Stochastic Identification: $u_k = 0 \quad \forall k.$

Problem: $x_{k+1} = Ax_k + w_k$

$y_k = Cx_k + v_k$

$E w_k = E v_k = 0$

$$E \begin{bmatrix} w_k \\ v_k \end{bmatrix} \begin{bmatrix} w_j \\ v_j \end{bmatrix}^T = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \delta_{kj}$$

Determine 1) $n \to$ order

2) $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{l\times n}$ within sim. tr.

3) $Q \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times l}$, $R \in \mathbb{R}^{l\times l}$ s.t.

$E y_k y_k^T$ of data and model are equal.

Old K.F. notation

$x_{i+1} = Fx_i + Gu_i$

$y_i = Hx_i + v_i$

$$\left\langle \begin{bmatrix} x_0 \\ u_i \\ v_i \end{bmatrix}, \begin{bmatrix} x_0 \\ u_j \\ v_j \\ 1 \end{bmatrix} \right\rangle = \begin{bmatrix} \pi_0 & 0 & 0 & 0 \\ 0 & \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}\delta_{ij} & & 0 \\ 0 & & & 0 \end{bmatrix}$$

$E[x_k x_k^T] = \Sigma \quad \longleftrightarrow \quad \langle x_i, x_i \rangle =: \overline{\pi}$

$\Sigma = A\Sigma A^T + Q \quad \longleftrightarrow \quad \overline{\pi} = F\overline{\pi}F^* + GQG^*$

$\Lambda_i = E[y_{k+i} \, y_k^T]$

$\Lambda_0 = C\Sigma C^T + R$

$$R_y(i) = \begin{cases} HF^{i-1}\overline{N} & i > 0 \\ H\overline{\pi}H^* + R & i = 0 \\ \overline{N}^* F^{*(-i-1)} H^* & i < 0 \end{cases}$$

where $\overline{N} = F\overline{\pi}H^* + GS$

$G := E[x_{k+1} y_k^T]$ (Exercise)

$= A\Sigma C^T + S$

For $i = 1, 2, \ldots$,

$\Lambda_i = CA^{i-1}G$

$\Lambda_{-i} = G^T (A^{i-1})^T C^T$

looks like Markov parameters

Innovations Model → Notation Matching

$$\hat{x}_{k+1} = A\hat{x}_k + K e_k$$
$$y_k = C\hat{x}_k + e_k$$
$$R_{e,k} = \langle e_k, e_k \rangle$$
$$= \Lambda_0 - C P_k C^T$$
$$\boxed{\begin{aligned} &C \Sigma C^T + R - CPC^T \\ &= C[\Sigma - P]C^T + R \end{aligned}}$$
$$\langle \hat{x}_k, \hat{x}_k \rangle =: P_k$$
$$P_{k+1} = AP_kA^T + (G - AP_kC^T) \times (\Lambda_0 - CP_kC^T)^{-1}(G - AP_kC^T)^T$$
$$K_k = (G - AP_kC^T)(\Lambda_0 - CP_kC^T)^{-1}$$

$$\hat{x}_{i+1} = F\hat{x}_i + K_p e_i \quad \Big| \; \hat{x}_0 = 0$$
$$y_i = H\hat{x}_i + e_i$$
$$K_p = (FPH^* + GS) R_e^{-1}$$
$$R_e = R + HPH^*$$
$$P = FPF^* + GQG^* - K_p R_e K_p^*$$

Alt. form

$$\Sigma_i = \langle \hat{x}_i, \hat{x}_i \rangle$$
$$R_{e,i} = R_y(i,i) - H_i \Sigma_i H_i^*$$
$$K_{p,i} = [N_i - G_i S_i - F_i \Sigma_i H_i^*] R_{e,i}^{-1}$$
$$\Sigma_{i+1} = F_i \Sigma_i F_i^* + K_{p,i} R_{e,i} K_{p,i}^*$$
$$\Sigma_0 = 0$$
$$[\Pi_i = \Sigma_i + P_i]$$

Recall notation for

$$\begin{array}{c} Y_p^+ \\ \\ Y_f^- \end{array} \begin{bmatrix} y_0 & \cdots & y_{j-1} \\ \vdots & & \\ y_{i-1} & \cdots & y_{i+j-2} \\ \hline y_i & \cdots & y_{i+j-1} \\ \hline y_{2i-1} & \cdots & y_{2i+j-2} \end{bmatrix} \begin{array}{c} Y_p \\ \\ Y_f \end{array} \qquad \begin{array}{c} \to Y_{0|0} \\ \\ \to Y_{i|i} \end{array}$$

$$\Delta_i^c = [A^{i-1}G \;\; A^{i-2}G \; \cdots \; AG \;\; G] \in \mathbb{R}^{n \times li}$$

$$C_i = \begin{bmatrix} \Lambda_i & \Lambda_{i-1} & \cdots & \Lambda_1 \\ \Lambda_{i+1} & \cdots & & \Lambda_2 \\ \vdots & & & \\ \Lambda_{2i-1} & & \cdots & \Lambda_i \end{bmatrix} \in \mathbb{R}^{li \times li}$$

$$L_i = \begin{bmatrix} \Lambda_0 & \Lambda_{-1} & \cdots & \Lambda_{1-i} \\ \Lambda_1 & \Lambda_0 & & \Lambda_{2-i} \\ & & \ddots & \\ \Lambda_{i-1} & \Lambda_{i-2} & \cdots & \Lambda_0 \end{bmatrix} \in \mathbb{R}^{li \times li}$$

Ergodic data: Given two jointly stationary ergodic processes a_k & e_k $k = 0, 1, \ldots, j, \ldots$

$$E[a_k e_k^T] = \lim_{j \to \infty} \left[\frac{1}{j} \sum_{i=0}^{j} a_i e_i^T \right]$$

due to ergodicity

$$=: E_j \left[\sum_{i=0}^{j} a_i e_i^T \right]$$

$$= E_j \left\{ \underbrace{[a_0 \; a_1 \cdots a_j]}_{a} \underbrace{\begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_j \end{bmatrix}}_{e^T} \right\}$$

$$= E_j [a e^T]$$

If $A \in \mathbb{R}^{p \times j}$, $B \in \mathbb{R}^{q \times j}$

$$\Phi[A, B] := E_j [A . B^T]$$

iid $\Rightarrow$ ergodic

stationary $\not\Rightarrow$ erg.

Note: $\Lambda_i = E_j \left[\sum_{k=0}^{j-1} y_{k+i} y_k^T \right]$

$$= E_j [Y_{i|i} Y_{0|0}^T]$$

Then $C_i = E_j [Y_f Y_p^T]$

$$L_i = E_j [Y_p Y_p^T] = E_j [Y_f Y_f^T]$$

[Ex: verify]

Theorem: If $\hat{x}_0 = 0$, $E[\hat{x}_0 x_0^T] =: P_0 = 0$ then the Kalman filter eqns:

$$\begin{cases} \hat{x}_k = A\hat{x}_{k-1} + K_{k-1}(y_{k-1} - C\hat{x}_{k-1}) \\ K_{k-1} = [G - AP_{k-1}C^T][\Lambda_0 - CP_{k-1}C^T]^{-1} \\ P_k = AP_{k-1}A^T + (G - AP_{k-1}C^T)\times(\Lambda_0 - CP_{k-1}C^T)^{-1} \\ \qquad \times (G - AP_{k-1}C^T)^T \end{cases}$$

are equivalent to:

$$\begin{cases} \hat{x}_k = \Delta_k^c L_k^{-1} \begin{bmatrix} y_0 \\ \vdots \\ y_{k-1} \end{bmatrix} & \text{—— (1)} \\ P_k = \Delta_k^c L_k^{-1} (\Delta_k^c)^T & \text{—— (2)} \end{cases}$$

Note: From (1):

$$\hat{X}_i^o = (\hat{x}_i^o \quad \hat{x}_{i+1} \quad \ldots \quad \hat{x}_{i+j-1})$$
$$= \Delta_i^{c\,o} L_i^{o\,-1} Y_p$$

Proof: By induction: $k = 1$

$$\hat{x}_1 = A\hat{x}_0 + K_0(y_0 - C\hat{x}_0) \quad \Big| \quad \hat{x}_0 = 0$$
$$= K_0 y_0$$

$$K_0 = (G - AP_0C^T)(\Lambda_0 - CP_0C^T)^{-1} \quad \Big| \quad P_0 = 0$$
$$= G\Lambda_0^{-1}$$

$$\Rightarrow \hat{x}_1 = G\Lambda_0^{-1} y_0 = \Delta_1^c L_1^{-1} y_0 \rightarrow \text{same as (1) for } k = 1$$

Let (1) be true for $k = p$

$$\hat{x}_p = \Delta_p^C L_p^{-1} \begin{bmatrix} y_0 \\ \vdots \\ y_{p-1} \end{bmatrix}$$

Then we claim that $\hat{x}_{p+1} = \Delta_{p+1}^C L_{p+1}^{-1} \begin{bmatrix} y_0 \\ \vdots \\ y_p \end{bmatrix}$

i.e

$$\hat{x}_{p+1} = \begin{bmatrix} A\Delta_p^C & G \end{bmatrix} \begin{bmatrix} L_p & (\Delta_p^C)^T C^T \\ C\Delta_p^C & \Lambda_0 \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ \vdots \\ y_p \end{bmatrix}.$$

Recall the matrix inversion lemma:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \equiv \begin{bmatrix} (A - BD^{-1}C)^{-1} & A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix} \longrightarrow (\#)$$

Also recall:

$$[A + BCD]^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$

$$\Rightarrow (A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B[D - CA^{-1}B]^{-1}CA^{-1}$$

Then (#) reduces to:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B\Phi^{-1}CA^{-1} & -A^{-1}B\Phi^{-1} \\ -\Phi^{-1}CA^{-1} & \Phi^{-1} \end{bmatrix} \longrightarrow (\#\#)$$

where $\Phi = D - CA^{-1}B$

$$\hat{x}_{p+1} = \begin{bmatrix} A\Delta_p^c & G \end{bmatrix} \begin{bmatrix} L_p^{-1} + L_p^{-1}(\Delta_p^c)^T C^T \Phi^{-1} C \Delta_p^c L_p^{-1} & -L_p^{-1}(\Delta_p^c)^T C^T \Phi^{-1} \\ -\Phi^{-1} C \Delta_p^c L_p^{-1} & \Phi^{-1} \end{bmatrix} \begin{bmatrix} y_0 \\ \vdots \\ y_p \end{bmatrix}$$

---

where $\Phi = \Lambda_0 - C \underbrace{\Delta_p^c L_p^{-1} (\Delta_p^c)^T}_{P_p} C^T$

$= \Lambda_0 - C P_p C^T$

$$= \left[A - \big(G - A \overbrace{\Delta_p^c L_p^{-1} (\Delta_p^c)^T}^{P_p} C^T\big)\Phi^{-1} C\right] \underbrace{\Delta_p^c L_p^{-1} \begin{bmatrix} y_0 \\ \vdots \\ y_{p-1} \end{bmatrix}}_{\hat{x}_p}$$

$$+ \left[G - A \underbrace{\Delta_p^c L_p^{-1} (\Delta_p^c)^T}_{P_p} C^T\right]\Phi^{-1} y_p$$

$$= A\hat{x}_p - \overbrace{\left(G - A P_p C^T\right)\Phi^{-1}}^{K_p} C \hat{x}_p$$
$$+ \underbrace{\left[G - A P_p C^T\right]\Phi^{-1}}_{K_p} y_p$$

$$= A\hat{x}_p + K_p\left[y_p - C\hat{x}_p\right]$$

$P_{p+1} \rightarrow$ proof skipped.

Thm: Assume 1) w_k & v_k are not identically zero
2) $j^o \to \infty$

Then: 1) $\xi_{i^o} := \mathcal{Y}_f / \mathcal{Y}_p = \Gamma_{i^o} \widehat{X}_{i^o}$

2) $\xi_{i^o} = [U_1 \quad U_2] \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$
$= U_1 S_1 V_1^T$

$n = \text{size}(S_1) \to$ i.e. no of non-zero singular values

3) $\Gamma_{i^o} = U_1 S_1^{1/2} T$
$\Delta_{i^o}^c = \Gamma_{i^o}^{+} E_j[\mathcal{Y}_f \mathcal{Y}_p^T]$
$\widehat{X}_{i^o} = T^{-1} S_1^{1/2} V_1^T$

Proof: $\mathcal{Y}_f / \mathcal{Y}_p = \langle \mathcal{Y}_f, \mathcal{Y}_p \rangle \langle \mathcal{Y}_p, \mathcal{Y}_p \rangle^{-1} \mathcal{Y}_p$
$= C_{i^o} L_{i^o}^{-1} \mathcal{Y}_p$

Now $C_{i^o} = \begin{bmatrix} \Lambda_{i^o} & \Lambda_{i-1} & \cdots & \Lambda_1 \\ \vdots & \ddots & & \vdots \\ \Lambda_{2i^o-1} & \cdots & \cdots & \Lambda_{i^o} \end{bmatrix}$

$= \begin{bmatrix} CA^{i^o-1}G_2 & CA^{i-2}G_2 & \cdots & CG_2 \\ \vdots & \ddots & & \vdots \\ CA^{2i^o-2}G_2 & \cdots & \cdots & CA^{i^o-1}G_2 \end{bmatrix}$

$$= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix} \begin{bmatrix} A^{i-1}G & A^{i-2}G & \cdots & AG & G \end{bmatrix}$$

$$= \Gamma_i^o \Delta_i^c \longrightarrow \left[\begin{aligned} \Delta_i^c &= \Gamma_i^{o+} C_i \\ &= \Gamma_i^{+} \mathcal{E}_j(y_f y_p^T) \end{aligned} \right]$$

Then $Y_f / Y_p = \Gamma_i^o \Delta_i^c L_p^{-1} Y_p = \Gamma_i^o \hat{X}_i^o$

[Since $\hat{X}_i^o = \Delta_i^c L_i^{o-1} Y_p$, $\rightarrow$]

<u>Computing System Matrices</u> : [$n, \Gamma_i^o, \hat{X}_i^o$ are known]

\# $\underline{\xi}_i^o := Y_f^- / Y_p^+ = \Gamma_{i-1}^o \hat{X}_{i+1}^o$

$$\Gamma_i^o = \begin{bmatrix} \\ \\ \hline \\ \end{bmatrix} \begin{matrix} \}\, \Gamma_{i-1}^o \\ \\ \}\, l \end{matrix} \Rightarrow \hat{X}_{i+1}^o = \Gamma_{i-1}^{o+} \underline{\xi}_i^o$$

\# $\begin{bmatrix} \hat{X}_{i+1}^o \\ Y_{i|i}^o \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} \hat{X}_i^o + \begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix}$ residuals & innovations

$$\Rightarrow \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} \hat{X}_{i+1}^o \\ Y_{i|i}^o \end{bmatrix} \cdot \hat{X}_i^{o+}$$

Combined Stochastic - Deterministic I.D.

Problem: Given $\{u_k\}_1^s \in \mathbb{R}^m$ & $\{y_k\}_{k=1}^s \in \mathbb{R}^l$ generated from:

$$x_{k+1} = Ax_k + Bu_k + w_k$$
$$y_k = Cx_k + Du_k + v_k$$

$$E\,w_k = E\,v_k = 0 \quad , \quad \left\langle \begin{bmatrix} w_p \\ v_p \end{bmatrix}, \begin{bmatrix} w_q \\ v_q \end{bmatrix} \right\rangle = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \delta_{pq}$$

Determine: 1) order n

2) A, B, C, D upto sim. tr.

3) Q, R, S so that the cov. of the stochastic parts match.

Notation: $x_k = x_k^d + x_k^s$
$y_k = y_k^d + y_k^s$

$$x_{k+1}^d = Ax_k^d + Bu_k$$
$$y_k^d = Cx_k^d + Du_k$$

$$x_{k+1}^s = Ax_k^s + w_k$$
$$y_k^s = Cx_k^s + v_k$$

For simplicity assume A stable, $\{A, B\}$ & $\{A, Q^{1/2}\}$ controllable.

$$X_i := \begin{bmatrix} x_i & x_{i+1} & \cdots & x_{i+j-1} \end{bmatrix} \in \mathbb{R}^{n \times j}$$
$$= X_i^d + X_i^s$$

$X_p^d = X_0^d$ | $X_f^d = X_i^d$ | Δ_i^s, L_i, C_i as defined previously

$X_p^s = X_0^s$ | $X_f^s = X_i^s$

$$\Delta_i^d = \begin{bmatrix} A^{i-1}B & A^{i-2}B & \cdots & AB & B \end{bmatrix}$$

$$H_i^d = \begin{bmatrix} D & & & 0 \\ CB & \ddots & & \\ \vdots & & \ddots & \\ CA^{i-2}B & \cdots & CB & D \end{bmatrix} \in \mathbb{R}^{li \times mi}$$

Combined I/O Equations

$$\left.\begin{aligned} Y_p &= \Gamma_i X_p^d + H_i^d U_p + Y_p^s \\ Y_f &= \Gamma_i X_f^d + H_i^d U_f + Y_f^s \\ X_f^d &= A^i X_p^d + \Delta_i^d U_p \end{aligned}\right\}$$

Thm: Given: $\hat{x}_0$, P_0 and $\{u_0, \cdots, u_{k-1}\}$, $\{y_0, \cdots y_{k-1}\}$ the non-steady state K.F. estimate $\hat{x}_k$

$$\hat{x}_k = A\hat{x}_{k-1} + Bu_k + K_{k-1}(y_{k-1} - C\hat{x}_{k-1} - Du_{k-1})$$

$$K_{k-1} = (G - AP_{k-1}C^T)(\Lambda_0 - CP_{k-1}C^T)^{-1}$$

$$P_k = AP_{k-1}A^T + (G - AP_{k-1}C^T)(\Lambda_0 - CP_{k-1}C^T)^{-1}(G - AP_{k-1}C^T)^T$$

can be explicitly written as:

$$\hat{x}_k = \left[A^k - \Omega_k \Gamma_k \;\middle|\; \Delta_k^d - \Omega_k H_k^d \;\middle|\; \Omega_k \right] \begin{bmatrix} \hat{x}_0 \\ \hline u_0 \\ \vdots \\ u_{k-1} \\ \hline y_0 \\ \vdots \\ y_{k-1} \end{bmatrix}$$

where,

$$\Omega_k := \left[\Delta_k^c - A^k P_0 \Gamma_k^T\right]\left[L_k - \Gamma_k P_0 \Gamma_k^T\right]^{-1}$$

$$P_k = A^k P_0 (A^T)^k + (\Delta_k^c - A^k P_0 \Gamma_k^T)(L_k - \Gamma_k P_0 \Gamma_k^T)^{-1}(\Delta_k^c - A^k P_0 \Gamma_k^T)^T$$

Proof: skipped (probably)

\# $\hat{X}_{i^o} = [\hat{x}_{i^o} \; \cdots \; \hat{x}_{i^o+j-1}]$

$$= \left[A^{i^o} - \Omega_{i^o}\Gamma_{i^o} \,\middle|\, \Delta_{i^o}^{d} - \Omega_{i^o}H_{i^o}^{d} \,\middle|\, \Omega_{i^o} \right] \begin{bmatrix} \hat{X}_0 \\ \hline U_p \\ \hline Y_p \end{bmatrix}$$

$$= \left[A^{i^o} - \Omega_{i^o}\Gamma_{i^o} \,\middle|\, \left(\Delta_{i^o}^{d} - \Omega_{i^o}H_{i^o}^{d} \;\; \Omega_{i^o}\right) \right] \begin{bmatrix} \hat{X}_0 \\ W_p \end{bmatrix}$$

initial

$$P_0, \hat{X}_0 = [\hat{x}_0^{i^o} \; \cdots \; \hat{x}_q^{i^o} \; \cdots \; \hat{x}_{j-1}^{i^o}]$$

$$W_p = \begin{bmatrix} U_p \\ Y_p \end{bmatrix} \begin{bmatrix} u_0 & u_q & u_{j-1} \\ \vdots & \vdots & \vdots \\ u_{i-1} & u_{i+q-1} & \vdots \\ y_0 & y_q & \vdots \\ \vdots & \vdots & \vdots \\ y_{i-1} & y_{i+q-1} & y_{i+j-2} \end{bmatrix}$$

K.F

$$\hat{X}_{i^o} \; [\hat{x}_{i^o} \; \cdots \; \hat{x}_{i+q} \; \cdots \; \hat{x}_{i+j-1}]$$

\#Since $\hat{X}_{i^o}$ is dependent on $\hat{X}_0$ & P_0

$\longrightarrow \hat{X}_{i^o}[\hat{X}_0, P_0]$

Thm: Orthogonal Projection:

Assume
1) $\langle u_k, \omega_k \rangle = 0$, $\langle u_k, v_k \rangle = 0$
2) u_k is persistently exciting of order $2i$
3) $i \to \infty$

4) $\omega_k, v_k \not\equiv 0$

Then:

$$\beta_i^o := Y_f / \begin{bmatrix} W_p \\ U_f \end{bmatrix} = \Gamma_i^o \hat{X}_i^o + H_i^d U_f$$

where $\hat{X}_i^o := X_i^o[\hat{X}_0, P_0]$

$$\hat{X}_0 = X_p^d / \begin{bmatrix} U_p \\ U_f \end{bmatrix}$$

$$P_0 = -\left\langle X_p^d / \begin{bmatrix} U_p \\ U_f \end{bmatrix}^{\perp}, X_p^d / \begin{bmatrix} U_p \\ U_f \end{bmatrix}^{\perp} \right\rangle$$

<u>Proof sketch:</u> $\beta_i^o = Y_f / \begin{bmatrix} U_p \\ U_f \\ Y_p \end{bmatrix}$

$$= \left\{ Y_f \cdot [U_p^T \; U_f^T \; Y_p^T] \right\} \left\{ \begin{bmatrix} U_p \\ U_f \\ Y_p \end{bmatrix} \cdot [U_p^T \; U_f^T \; Y_p^T] \right\}^{-1} \begin{bmatrix} U_p \\ U_f \\ Y_p \end{bmatrix}$$

$$\boxed{\beta_i = \Gamma_i^o \hat{X}_i^o + H_i^d U_f}$$

Extensive computation & using expressions for $\hat{X}_0$, P_0, $\hat{X}_i^o$

<u>Theorem:</u> Assume: 1) $\langle u_k, \omega_k \rangle = 0$, $\langle u_k, v_k \rangle = 0$

2) u_k is persistently exciting of order $2i^o$

3) $j \to \infty$

4) $\omega_k \not\equiv 0$, $v_k \not\equiv 0$

Then 1) $\xi_i^o := Y_f / _{U_f} W_p = \Gamma_i^o \tilde{X}_i^o$

where

$\tilde{X}_i^o$ is a K.F seq. with initial conditions $\hat{X}_0 = X_p^d /_{U_f} U_p$

and $P_0 = -\left\langle X_p^d / \begin{bmatrix} U_p \\ U_f \end{bmatrix}^\perp , X_p^d / \begin{bmatrix} U_p \\ U_f \end{bmatrix}^\perp \right\rangle$

2) $\xi_i^o = [U_1 \;\; U_2] \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 S_1 V_1^T$

$n = \text{size}(S_1)$

3) $\Gamma_i^o = U_1 S_1^{1/2} T$, $\tilde{X}_i^o = T^{-1} S_1^{1/2} V_1^T$

The obvious path does not work

Clearly, $\tilde{X}_i^o = \Gamma_i^{o+} \xi_i^o$

Also as earlier, $\xi_{i+1}^o = Y_f^- /_{U_f^-} W_p^+ = \Gamma_{i-1}^o \tilde{X}_{i-1}$

Recall $\Gamma_i^o = \begin{bmatrix} \Gamma_{i-1}^o \\ \hline \ \end{bmatrix} \begin{matrix} \} \Gamma_{i-1}^o \\ \} \ell \end{matrix} \Rightarrow \tilde{X}_{i-1}^o = \Gamma_{i-1}^{o+} \xi_{i-1}^o$

However, $\begin{bmatrix} \tilde{X}_{i+1}^o \\ Y_{i|i}^o \end{bmatrix} \neq \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \tilde{X}_i^o \\ U_{i|i}^o \end{bmatrix} + \begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix}$

Reason : $\tilde{X}_i^o \leftarrow \hat{X}_0 = X_p^d /_{U_f} U_p$
$\tilde{X}_{i+1}^o \leftarrow \hat{X}_0 = X_p^d /_{U_f^-} U_p^+$ clearly not equal

Proof: From last thm: $\beta_i^o = \Gamma_i^o \hat{X}_i^o + \underline{H_i^d U_f} \quad (1)$

Recall $\left\{ \begin{aligned} &\hat{X}_0 = X_p^d \Big/ \begin{bmatrix} U_p \\ U_f \end{bmatrix} = X_p^d/_{U_p}\, U_f + X_p^d/_{U_f}\, U_p \\ &\hat{X}_i = \left[A_i^o - \Omega_i \Gamma_i^o \;\middle|\; \left(\Delta_i^d - \Omega_i H_i^d \quad \Omega_i \right) \right] \begin{bmatrix} \hat{X}_0 \\ \hline W_p \end{bmatrix} \end{aligned} \right.$ &

Then, $\beta_i^o = \Gamma_i^o \left[\Delta_i^d - \Omega_i H_i^d \quad \Omega_i \right] W_p + H_i^d U_f$

$\qquad + \Gamma_i \left(A_i^o - \Omega_i F_i \right) \left[X_p^d/_{U_p}\, U_f + X_p^d/_{U_f}\, U_p \right]$

But $\beta_i^o = Y_f \Big/ \begin{bmatrix} U_f \\ W_p \end{bmatrix} = Y_f/_{W_p}\, U_f + \underbrace{Y_f/_{U_f}\, W_p}$

$\qquad = Y_f/_{W_p}\, U_f + \xi_i^o$

Hence:

$$\xi_i^o = \Gamma_i^o \underbrace{\left[\left(A_i^o - \Omega_i F_i^o \right) \middle| \left(\Delta_i^d - \Omega_i H_i^d \quad \Omega_i \right) \right] \begin{bmatrix} X_p^d/_{U_f}\, U_p \\ W_p \end{bmatrix}}_{\tilde{X}_i^o}$$

$$= \Gamma_i^o \tilde{X}_i^o$$

Algo 2: Assume equality. [True under some conditions such as 1) $i^o \to \infty$. Then $\hat{X}_{i^o} - \tilde{X}_{i^o} \to 0$ as $i^o \to \infty$.
2) u_k is white noise $\Rightarrow X_p^d \perp \begin{bmatrix} U_p \\ U_f \end{bmatrix}$
$\Rightarrow \hat{X}_0 = 0$ for all $\tilde{X}_{i^o}$

And solve for A, B, C, D.

Algo 1: From previous thms. we have:

$n,\ \Gamma_{i^o},\ \Gamma_{i^o-1},\ \beta_{i^o},\ \beta_{i^o+1}$

Recall,
$$\beta_{i^o} = \Gamma_{i^o} \hat{X}_{i^o} + H_{i^o}^d U_f$$
$$\beta_{i^o+1} = \Gamma_{i^o-1} \hat{X}_{i^o+1} + H_{i^o-1}^d U_f^-.$$

Hence
$$\hat{X}_{i^o} = \Gamma_{i^o}^+ [\beta_{i^o} - H_{i^o}^d U_f] \quad \text{--- (1)}$$
$$\hat{X}_{i^o+1} = \Gamma_{i^o-1}^+ [\beta_{i^o+1} - H_{i^o-1}^d U_f] \quad \text{--- (2)}$$

Here,
$$\begin{bmatrix} \hat{X}_{i^o+1} \\ Y_{i^o|i^o} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{X}_{i^o} \\ U_{i^o|i^o} \end{bmatrix} + \begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix} \quad \text{--- (3)}$$

since $\hat{X}_0 = X_p^d / \begin{bmatrix} U_p \\ U_f \end{bmatrix} = X_p^d / \begin{bmatrix} U_p^+ \\ U_f^- \end{bmatrix}$

Substituting (1) & (2) in (3)

$$\begin{bmatrix} \Gamma_{i-1}^{o+} \beta_{i+1}^{o} \\ Y_{i/i}^{o} \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} \Gamma_i^{o+} \beta_i^{o} + K U_f + \begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix} \quad (*)$$

$$K = \begin{bmatrix} \left[B \mid \Gamma_{i-1}^{o+} H_{i-1}^{d} \right] - A \Gamma_i^{o+} H_i^{d} \\ \left[D \mid 0 \right] - C \Gamma_i^{o+} H_i^{d} \end{bmatrix} \quad (**)$$

1) solve (*) for A, C, K

2) solve (**) for B, D [(**) is linear in B, D]

Computing Techniques

1) Orthogonal projection: A/B

$A \in \mathbb{R}^{p \times j}$
$B \in \mathbb{R}^{q \times j}$
$R \in \mathbb{R}^{r \times j}$
$j \gg \max(p, q, r)$

$$\begin{bmatrix} B \\ A \end{bmatrix} = LQ^T = \begin{bmatrix} \overset{q\times q}{L_{11}} & 0 \\ \underbrace{L_{21}}_{p\times q} & \underbrace{L_{22}}_{p\times p} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ 0 & I_p \end{bmatrix} ; \quad Q \in \mathbb{R}^{j \times (p+q)}$$

clearly: $B = L_{11} Q_1^T$
$A = L_{21} Q_1^T + L_{22} Q_2^T$

Then $A/B = L_{21} Q_1^T$; $A/B^\perp = L_{22} Q_2^T$

2) Oblique projection: $A /_B C$

$$\begin{bmatrix} B \\ C \\ A \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \end{bmatrix}$$

$B = L_{11} Q_1^T$
$C = L_{21} Q_1^T + L_{22} Q_2^T$ — (A)
$A = L_{31} Q_1^T + L_{32} Q_2^T + L_{33} Q_3^T$

$$A / \begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} L_{31} & L_{32} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} \quad \text{— (1)}$$

Let $A / \begin{bmatrix} B \\ C \end{bmatrix} = L_B B + L_C C$

$$= \begin{bmatrix} L_B & L_C \end{bmatrix} \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} \quad \text{— (2)}$$

← from (A)

From ① & ②:

$$\begin{bmatrix} L_B & L_C \end{bmatrix} \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} L_{31} & L_{32} \end{bmatrix}$$

$$\Rightarrow L_C L_{22} = L_{32} \Rightarrow L_C = L_{32} L_{22}^{-1}$$

bothersome bit

Then $A/_B C = L_C C = L_{32} L_{22}^{-1} \begin{bmatrix} L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}$

Stochastic Sub. Id. Algo

$$\begin{bmatrix} Y_{0/i-1} \\ Y_{i/i} \\ Y_{i+1/2i-1} \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} li & l & l(i-1) \end{array} \\ \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \end{array} \begin{array}{c} j \\ \begin{bmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \end{bmatrix} \end{array}$$

$$\mathcal{O}_i = Y_f / Y_p = \begin{bmatrix} L_{21} \\ L_{31} \end{bmatrix} Q_1^T$$

$$\mathcal{O}_{i-1} = Y_f^- / Y_p^+ = \begin{bmatrix} L_{31} & L_{32} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}$$

Calculate S.V.D of $\begin{bmatrix} L_{21} \\ L_{31} \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$

$$= U_1 S_1 V_1^T$$

$U_1 \in li \times n$ $V_1 \in li \times n$	$S_1 \in n \times n$

Recall: $\xi_i = \Gamma_i^o \hat{X}_i^o = U_1 S_1 V_1^T Q_1^T$

$\xi_{i-1}^o = \Gamma_{i-1}^o \hat{X}_{i+1}^o$

#Choose $\Gamma_i^o = U_1 S_1^{1/2}$; $\hat{X}_i^o = S_1^{1/2} V_1^T Q_1^T$

#
$$\hat{X}_{i+1} = \Gamma_{i-1}^{+} \xi_{i-1} = \underline{\Gamma}_i^{o+} \xi_{i-1} = (\underline{U}_1 S_1^{1/2})^{+} [L_{31} \;\; L_{32}] \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}$$

Also
$$Y_{i|i} = [L_{21} \;\; L_{22}] \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}$$

Then
$$\begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix} = \begin{bmatrix} \hat{X}_{i+1} \\ Y_{i|i} \end{bmatrix} \hat{X}_i^{+}$$

$$= \begin{bmatrix} (\underline{U}_1 S_1^{1/2})^{+} [L_{31} \;\; L_{32}] \\ [L_{21} \;\; L_{22}] \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} Q_1 V_1 S_1^{-1/2}$$

$$= \begin{bmatrix} (\underline{U}_1 S_1^{1/2})^{+} L_{31} \\ L_{21} \end{bmatrix} V_1 S_1^{-1/2}$$

$\begin{bmatrix} \hat{Q}_i & \hat{S}_i \\ \hat{S}_i^T & \hat{R}_i \end{bmatrix} \rightarrow$ can also be derived similarly in terms of L-factors & U_1, V_1.

Combined Stoch + Det. I.D.

$$\begin{bmatrix} U_{0/i-1} \\ U_{i/i} \\ U_{i+1/2i-1} \\ Y_{0/i-1} \\ Y_{i/i} \\ Y_{i+1/2i-1} \end{bmatrix} = \begin{bmatrix} L_{11} & & & & & 0 \\ L_{21} & L_{22} & & & & \\ L_{31} & L_{32} & L_{33} & & & \\ L_{41} & L_{42} & L_{43} & L_{44} & & \\ L_{51} & - & - & - & L_{55} & \\ L_{61} & - & - & - & - & L_{66} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \\ Q_4^T \\ Q_5^T \\ Q_6^T \end{bmatrix}$$

$$Y_f / \begin{bmatrix} U_f \\ W_p \end{bmatrix} = L_{U_p} U_p + L_{Y_p} Y_p + L_{U_f} U_f \longrightarrow (*)$$

Also from LQ: $Y_f / \begin{bmatrix} U_f \\ W_p \end{bmatrix} = \begin{bmatrix} L_{51} & L_{52} & L_{53} & L_{54} \\ L_{61} & L_{62} & L_{63} & L_{64} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \\ Q_4^T \end{bmatrix}$

Using $U_p = L_{11} Q_1^T$, $U_f = [\quad] \begin{bmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \end{bmatrix}$
& $Y_p = \ldots\ldots$, in $(*)$,

$$\begin{bmatrix} L_{U_p} \,|\, L_{Y_p} \,|\, L_{U_f} \end{bmatrix} \left[\begin{array}{c|cc|c} L_{11} & 0 & 0 & 0 \\ \hline L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ \hline L_{41} & L_{42} & L_{43} & L_{44} \end{array}\right] = \left[\begin{array}{c|cc|c} L_{51} & L_{52} & L_{53} & L_{54} \\ L_{61} & L_{62} & L_{63} & L_{64} \end{array}\right] \longrightarrow (**)$$

From $(**)$, $L_{U_p}, L_{Y_p}, L_{U_f}$ can be calculated.
Using these values of L_{U_p}, L_{Y_p}:

$$\xi_1^o = Y_f / U_f \, W_p = L_{u_p} \underbrace{L_{11} Q_1^T}_{U_p} + L_{y_p} \underbrace{\begin{bmatrix} L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \\ Q_4^T \end{bmatrix}}_{Y_p}$$

Similarly for calculating ξ_{i-1}:

$$\begin{bmatrix} L_{u_p}^+ & L_{u_f}^- & L_{y_p}^+ \end{bmatrix} \left[\begin{array}{cc|c|cc} L_{11} & 0 & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 & 0 \\ \hline L_{31} & L_{32} & L_{33} & 0 & 0 \\ \hline L_{41} & L_{42} & L_{43} & L_{44} & 0 \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{array}\right]$$

$$= \left[\begin{array}{cc|c|cc} L_{61} & L_{62} & L_{63} & L_{64} & L_{65} \end{array}\right]$$

$\rightarrow \begin{bmatrix} L_{u_p}^+ & L_{u_f}^- & L_{y_p}^+ \end{bmatrix}$ can be calculated.

$$\xi_{i-1}^o = L_{u_p}^+ \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}$$

$$+ L_{y_p}^+ \begin{bmatrix} L_{41} & L_{42} & L_{43} & L_{44} & 0 \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix} \begin{bmatrix} Q_1^T \\ \vdots \\ Q_5^T \end{bmatrix}$$

S.V.D. of ξ_1^o:

Let $L_{u_p} \begin{bmatrix} L_{11} & 0 & 0 & 0 \end{bmatrix} + L_{y_p} \begin{bmatrix} L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix}$

$$= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 S_1 V_1^T$$

$$\# \ \Gamma_o = U_1 S_1^{1/2}, \quad \tilde{X}_i = S_1^{1/2} V_1^T \begin{bmatrix} Q_1^T \\ \vdots \\ Q_4^T \end{bmatrix}$$