Kalman Filter

State space model: $x_{i+1} = F_i x_i + G_i u_i \quad i \geq 0$

$$y_i = H_i x_i + v_i \quad i \geq 0$$

$u_i \to$ process noise $\in \mathbb{R}^{m\times 1}$ | $x_i \in \mathbb{R}^n$

$v_i \to$ measurement noise $\in \mathbb{R}^{p\times 1}$

Assumption:

$$\left\langle \begin{bmatrix} u_i \\ v_i \\ x_0 \end{bmatrix}, \begin{bmatrix} u_j \\ v_j \\ x_0 \\ 1 \end{bmatrix} \right\rangle = \underbrace{\begin{bmatrix} Q_i\delta_{ij} & S_i\delta_{ij} & 0 & 0 \\ S_i^*\delta_{ij} & R_i\delta_{ij} & 0 & 0 \\ 0 & 0 & \Pi_0 & 0 \end{bmatrix}}_{\text{known}}$$

$F_i \to n\times n$ | $Q_i \to m\times m$ | $\Pi_0 \to n\times n$

$G_i \to n\times m$ | $R_i \to p\times p$

$H_i \to p\times n$ | $S_i \to m\times p$

$\{F_i, G_i, H_i\} \to$ known.

1) FACT: $\langle u_i, x_j\rangle = 0$, $\langle v_i, x_j\rangle = 0 \quad j \leq i$

Proof: $x_j \in \mathcal{L}\{x_0, u_k, k \leq j-1\}$. But $u_i \perp \mathcal{L}\{x_0, u_k\}_{k\leq j-1}$

$v_i \perp \mathcal{L}\{x_0, u_k\}$

2) FACT: $u_i \perp y_j$, $v_i \perp y_j \quad j \leq i-1$

$\hookrightarrow$ same proof as above

3) FACT: $\langle u_i, y_i\rangle = \langle u_i, x_i\rangle H_i^* + \langle u_i, v_i\rangle = 0 + S_i$

$\langle v_i, y_i\rangle = \langle v_i, x_i\rangle H_i^* + \langle v_i, v_i\rangle = 0 + R_i$

4) FACT: Define $\Pi_i := \langle x_i, x_i\rangle$

$$\Pi_{i+1} = F_i \Pi_i F_i^* + G_i Q_i G_i^* \quad i \geq 0$$

$$\left(\Pi_{i+1} = F_i\langle x_i, x_i\rangle F_i^* + F_i\langle x_i, u_i\rangle G_i^* + G_i\langle u_i, x_i\rangle F_i^* + G_i\langle u_i, u_i\rangle G_i^*\right)$$

Innovations: (Recursion)

$$e_i = y_i - \hat{y}_{i/i-1}$$

$\uparrow$ Projection of x_i onto $\mathcal{L}\{y_0, \dots, y_{i-1}\}$

Using: $y_i = H_i x_i + v_i$, $\hat{y}_{i/i-1} = H_i \hat{x}_{i/i-1} + \hat{v}_{i/i-1}$

But $\hat{v}_{i/i-1} = 0$ since $v_i \perp y_j \quad j \leq i-1$

$\Rightarrow \quad e_i = y_i - \hat{y}_{i/i-1} = y_i - H_i \hat{x}_{i/i-1}$

So to find e_i it seems $\hat{x}_{i/i-1}$ is req. first.

Recall, we had derived such a general formula:

$$\hat{x}_{i+1/i} = \sum_{j=0}^{i} \langle x_{i+1}, e_j \rangle R_{e_j}^{-1} e_j$$

$$= \underbrace{\hat{x}_{i+1/i-1}} + \langle x_{i+1}, e_i \rangle R_{e,i}^{-1} (y_i - H_i \hat{x}_{i/i-1})$$

\# Unfortunately this is not recursive. Would have been perfect if the first term can be expressed as $\hat{x}_{i/i-1}$, & e_i

\# We can now do this using the state eqn:

$$\hat{x}_{i+1/i-1} = F_i \hat{x}_{i/i-1} + G_i \hat{u}_{i/i-1} = F_i \hat{x}_{i/i-1} + 0$$

Thus we get a recursion: (since $u_i \perp y_j, j \leq i-1$)

$$(1) \quad \boxed{\begin{aligned} e_i &= y_i - H_i \hat{x}_{i/i-1} \qquad e_0 = y_0 \\ \hat{x}_{i+1/i} &= F_i \hat{x}_{i/i-1} + K_{p,i} e_i \qquad i \geq 0 \end{aligned}}$$

where $K_{p,i} := \langle x_{i+1}, e_i \rangle R_{e,i}^{-1}$

\# Note: Its starting to look familiar: (2nd eqn)

$$\hat{x}_{i+1/i} = F_i \hat{x}_{i/i-1} + K_{p,i}(y_i - H_i \hat{x}_{i/i-1})$$

Q. How to compute $R_{e,i}$ & $K_{p,i}$ (iteratively)

Define the error covariance matrix:

$$P_{i|i-1} := \left\langle \underbrace{(x_i - \hat{x}_{i|i-1})}_{\tilde{x}_{i|i-1}}, (x_i - \hat{x}_{i|i-1}) \right\rangle$$

Notation: $\hat{x}_i := \hat{x}_{i|i-1}$, $\tilde{x}_i := \tilde{x}_{i|i-1}$, $P_i := P_{i|i-1}$

Clearly, $e_i = y_i - H\hat{x}_i = Hx_i + v_i - H\hat{x}_i = H\tilde{x}_i + v_i$

$\boxed{R_{e,i} := \langle e_i, e_i \rangle = H_i P_i H_i^* + R_i}$ (since $v_i \perp \tilde{x}_i$)

$(*)$

Similarly for $K_{p,i}$

Recall $K_{p,i} = \langle \underbrace{x_{i+1}, e_i}_{\text{has to be found.}} \rangle R_{e,i}^{-1}$

$$\langle x_{i+1}, e_i \rangle = F_i \langle x_i, e_i \rangle + G_i \langle u_i, e_i \rangle$$

Now
$$\langle x_i, e_i \rangle = \langle x_i, \tilde{x}_i \rangle H_i^* + \langle x_i, v_i \rangle$$
$$= P_i H_i^* + 0 \quad \left[\text{Since } \langle x_i, \tilde{x}_i \rangle = \langle \hat{x}_i + \tilde{x}_i, \tilde{x}_i \rangle = 0 + P_i\right]$$
$$\langle u_i, e_i \rangle = \langle u_i, \tilde{x}_i \rangle H_i^* + \langle u_i, v_i \rangle = 0 + S_i$$

where $\tilde{x}_i \in \mathcal{L}\{x_i; y_0, \ldots, y_{i-1}\} \subset \mathcal{L}\{x_0; u_0, \ldots, u_{i-1}; v_0, \ldots, v_{i-1}\} \perp u_i$

Here $\boxed{K_{p,i} = (F_i P_i H_i^* + G_i S_i) R_{e,i}^{-1}}$

\# So if we knew P_i we can calculate $R_{e,i}, K_{p,i}$.

Q. How to compute P_i?

$$\hat{x}_{i+1} = F_i \hat{x}_i + K_{p,i} e_i = F_i \hat{x}_i + K_{p,i}(H_i \tilde{x}_i + v_i)$$

Here $\tilde{x}_{i+1} = F_i \tilde{x}_i + G_i u_i - K_{p,i} H_i \tilde{x}_i - K_{p,i} v_i$

$$P_{i+1} = \langle \tilde{x}_{i+1}, \tilde{x}_{i+1} \rangle = [F_i - K_{p,i} H_i]\, P_i\, [F_i - K_{p,i} H_i]^* + [G_i \quad -K_{p,i}] \begin{bmatrix} Q_i & S_i \\ S_i^* & R_i \end{bmatrix} \begin{bmatrix} G_i^* \\ -K_{p,i}^* \end{bmatrix}$$

$$\Leftrightarrow \boxed{P_{i+1} = F_i P_i F_i^* + G_i Q_i G_i^* - K_{p,i} R_{e,i} K_{p,i}^* \qquad i \geq 0}$$

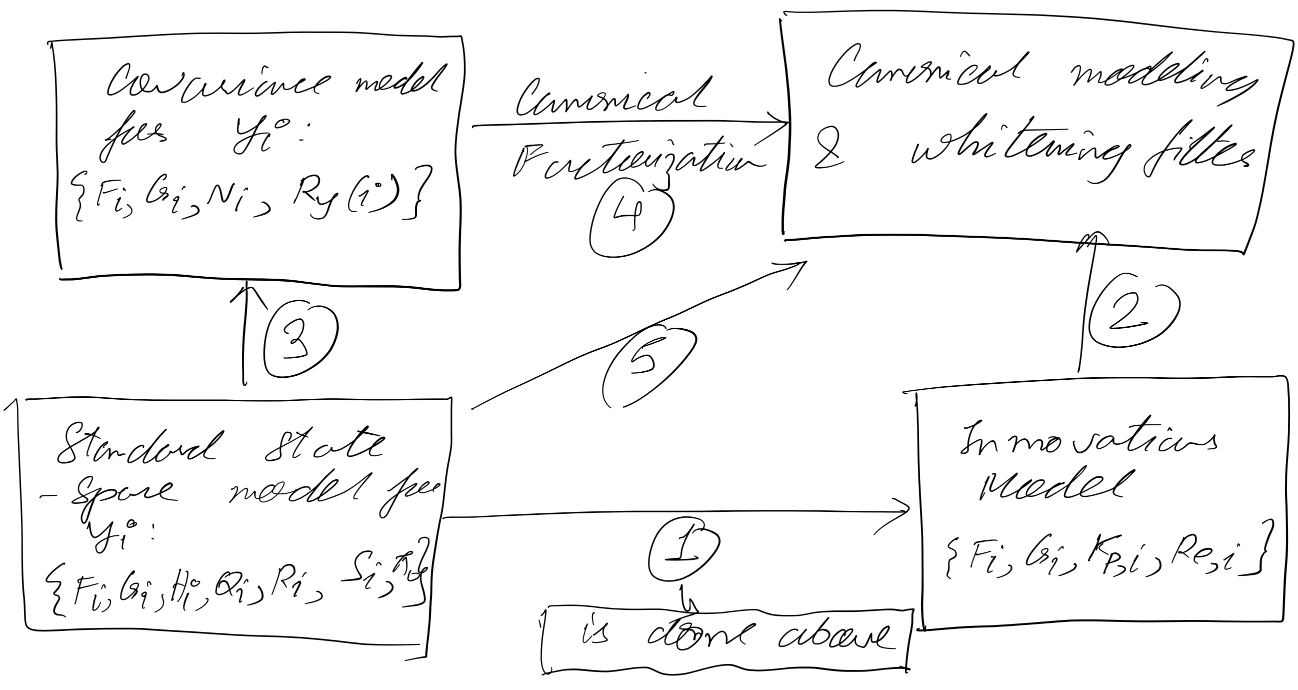
with $P_0 = \langle \tilde{x}_0, \tilde{x}_0 \rangle = \langle (x_0 - \hat{x}_0), (x_0 - \hat{x}_0) \rangle = \langle x_0, x_0 \rangle = \Pi_0$

Discrete time Riccatti Eqn

\# P_i & $K_{p,i}$, $R_{e,i}$ recursions depend only on model parameters & not on y_i. In principle, "off-line" computation is possible.

\# Thus we have derived the llms estimates for x_i given y_i & the model ≡ the Kalman filter.

\# The observer model $\hat{x}_{i+1} = F_i \hat{x}_i + K_{p,i}(y_i - H_i \hat{x}_i)$ is thus proved to be optimal (and not ad-hoc).



Causal & Causally Invertible relationship between y_i & e_i (② in the Diag. above)

From ① above: $(*)\begin{cases} \hat{x}_{i+1} = F_i \hat{x}_i + K_{p,i} e_i \\ y_i = H_i \hat{x}_i + e_i \end{cases}$ $\Big|\ \hat{x}_0 = 0$

$$y_i \longrightarrow \boxed{(**)} \xrightarrow{\;e_i\;} \boxed{(*)} \longrightarrow y_i$$

From $(*)$, $\hat{x}_{i+1} = F_i \hat{x}_i + K_{p,i}(y_i - H_i \hat{x}_i)$
$y_i = H_i \hat{x}_i + e_i$

$$\Leftrightarrow (**)\left[\begin{array}{l} \hat{x}_{i+1} = (F_i - K_{p,i} H_i)\hat{x}_i + K_{p,i} y_i \\ e_i = -H_i \hat{x}_i + y_i \end{array}\right. \Big|\ \hat{x}_0 = 0$$

Note: 1) The original model might not be causally invertible.
2) The innovations model is more convenient for system identification (lesser no of parameters)

Q. Is the innovations model (not the process) unique?

Relationship with $R_y = LDL^T$ (⑤ in Diag. above)

Recall $\left.\begin{array}{l} y_i = H_i \hat{x}_i + e_i \\ \hat{x}_{i+1} = F_i \hat{x}_i + K_{p,i} e_i \end{array}\right\}$ $y_0 = e_0$, $\hat{x}_0 = 0$

Define: $\Phi(i,j) := F_{i-1} F_{i-2} \cdots F_j$ for $i > j$
$\Phi(i,i) := I$

Then:

Let $L :=$

$$y = \begin{bmatrix} I & & & & \\ H_1 K_{p,0} & I & & \cdots & 0 \\ H_2 \Phi(2,1) K_{p,0} & H_2 K_{p,1} & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ H_N \Phi(N,1) K_{p,0} & H_N \Phi(N,2) K_{p,1} & H_N \Phi(N,3) K_{p,2} & \cdots & I \end{bmatrix} e$$

Then ; $y = Le$

Q) How about L^{-1} ? → can be derived from the inv. model (2R).

$$\hat{x}_{i+1} = F_{p,i}\hat{x}_i + K_{p,i} y_i \qquad \Big| \quad \hat{x}_0 = 0$$
$$e_i = -H_i \hat{x}_i + y_i$$

$$L^{-1} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ -H_1 K_{p,0} & I & 0 & \cdots & 0 \\ -H_2 \Phi_p(2,1) K_{p,0} & -H_2 K_{p,1} & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -H_N \Phi_p(N,1) K_{p,0} & -H_N \Phi_p(N,2) K_{p,1} & \cdots & \cdots & I \end{bmatrix}$$

$$\Phi_p(i,j) := F_{p,i-1} F_{p,i-2} \cdots F_{p,j} \quad \text{for } i > j$$
$$= I \quad \text{for } i = j$$

Covariance Model for y_i (B in diag. above)

$y = Le$ so $R_y = \langle y, y \rangle = L R_e L^T$

Using the std. state space model:

$$x_{i+1} = F_i x_i + G_i u_i \qquad i \geq 0$$
$$y_i = H_i x_i + v_i \,,$$

$$R_y(i,j) = \langle y_i, y_j \rangle = \begin{cases} H_i \Phi(i, j+1) N_j & i > j \\ H_i \Pi_i H_i^* + R_i & i = j \\ N_i^* \Phi^*(j, i+1) H_j^* & i < j \end{cases}$$

direct calculation (exercise)

where $\langle x_{i+1}, x_{i+1} \rangle := \Pi_{i+1} = F_i \Pi_i F_i^* + G_i Q_i G_i^*$

$N_i := F_i \Pi_i H_i^* + G_i S_i$

<u>Claim</u>: The canonical modeling & whitening filters can be derived solely in terms of R_y (~~true~~ (4) in diag above)

Recall (already derived formula for $R_{e,i}$, $K_{p,i}$ & P_i):

(1) — $\hat{x}_{i+1} = F_i \hat{x}_i + K_{p,i}(y_i - H_i \hat{x}_i) \qquad \hat{x}_0 = 0$

(2) — $K_{p,i} = F_i P_i H_i^* R_{e,i}^{-1}$

(3) → $R_{e,i} = H_i P_i H_i^* + R_i$

(4) → $P_{i+1} = F_i P_i F_i^* + G_i Q_i G_i^* - K_{p,i} R_{e,i} K_{p,i}^*$

$P_0 = \Pi_0$

Using (1), $\langle \hat{x}_{i+1}, \hat{x}_{i+1} \rangle =: \Sigma_{i+1}$

$= F_i \Sigma_i F_i^* + K_{p,i} R_{e,i} K_{p,i}^*$ [$\because e_i$ is white & e_i & $\hat{x}_i$ are uncorrelated)

Recall, $P_i = \langle \tilde{x}_i, \tilde{x}_i \rangle$, & $\tilde{x}_i \perp \hat{x}_i$

$$\Rightarrow \Pi_i = \Sigma_i + P_i$$

Here from (3), $R_{e,i} = R_i + H_i P_i H_i^* = R_i + H_i(\Pi_i - \Sigma_i)H_i^*$

$$= R_y(i,i) - H_i \Sigma_i H_i^*$$

Fm (2), $K_{p,i} = F_i P_i H_i^* R_{e,i}^{-1} = \left[N_i - F_i \Sigma_i H_i^* \right] R_{e,i}^{-1}$

So: given R_y in state space form:

$y_i \rightarrow$

$$\begin{aligned} \theta_{i+1} &= (F_i - K_{p,i} H_i)\,\theta_i + K_{p,i}\, y_i \\ e_i &= H_i \theta_i - y_i \\ &\text{with } \theta_0 = 0 \end{aligned}$$

Canonical whitening filter

$\rightarrow e_i \rightarrow$

$$\begin{aligned} \theta_{i+1} &= F_i \theta_i + K_{p,i}\, e_i \\ y_i &= H_i \theta_i + e_i \\ &\text{with } \theta_0 = 0 \end{aligned}$$

$\rightarrow y_i$

Canonical Modeling Filter

$$\langle e_i, e_j \rangle =: R_{e,i}\,\delta_{ij}$$

$$\left[\begin{aligned} K_{p,i} &= \left[N_i - F_i \Sigma_i H_i^* \right] R_{e,i}^{-1} \\ R_{e,i} &= R_y(i,i) - H_i \Sigma_i H_i^* \\ \Sigma_{i+1} &= F_i \Sigma_i F_i^* + K_{p,i} R_{e,i} K_{p,i}^* \quad \Big|\ \Sigma_0 = 0 \end{aligned}\right.$$

Note: For LTI S.S. models, nice z-domain factorizations are possible (DARE): not done due to limited time.

Spectral Factorization & ARE for LTI models (NOT INCLUDED in syllabus)

$$x_{i+1} = Fx_i + Gu_i \; ; \; i \geq 0$$
$$y_i = Hx_i + v_i$$

$$\left\langle \begin{bmatrix} x_0 \\ u_i \\ v_i \end{bmatrix}, \begin{bmatrix} x_0 \\ u_j \\ v_j \\ 1 \end{bmatrix} \right\rangle = \begin{bmatrix} \Pi_0 & 0 & 0 & 0 \\ 0 & \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}\delta_{ij} & & 0 \\ 0 & & & 0 \end{bmatrix}$$

\# All Formulae derived above are valid.

\# In general $\{x_i\}, \{y_i\}$ $i \geq 0$ are NOT stationary because Π_i is time dependent.

Lemma: (without proof): Consider the LTI s.s. model & let F be stable & let $\Pi_0 = \bar{\Pi}$ where $\bar{\Pi}$ is the unique soln of the Lyapunov eqn.

$$\bar{\Pi} = F\bar{\Pi}F^* + GQG^*.$$

Then the processes $\{x_i, y_i\}$ are both stationary.

z-spectrum

$$x_{i+1} = Fx_i + \begin{bmatrix} I & 0 \end{bmatrix}\begin{bmatrix} Gu_i \\ v_i \end{bmatrix} ; \quad y_i = Hx_i + \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} Gu_i \\ v_i \end{bmatrix}$$

Taking z-transform: (Does it exist? Is stationarity req?)

$$Y(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix}\begin{bmatrix} Gu(z) \\ V(z) \end{bmatrix}$$

Recall

$$\xrightarrow{r(t)} \boxed{H(z)} \xrightarrow{o(t)} \quad R_o(z) = H(z)S_r(z)H^*\left(\frac{1}{z^*}\right)$$

white noise

So:

$$S_y(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \underbrace{\begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix}}_{\geq 0} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} \quad (*)$$

$$\Rightarrow S_y(e^{j\omega}) \geq 0$$

$\Updownarrow$ (Big jump)

(complications due to $S \neq 0$)

<u>Thm 1</u> (DARE): Define: $F^S = F - GSR^{-1}H$ & $Q^S = Q - SR^{-1}S^*$

Assume F is stable (or (F,H) is detectable),

$\{F^S, GQ^{S/2}\}$ stabilizable, $\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0$ & $R > 0$.

Then the Discrete time Riccati eqn

$$P = FPF^* + GQG^* - K_p R_e K_p^*$$

where K_p & R_e are given by

$$R_e = R + HPH^* \quad \text{and} \quad K_p = (FPH^* + GS)R_e^{-1},$$

has a unique psd sol^{n} $\bar{P}$ s.t. $F - K_{\bar{p}}H$ is stable. Moreover the resulting $R_e > 0$.

<u>Thm 2</u>: (Canonical Spectral Factorization)

Consider $S_y(z)$ of the form $(*)$ with $\{F, G, H, Q, S, R\}$ satisfying the conditions of Thm 1. Then the canonical spectral fact. is

$$S_y(z) = L(z) R_e L^*\left(\frac{1}{z^*}\right), \quad L(\infty) = I, \; R_e > 0$$

where

$$L(z) = I + H(zI - F)^{-1}K_p$$

$$L^{-1}(z) = I - H(zI - F + K_pH)^{-1}K_p$$

$K_p = (FPH^* + GS) R_e^{-1}$; $R_e = R + HPH^*$

& P is the unique psd stabilizing solution to the PARE.

Here F stable + $(F - K_p H)$ stable

$\Rightarrow$ $L(z)$ is minimum phase

The Kalman Filter can be derived straight-away:

$$\hat{y}(z) = [I - L^{-1}(z)] y(z)$$ [Recall example from prev. chap]

$$= H (zI - F + K_p H)^{-1} K_p y(z)$$

after steps similar to the time varying case,

Similarly, $\hat{x}(z) = (zI - F)^{-1} K_p e(z)$

Measurement and Time Update Form

$$0 = \hat{x}_{0/-1} \xrightarrow{m.u.} \hat{x}_{0/0} \xrightarrow{t.u.} \hat{x}_{1/0} \xrightarrow{m.u} \hat{x}_{1/1} \xrightarrow{t.u.} \hat{x}_{2/1} \xrightarrow{m.u.} \hat{x}_{2/2} \xrightarrow{t.u.} \hat{x}_3 \cdots$$

$$\Pi_0 = P_{0/-1} \xrightarrow{m.u.} P_{0/0} \xrightarrow{t.u.} P_{1/0} \xrightarrow{m.u.} P_{1/1} \cdots$$

Recall (already derived formula for $R_{e,i}$, $K_{p,i}$ & P_i):

(1) — $\hat{x}_{i+1/i} = F_i \hat{x}_{i/i-1} + K_{p,i}(y_i - H_i \hat{x}_{i/i-1})$, $\quad \hat{x}_0 = 0$

(2) — $K_{p,i} = F_i P_{i/i-1} H_i^* R_{e,i}^{-1}$

(3) → $R_{e,i} = H_i P_{i/i-1} H_i^* + R_i$

(4) → $P_{i+1/i} = F_i P_{i/i-1} F_i^* + G_i Q_i G_i^* - K_{p,i} R_{e,i} K_{p,i}^*$

$P_{0/-1} = \Pi_0$

\# So we need to derive formula for $\hat{x}_{i/i}$ and $P_{i/i}$

Measurement Update

$$\hat{x}_{i/i} = \sum_{j=0}^{i} \langle x_i, e_j \rangle R_{e,j}^{-1} e_j = \hat{x}_{i/i-1} + \langle x_i, e_i \rangle R_{e,i}^{-1} e_i$$

Now, $\langle x_i, e_i \rangle = \langle x_i, H_i \tilde{x}_{i/i-1} + v_i \rangle = \langle x_i, \tilde{x}_{i/i-1} \rangle H_i^* + \langle x_i, v_i \rangle$
$= P_{i/i-1} H_i^*$

Define $\boxed{K_{f,i} = P_{i/i-1} H_i^* R_{e,i}^{-1}}$. Then (A1)

$$\boxed{\begin{aligned} \hat{x}_{i/i} &= \hat{x}_{i/i-1} + K_{f,i} e_i \\ &= \hat{x}_{i/i-1} + K_{f,i}(y_i - H_i \hat{x}_{i/i-1}) \end{aligned}}$$

(A2)

Next note, $\tilde{x}_{i|i} = x_i - \hat{x}_{i|i}$

$$= x_i - \hat{x}_{i|i-1} - K_{f,i} e_i$$

$$= \tilde{x}_{i|i-1} - K_{f,i} e_i$$

$$\therefore P_{i|i} = \langle \tilde{x}_{i|i}, \tilde{x}_{i|i} \rangle$$

$$= \langle \tilde{x}_{i|i-1}, \tilde{x}_{i|i-1} \rangle - K_{f,i} \langle e_i, \tilde{x}_{i|i-1} \rangle - \langle \tilde{x}_{i|i-1}, e_i \rangle K_{f,i}^* + K_{f,i} \langle e_i, e_i \rangle K_{f,i}^* \quad (2)$$

Now, $\langle e_i, \tilde{x}_{i|i-1} \rangle = H_i \langle \tilde{x}_{i|i-1}, \tilde{x}_{i|i-1} \rangle + \langle v_i, \tilde{x}_{i|i-1} \rangle$

$$= H_i P_{i|i-1} + 0$$

Then, $K_{f,i} \langle e_i, \tilde{x}_{i|i-1} \rangle = P_{i|i-1} H_i^* R_{e,i}^{-1} H_i P_{i|i-1}$

$$= \langle \tilde{x}_{i|i-1}, e_i \rangle K_{f,i}^* \quad (1)$$

Using (1) in (2),

$$P_{i|i} = P_{i|i-1} - 2 P_{i|i-1} H_i^* R_{e,i}^{-1} H_i P_{i|i-1} + P_{i|i-1} H_i^* R_{e,i}^{-1} \cancel{R_{e,i} R_{e,i}^{-1}} H_i P_{i|i-1}$$

$$= P_{i|i-1} - P_{i|i-1} H_i^* R_{e,i}^{-1} H_i P_{i|i-1}$$

or $\boxed{P_{i|i} = P_{i|i-1} - K_{f,i} R_{e,i} K_{f,i}^*}$ — (A3)

Time Update ($S_i^o = 0$)

$$x_{i+1}^o = F_i x_i^o + G_i u_i^o$$

$$\Rightarrow \quad \hat{x}_{i+1|i}^o = F_i^o \hat{x}_{i|i}^o + G_i \hat{u}_{i|i}^o$$

Exercise: show $\hat{u}_{i|i}^o = 0$ if $S_i^o = 0$.

$$\Rightarrow \boxed{\hat{x}_{i+1|i}^o = F_i^o \hat{x}_{i|i}^o} \text{ --- (B1)}$$

$$\# \quad \tilde{x}_{i+1|i} = F_i x_i + G_i u_i - F_i \hat{x}_{i|i} = F_i \tilde{x}_{i|i}^o + G_i u_i^o$$

Hence $P_{i+1|i}^o = \langle \tilde{x}_{i+1|i}^o, \tilde{x}_{i+1|i} \rangle = F_i \langle \tilde{x}_{i|i}, \tilde{x}_{i|i}^o \rangle F_i^* + G_i^o \langle u_i, u_i \rangle G_i^*$

$$\boxed{P_{i+1|i} = F_i P_{i|i}^o F_i^{o*} + G_i Q_i G_i^{o*}} \text{ --- (B2)}$$

(A1) + (A2) + (A3) $\longrightarrow$ M. U.

(B1) + (B2) $\longrightarrow$ T. U.

Approximate Nonlinear Filtering / EKF

$$(*)\begin{cases} x_{i+1} = f_i(x_i) + g_i(x_i)\,u_i \\ y_i = h_i(x_i) + v_i \end{cases}$$

u_i, v_i are zero mean. $E x_0 = \bar{x}_0$

$$\left\langle \begin{bmatrix} u_i \\ v_i \\ x_0 - \bar{x}_0 \end{bmatrix}, \begin{bmatrix} u_j \\ v_j \\ x_0 - \bar{x}_0 \end{bmatrix} \right\rangle = \begin{bmatrix} Q\delta_{ij} & 0 & 0 \\ 0 & R\delta_{ij} & 0 \\ 0 & 0 & \Pi_0 \end{bmatrix}$$

The most obvious method is to linearize about a nominal solution / eq. pt. & apply the std K.F. eqns.

EKF : Notation : a particular estimate of x_i i.e. the value of $\hat{x}_{i/i}$ is denoted by $\hat{x}_{i/i}$

$$f_i(x_i) \approx f_i(\hat{x}_{i/i}) + F_i(x_i - \hat{x}_{i/i})$$
$$h_i(x_i) \approx h_i(\hat{x}_{i/i-1}) + H_i(x_i - \hat{x}_{i/i-1})$$
$$g_i(x_i) \approx g_i(\hat{x}_{i/i}) =: G_i$$

$$F_i = \left.\frac{\partial f_i(x)}{\partial x}\right|_{x=\hat{x}_{i/i}} \; ; \quad H_i = \left.\frac{\partial h_i(x)}{\partial x}\right|_{x=\hat{x}_{i/i-1}}$$

Then $(*)$ can be re-written as:

$$\left\{\begin{array}{l} x_{i+1} = F_i x_i + \underbrace{\left[f_i(\hat{x}_{i|i}) - F_i \hat{x}_{i|i} \right]}_{\text{known at time } i} + G_i u_i \\ y_i - \underbrace{\left(h_i(\hat{x}_{i|i-1}) - H_i \hat{x}_{i|i-1} \right)}_{\text{known at time } i-1} = H_i x_i + v_i \end{array}\right\} \quad (*)$$

From $(*)$,

$$\hat{x}_{i+1|i} = F_i \hat{x}_{i|i} + f_i(\hat{x}_{i|i}) - F_i \hat{x}_{i|i}$$

$$= f_i(\hat{x}_{i|i})$$

$$\hat{x}_{i|i} = \hat{x}_{i|i-1} + K_{f,i}\left[y_i - h_i(\hat{x}_{i|i-1}) + H_i \hat{x}_{i|i-1} - H_i \hat{x}_{i|i-1} \right]$$

$$= \hat{x}_{i|i-1} + K_{f,i}\left[y_i - h_i(\hat{x}_{i|i-1}) \right]$$

EKF Eqns:

$$\hat{x}_{i+1|i} = f_i(\hat{x}_{i|i}) \longrightarrow \text{T.U.}$$

$$\left.\begin{array}{l} \hat{x}_{i|i} = \hat{x}_{i|i-1} + K_{f,i}\left[y_i - h_i(\hat{x}_{i|i-1}) \right] \\ K_{f,i} = P_{i|i-1} H_i^* (H_i P_{i|i-1} H_i^* + R_i)^{-1} \\ P_{i|i} = (I - K_{f,i} H_i) P_{i|i-1} \end{array}\right\} \text{M.U.}$$

$$P_{i+1|i} = F_i P_{i|i} F_i^* + G_i Q_i G_i^* \longrightarrow \text{T.U}$$

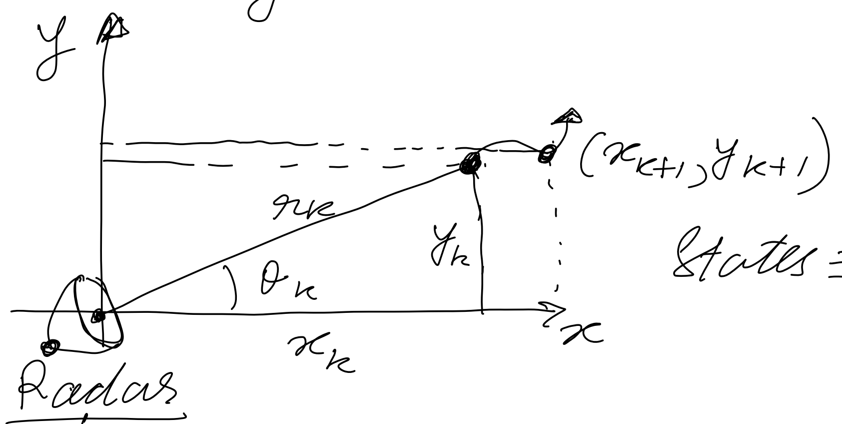
$$\hat{x}_{0|-1} = \bar{x}_0, \quad P_{0|-1} = \Pi_0$$

Note: ① F_i, G_i, H_i depend on the measurements so $P_i, K_{f,i}$ cannot be pre-computed.

② $K_{f,i}$ depends non-linearly on prior measurements.

Kalman Filter Modelling Example

Track a point in 2D space with range & bearing measurement



States $= \bar{x}_k := \begin{bmatrix} \dot{x}_k \\ x_k \\ \dot{y}_k \\ y_k \end{bmatrix}$

(1) Noisy range measurement : $r_k + e_{r_k}$

(2) Noisy bearing measurement : $\theta_k + e_{\theta_k}$

We assume (from Radar characteristics) that $E e_{r_k} = E e_{\theta_k} = 0$ and

$$\left\langle \begin{bmatrix} e_{r_k} \\ e_{\theta_k} \end{bmatrix}, \begin{bmatrix} e_{r_j} \\ e_{\theta_j} \end{bmatrix} \right\rangle = \begin{bmatrix} \sigma_r \delta_{ij} & 0 \\ 0 & \sigma_\theta \delta_{ij} \end{bmatrix}$$

We want to model the output/measurement eqn as:

$$z_k := \begin{bmatrix} r_k \sin\theta_k \\ r_k \cos\theta_k \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_k \\ x_k \\ \dot{y}_k \\ y_k \end{bmatrix} + v_k$$

i.e $\quad z_k = H\bar{x}_k + v_k$

Q. How to model v_k ?

$$v_k = \begin{bmatrix} (r_k + e_{r_k})\sin(\theta_k + e_{\theta_k}) - r_k \sin\theta_k \\ (r_k + e_{r_k})\cos(\theta_k + e_{\theta_k}) - r_k \cos\theta_k \end{bmatrix}$$

$$\approx \begin{bmatrix} e_{r_k}\sin\theta_k + r e_{\theta_k}\cos\theta_k \\ e_{r_k}\cos\theta_k - r e_{\theta_k}\sin\theta_k \end{bmatrix} \quad \begin{bmatrix}\text{Assuming small errors}\end{bmatrix}$$

Hence $R_k := \langle v_k, v_k \rangle =$

$$= \begin{bmatrix} \sigma_r^2 \sin^2\theta_k + r_k^2\sigma_\theta^2\cos^2\theta_k & (\sigma_r^2 - r_k^2\sigma_\theta^2)\sin\theta_k\cos\theta_k \\ (\sigma_r^2 - r_k^2\sigma_\theta^2)\sin\theta_k\cos\theta_k & \sigma_r^2\cos^2\theta_k + r_k^2\sigma_\theta^2\sin^2\theta_k \end{bmatrix}$$

<u>Note</u>: R_k is dependent on states. So heuristically the last state estimates are used to calculate R_k.

Q. How to model the state eqn:

$$\bar{x}_{k+1} = F\bar{x}_k + \omega_k \qquad ?$$

Since we do not know how the target

is going to maneuver we try to model the uncertainty in motion by noise.

e.g. we can try $\dot{x}_{k+1} = \dot{x}_k + w_k^1$

But we don't have any way of measuring w_k^1.

\# Hence assume that we have observed r_k & θ_k and have been able to estimate the capabilities of the target in changing speed (i.e. $\dot{r}_k$) and course (i.e. θ_k). $\Rightarrow$ We have estimated

1> Mean square change in speed $=: \sigma_s^2$

2> " " " " course $=: \sigma_c^2$

that occurs in the sampling interval $=: \Delta$

Velocity Modeling:

$$\text{(I)}\quad \begin{cases} \dot{x}_{k+1} = \alpha \dot{x}_k + w_k^1 \\ \dot{y}_{k+1} = \alpha \dot{y}_k + w_k^3 \end{cases}$$

α - explained below. Assume $\alpha < 1$ for now

Recalling $\begin{cases} \dot{x}_k = \dot{r}_k \cos \theta_k \\ \dot{y}_k = \dot{r}_k \sin \theta_k \end{cases}$, we can derive

$$\langle w_k^1, w_k^1 \rangle \approx \sigma_s^2 \sin^2\theta_k + \sigma_c^2 \dot{r}_k^2 \cos^2\theta_k$$
$$\langle w_k^3, w_k^3 \rangle \approx \sigma_s^2 \cos^2\theta_k + \sigma_c^2 \dot{r}_k^2 \sin^2\theta_k$$
$$\langle w_k^1, w_k^3 \rangle \approx \left(\sigma_s^2 - \sigma_c^2 \dot{r}_k^2\right) \sin\theta_k \cos\theta_k$$

Determining α :

From $\circledast$, $\dot{r}_{k+1} = \alpha \dot{r}_k + e_r$

If $\dot{r}_0 = 0$, $\langle \dot{r}_k, \dot{r}_k \rangle = k\sigma_s^2$

$\Rightarrow$ Unbounded variance in speed

So $\alpha < 1$ has to be chosen judiciously.

Position Modeling :

$$x_{k+1} = x_k + \frac{1}{2}\Delta\left(\dot{x}_k + \dot{x}_{k+1}\right)$$
$$= x_k + \Delta\left(\dot{x}_k + \frac{1}{2} w_k^1\right)$$

Combined eqn:

$$\begin{bmatrix} \dot{x}_{k+1} \\ x_{k+1} \\ \dot{y}_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ \Delta & 1 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & \Delta & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_k \\ x_k \\ \dot{y}_k \\ y_k \end{bmatrix} + \begin{bmatrix} w_k^1 \\ \frac{\Delta}{2} w_k^1 \\ w_k^3 \\ \frac{\Delta}{2} w_k^4 \end{bmatrix}$$

Remaining Questions: $P_0 = ?$, $\bar{x}_0 = ?$