2 = t + c$. but since at $t = t_f$ $x_2(t_f) = 0$

$x_2 = t - t_f$.

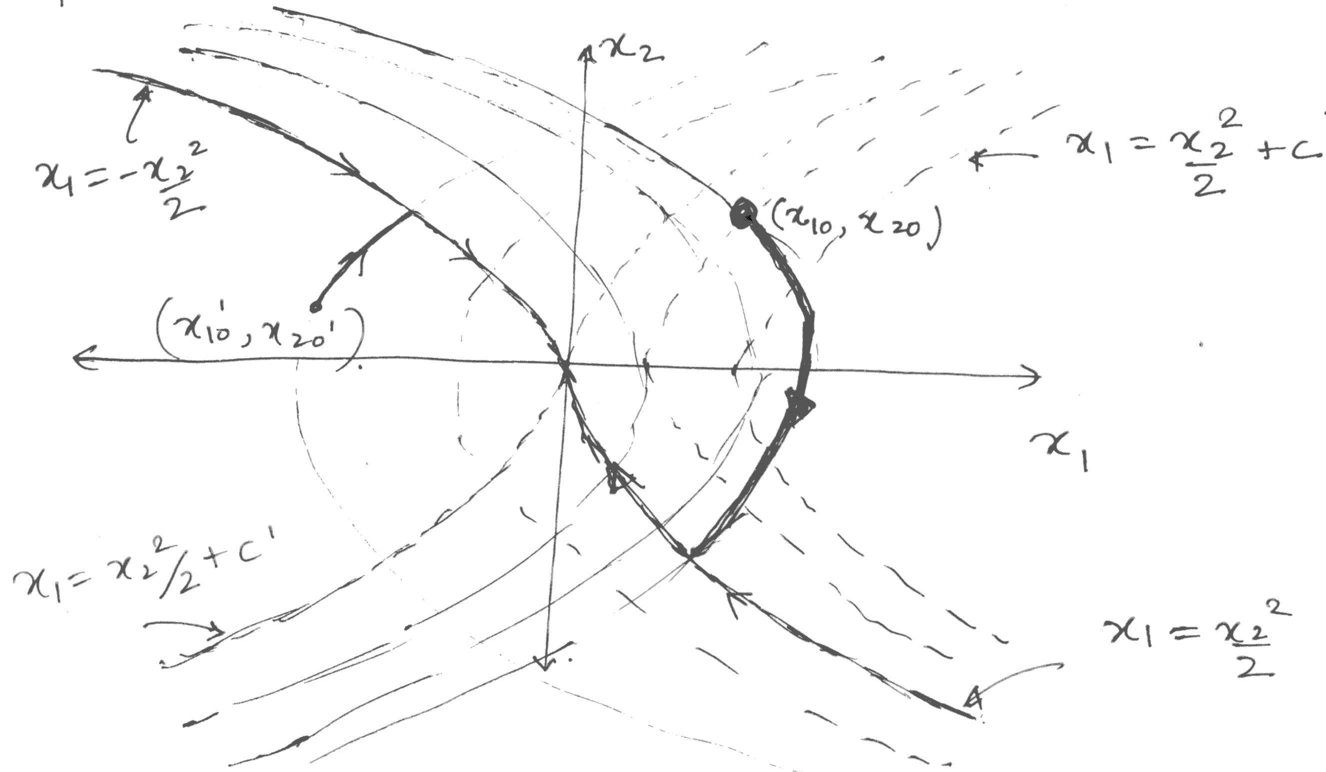
and $x_1 = \frac{(t - t_F)^2}{2}$. Thus, $x_1 = \frac{x_2^2}{2}$.

Also. $\frac{dx_1}{dt} = x_2$, $\frac{dx_2}{dt} = u$. gives $\frac{dx_1}{dx_2} = \frac{x_2}{u}$.

Integrating $x_1 = \frac{x_2^2}{2} + c$. and $x_1 = -\frac{x_2^2}{2} + c'$

for $u = +1$ and -1 respectively

Thus State trajectory (x_1, x_2) moves along parabolic arcs. Shown in fig.



~~If one~~ The switching law can be stated as follows.

$$u = \text{sign}\left(x_1 \mp \frac{|x_2|\, x_2}{2}\right).$$

Minimum ~~Fuel~~ Control Effort Problems :

Find a Control $u^*(t)$ Satisfying

$$M_{i-} \leq u_i(t) \leq M_{i+} \quad i = 1, 2, \ldots, m.$$

which transfers system

$$\dot{x}(t) = a(x(t), u(t), t)$$

from any initial Condition x_0 to Specified target set $S(t)$ with minimum Expenditure of Control Effort.

We consider two performance indices

$$\text{(i)}\ J(u) = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} \beta_i |u_i(t)| \, dt \right]$$ {magnitude of input is proportional to fuel consumption}

and (ii) $$J(u) = \int_{t_0}^{t_1} \left[\sum_{i=1}^{m} r_i u_i^2(t) \right] dt.$$ {mini. Energy Consumption}

where β_i and r_i are non-negative

(i) minimum fuel problem.

$$\dot{x}(t) = a(x(t), t) + B(x(t), t)\, u(t).$$

$$\text{min.} \quad J(u) = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} |u_i(t)| \right] dt$$

Control Constraints.

$$-1 \leq u_i(t) \leq 1, \qquad i = 1, 2, \dots, m \quad t \in [t_0, t_f]$$

Hamiltonian is

$$\mathcal{H}(x(t), u(t), p(t), t) = \sum_{i=1}^{m} |u_i(t)| + p^T(t)\, a(x(t), t) + p^T(t)\, B(x(t), t)\, u(t)$$

By PMP,

$$\left\{ \sum_{i=1}^{m} |u_i^*(t)| + p^{*T}(t)\, B(x^*(t), t)\, u^*(t) + p^{*T}(t)\, a(x^*(t), t) \right\} \leq$$

$$\left\{ \sum_{i=1}^{m} |u_i(t)| + p^{*T}(t)\, a(x^*(t), t) + p^*(t)\, B(x^*(t), t)\, u(t) \right.$$

Or.

$$\sum_{i=1}^{m} |u_i^*(t)| + p^*(t)\, B(x^*(t), t)\, u^*(t)) \leq \sum_{i=1}^{m} |u_i(t)| + p^*(t)\, B(x^*(t), t)\, u(t).$$

for all admissible $u(t)$ and for all $t \in [t_0, t_f]$

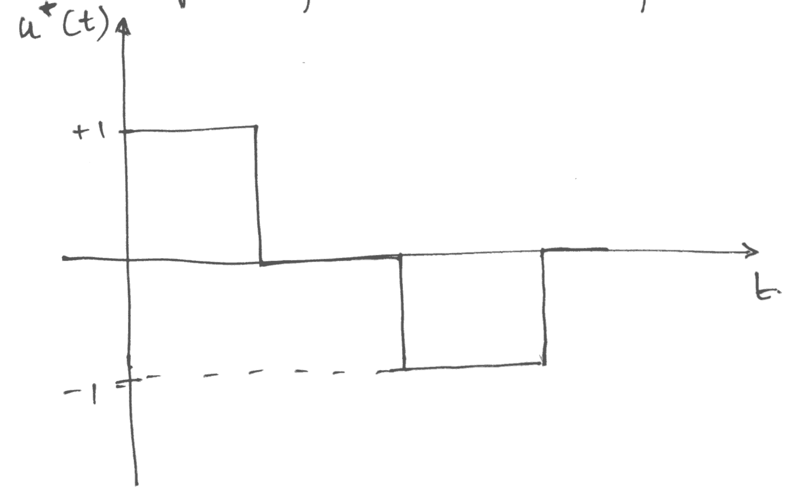
Assuming components of u are independent of one another.

$$|u_i^*(t)| + p^{*T}(t)\, b_i(x^*(t), t)\, u_i^*(t)$$
$$\leq |u_i(t)| + p^{*T}(t)\, b_i(x^*(t), t)\, u_i(t)$$

Once again we see that.

$$u_i^*(t) = \begin{cases} 1.0 & \text{if } p^*(t)\, b_i(x^*(t), t) < -1 \\ 0 & \text{if } |p^*(t)\, b_i(x^*(t), t)| < 1. \\ -1.0 & \text{if } p^*(t)\, b_i(x^*(t), t) > 1 \end{cases}$$

Example of such an input.



Example: Free final time [non-unique soln]

$$\dot{x} = u. \qquad x(0) = x_0$$
$$x(t_f) = 0.$$

$$J(u) = \int_{t_0}^{t_f} |u(t)|\,dt. \qquad |u| \leq 1.$$

Where t_f is free.

$$\mathcal{H} = |u(t)| + p(t)\,u(t).$$

$$\dot{p} = 0.$$

$$p^* = c$$

$$u^*(t) = \begin{cases} 1.0 & \text{if } c < -1 \\ 0 & \text{if } |c| < 1 \\ -1 & \text{if } c > 1 \end{cases}$$

Now,

$$x(t_f) = \int_{t_0}^{t_f} u(t)\,dt + x_0.$$

but $x(t_f) = 0$;

$$x_0 = -\int_{t_0}^{t_f} u(t)\,dt.$$

Consider a case when $x_0 = 5$.

$$\left.\begin{aligned} u(t) &= -1 \quad \text{for } t_f = 5 \\ u(t) &= -0.5 \quad \text{for } t_f = 10. \end{aligned}\right\} \quad \text{— (1)}$$

Similarly, one forms many other cases for $u(t)$ values

$|x_0| \leq \int_0^{t_f} |u(t)|\,dt = J$. But (1) satisfies $J = |x_0|$;

Thus each of these controls is ~~time~~ optimal.

min Energy Problems:

$$\min J = \frac{1}{2}\int_0^{t_f} r_i u^2 dt \qquad r_i > 0.$$

$$M_i^- < u_i < M_i^+$$

$$\dot{x} = A(x,t) + B(x,t)\,u(t)$$

$$H = \frac{1}{2}u^T R u + p^T \{ A(x,t) + B(x,t) u \}$$

- By PMP.

$$u^*(t) = \arg\min_{u(t)\in U}\left[\frac{1}{2}u^T \cancel{R_i} u + p^T B(x,t)\,u\right]$$

$\cancel{R =}$ $R = \mathrm{diag}(r_i)$

~~i~~
In unconstrained case.

$\cancel{\frac{1}{2} R\, u_i^2 +}$

$$\tilde{u} = R^{-1}p^T B \quad , \quad \tilde{u}_i = r_i^{-1} p^T b_i$$

But under constraints

$$u_i(t) = \begin{cases} M_i^- & \tilde{u}_i < M_i^- \\ \tilde{u}_i & M_i^- < u_i < M_i^+ \\ M_i^+ & u_i > M_i^+. \end{cases}$$