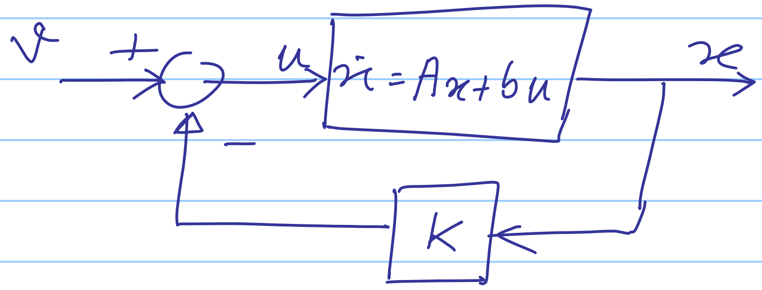
EE640 - 10 State Feedback

Note Title 24-07-2008

Given a realization $\dot{x} = Ax + bu$
$y = cx$



k is a row vector $= [k_1 \; k_2 \; \cdots \; k_n]$

$$kx = [k_1 \; \cdots \; k_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = k_1 x_1 + k_2 x_2 + \cdots + k_n x_n$$

$$u = v - kx$$

Hence, $\dot{x} = Ax + b(v - kx)$
$= Ax - bkx + bv$

$$\boxed{\begin{aligned} \dot{x} &= (A - bk)x + bv \\ y &= cx \end{aligned}}$$

$$bk = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} [k_1 \; \cdots \; k_n] = \begin{bmatrix} k_1 b_1 & k_2 b_1 & \cdots & k_n b_1 \\ k_1 b_2 & k_2 b_2 & & k_n b_2 \\ \vdots & & & \\ k_1 b_n & k_2 b_n & \cdots & k_n b_n \end{bmatrix}$$

$$= [k_1 b \quad k_2 b \quad \cdots \quad k_n b]_{n \times n}$$

Write $A = [A_1 \; A_2 \; \cdots \; A_n]$

Then $A - bk = [A_1 \; A_2 \cdots A_n] - [k_1 b \; k_2 b \cdots k_n b]$

$$= [A_1 - k_1 b \quad A_2 - k_2 b \; \cdots\cdots \; A_n - k_n b]$$

The effect of state feedback on Controllability

FACT: State Feedback does not affect controllability

Proof: Use the PBH rank criteria

Consider:

$$[sI - A \qquad b]$$

Perform the following elementary col. operations:

Add $k_1 \times$ column $(n+1)$ to col 1.
" $k_2 \times$ col $(n+1)$ " " 2
⋮
Add $k_n \times$ col $(n+1)$ to col n

This produces,

$$[\{sI - (A_1 - k_1 b \quad A_2 - k_2 b \cdots A_n - k_n b)\} \quad b]$$

$$= [\{sI - (A - bk)\} \quad b]$$

Being elementary column operations

there is no change in rank.

Hence, $\text{rank}[sI-(A-bk) \quad b]$
$= \text{rank}[sI-A \quad b]$

So controllability is unaffected.

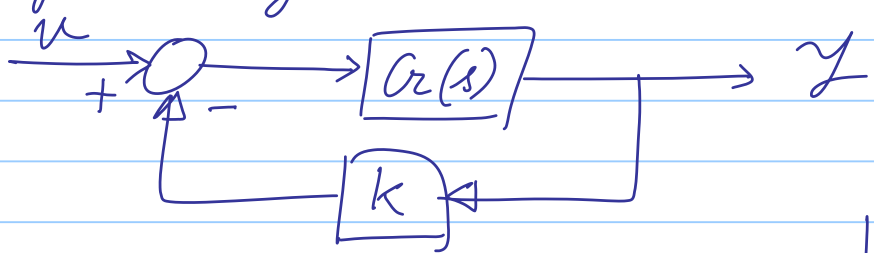
NOTE: State feedback CAN affect observability properties.

Q. How? $\longrightarrow$ Exercise.

Why State Feedback? Or why not static output feedback?

Consider the example: $G(s) = \dfrac{1}{s^2-s}$

Let us try to stabilize using static output feedback.



$$T(s) = \frac{G}{1+kG} = \frac{1}{s^2-s+k}$$

Poles: $s = \dfrac{1 \pm \sqrt{1-4k}}{2}$

So no k will stabilize this.

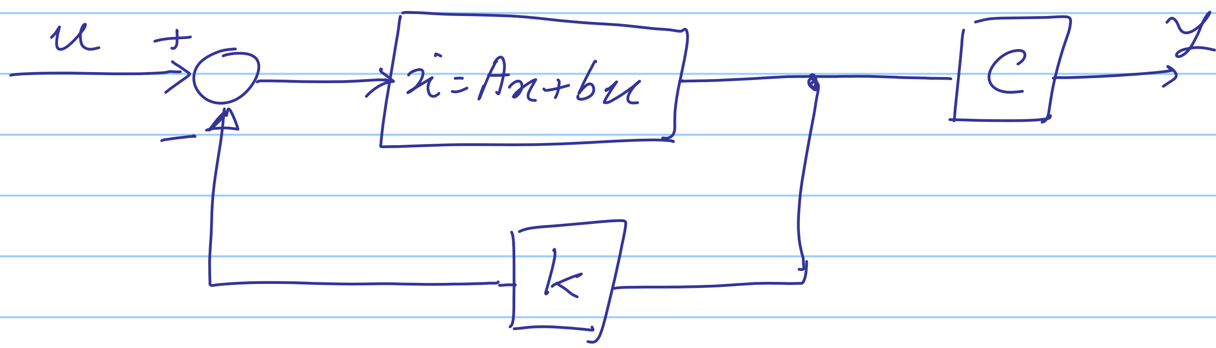
For $k \le \frac{1}{4}$, one root $\geq \frac{1}{2}$
for other k, we have $\text{Re}\, s = \frac{1}{2}$

Stabilizing Linear Systems by state feedback

Given the realization:

$$\dot{x} = Ax + bu$$
$$y = cx$$

Assume we have access to the states



The new system : $\begin{cases} \dot{x} = (A - bk)x + bu \\ y = cx \end{cases}$

The process of pole assignment

We assign desirable eigenvalues to the matrix $(A - bk)$

Step 1 : Select n eigenvalues $\lambda_1, \dots, \lambda_n$
If they are complex, their complex conjugate is included.

Step 2 : Compute the new characteristic polynomial

$$\alpha(s) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n)$$
$$= s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$$

$(\alpha_1, \ldots, \alpha_n)$ are real numbers.

Step 3: The characteristic polynomial of the system with feedback is

$$a_k(s) = \det(sI - (A - bk))$$

Find k such that

$$a_k(s) = \alpha(s).$$

A property of the determinant

Suppose we have two matrices

A is $n \times m$

B is $m \times n$

Then, $\det(I_n + AB) = \det(I_m + BA)$

Proof:

$$\left[\begin{array}{c|c} I_n & -A \\ \hline B & I_m \end{array}\right]_{(n+m)\times(n+m)}$$

Perform the following block column operations:

1) Multiply the first n-columns by A and add to the last m-columns

Result: $\begin{bmatrix} I_n & 0 \\ B & I_m + BA \end{bmatrix}$

Clearly, this column block operation can be achieved by multiplying on the right by $\begin{bmatrix} I_n & A \\ 0 & I_m \end{bmatrix}$

Check: $\begin{bmatrix} I_n & -A \\ B & I_m \end{bmatrix}\begin{bmatrix} I_n & A \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ B & I_m+BA \end{bmatrix}$

Elementary column operations does not affect the determinant

$$\det\begin{bmatrix} I_n & 0 \\ B & I_m+BA \end{bmatrix} = \det\begin{bmatrix} I_n & -A \\ B & I_m \end{bmatrix} \cdot \underbrace{\det\begin{bmatrix} I_n & A \\ 0 & I_m \end{bmatrix}}_{1}$$

$$= \det\begin{bmatrix} I_n & -A \\ B & I_m \end{bmatrix}$$

So $\det\begin{bmatrix} I_n & -A \\ B & I_m \end{bmatrix} = \det\begin{bmatrix} I_n & 0 \\ B & I_m+BA \end{bmatrix}$

$$= \det(I_m+BA)$$

Lets go back to our "trick" matrix, multiply the bottom m rows by A & add to the top n rows

Result : $\left[\begin{array}{c|c} I_n+AB & 0 \\ \hline B & I_m \end{array}\right]$

This operation can be achieved by left multiplication by $\begin{bmatrix} I_n & A \\ 0 & I_m \end{bmatrix}$

$$\therefore \det\left[\begin{array}{c|c} I_n+AB & 0 \\ \hline B & I_m \end{array}\right] = \det\left[\begin{array}{c|c} I_n & -A \\ \hline B & I_m \end{array}\right] \cdot 1$$

So, $\det \left[\begin{array}{c|c} I_n & -A \\ \hline B & I_m \end{array}\right] = \det(I_n + AB) \cdot \det(I_m)$

$$= \det(I_n + AB)$$

Hence proved.

Using this property we can get a useful simplification of the characteristic polynomial for systems with feedback

$$a_k(s) = \det[sI - (A - bk)]$$

$$= \det[(sI - A) + bk]$$

$$= \det[(sI - A)\{I + (sI-A)^{-1} bk\}]$$

$$= \underbrace{\det(sI - A)}\ \det[I_n + (sI - A)^{-1} bk]$$

$a(s) \equiv$ characteristic poly of the original realization.

Now, $\det[I_n + \underbrace{(sI-A)^{-1} b}_{\substack{\text{"A"} \\ n\times 1}} \underbrace{k}_{\substack{\text{"B"} \\ 1\times n}}]$

$$= \cancel{\det}[1 + k \cdot (sI - A)^{-1} b]$$

So, $a_k(s) = a(s)[1 + k(sI-A)^{-1} b]$

$$= a(s) + a(s)\, k\, (sI - A)^{-1} b$$

Recall, $a(s)(sI - A)^{-1} = \operatorname{adj}(sI - A)$

So, $a_k(s) - a(s) = k\,\mathrm{adj}(sI - A)\,b$

We will now use the resolvent formula for the adj:

$$\mathrm{adj}(sI-A) = s^{n-1}I + s^{n-2}(A + a_1 I) + \text{----}$$

Substituting:

$$a_k(s) - a(s) = k\left[s^{n-1}I + s^{n-2}(A + a_1 I) + s^{n-3}(A^2 + a_1 A + a_2 I) + \cdots\right] b$$

Recall that we wanted

$$a_k(s) = \alpha(s) = s^n + \alpha_1 s^{n-1} + \text{---} + \alpha_n$$
$$a(s) = s^n + a_1 s^{n-1} + \text{---} + a_n$$

$$a_k(s) - a(s) = \alpha(s) - a(s)$$
$$= s^{n-1}(\alpha_1 - a_1) + s^{n-2}(\alpha_2 - a_2) + \cdots$$
$$\text{----} + (\alpha_n - a_n)$$

Lets compare coefficients of corresponding powers of s.

Columns of controllability matrix

s^{n-1} : $\alpha_1 - a_1 = kb$

s^{n-2} : $\alpha_2 - a_2 = k(Ab) + a_1 kb$

s^{n-3} : $\alpha_3 - a_3 = k(A^2b) + a_1 kAb + a_2 kb$

We know: $\mathcal{C} = [b \;\; Ab \;\; A^2b \;\text{-----}]$

Let us try to write the above formulae in matrix form:

$$\left[b \; Ab \; A^2b \cdots A^{n-1}b\right] \underbrace{\begin{bmatrix} 1 & a_1 & a_2 & \cdots & \cdots \\ 0 & 1 & a_1 & \cdots & \\ 0 & 0 & 1 & \cdots & \\ \vdots & & & \ddots & \\ \vdots & & 0 & & \ddots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{(a_-)^T}$$

$$= \left[\, b \;\middle|\; a_1 b + Ab \;\middle|\; a_2 b + a_1 Ab + A^2 b \;\middle|\; \cdots \,\right]$$

Define the following quantities:

$$a_- := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & \\ a_2 & a_1 & 1 & 0 \cdots & 0 \\ & & & \ddots & 0 \\ a_{n-1} & a_{n-2} & \cdots & \cdots & 1 \end{bmatrix}$$

$$a = [a_1 \;\; a_2 \; \cdots \; a_n] \;;\; \alpha = [\alpha_1 \;\; \alpha_2 \cdots \alpha_n]$$

So the vector form for the coefficient equations is:

$$\alpha - a = k \, \mathcal{C} \, a_-^T$$

Now note that $\det(a_-^T) = 1$. So a_-^T is always invertible.

So when the realization is controllable we can write:

$$\boxed{K = (\alpha - a)(a_-^T)^{-1} \mathcal{C}^{-1}}$$

Conclusion: Arbitrary pole assignment by state feedback is possible if and only if the realization is controllable.

Example: $\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$

$y = (1 \quad 0) x$

Problem: Use state feedback to assign the eigenvalues: $(-2, -3)$

(1) The controllability matrix $\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so realization is controllable.

$$a(s) = \det(sI - A) = \det \begin{pmatrix} s & -1 \\ -1 & s \end{pmatrix} = s^2 - 1$$

$$= s^2 + 0.s - 1$$

So $a = [0 \quad -1]$

$$\alpha(s) = (s+2)(s+3) = s^2 + 5s + 6$$

$$\alpha = [5, 6]$$

$$a_-^T = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $K = (\alpha - a)(a_-^T)^{-1} \mathcal{C}^{-1}$

$$= [(5 \quad 6) - (0 \quad 1)] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

So $K = [5, 7]$

Check: $\dot{x} = (A - bk) x + bu$

$$A - bk = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 5 & 7 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -5 & -6 \\ 1 & 0 \end{pmatrix}$$

$$a_k(s) = (s+5)s \underline{+6} = s^2 + 5s + 6$$

We found the general formula for k.

Q. For which realizations this formula is particularly simple?

The Controller Canonical Form

$$\dot{x} = A_c x + b_c u$$
$$y = c_c x$$

$$A_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \end{bmatrix} ; \quad b_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$c_c = \left[c_{c_1}, c_{c_2} \cdots c_{c_n} \right] \Big\} \text{— Not special}$$

For a matrix in companion form

$$\det(sI - A_c) = s^n + a_1 s^{n-1} + \cdots + a_n$$

[Check this : Exercise]

Apply state feedback k_c in controller form. The new matrix $(A_c - b_c k_c)$

$$b_c k_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [k_{c_1} \ k_{c_2} \ \cdots \ k_{c_n}]$$

$$= \begin{bmatrix} k_{c_1} & k_{c_2} & \cdots & k_{c_n} \\ & & 0 & \\ & & & \end{bmatrix}$$

$$A_c - b_c k_c = \begin{bmatrix} -(a_1 + k_{c_1}) & -(a_2 + k_{c_2}) & \cdots & -(a_n + k_{c_n}) \\ 1 & \ddots & & \\ & 1 & \ddots & 0 \\ 0 & & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

So the matrix is in the same form with the new characteristic polynomial

$$a_{k_c}(s) = s^n + (a_1 + k_{c_1}) s^{n-1} + (a_2 + k_{c_2}) s^{n-2} + \cdots + (a_n + k_{c_n})$$

$$\equiv \alpha(s)$$

Then $\alpha = [(a_1 + k_{c_1}) \quad (a_2 + k_{c_2}) \ \cdots \ (a_n + k_{c_n})]$

Recall $a = [a_1 \ a_2 \ \cdots \ a_n]$

Then, $\alpha - a = [k_{c_1} \ k_{c_2} \ \cdots \ k_{c_n}] = k_c$

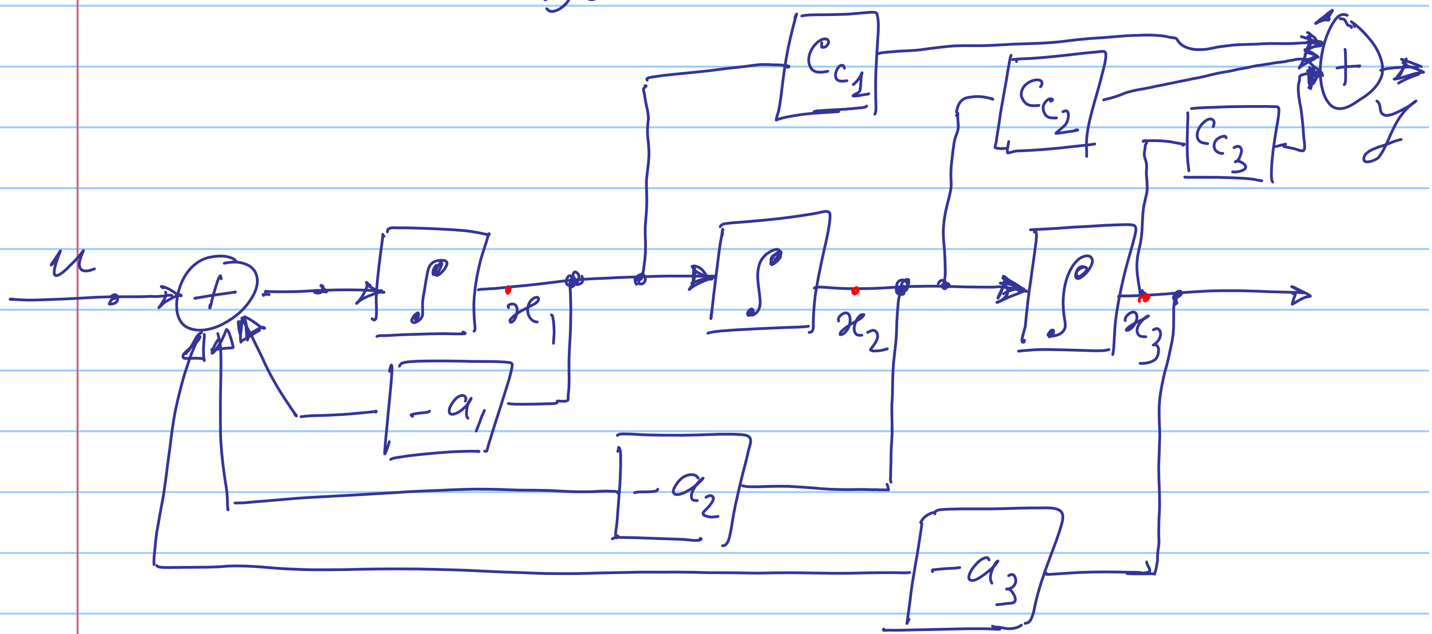
$\therefore \boxed{k_c = \alpha - a}$

The controller form also has other advantages:

<u>Connections to the transfer function:</u>

Recall: $$G(s) = \frac{c_{c_1} s^2 + c_{c_2} s + c_{c_3}}{s^3 + a_1 s^2 + a_2 s + a_3}$$

can be realized as:



$$\dot{x}_1 = -a_1 x_1 - a_2 x_2 - a_3 x_3 + u$$
$$\dot{x}_2 = x_1$$
$$\dot{x}_3 = x_2$$
$$y = b_1 x_1 + b_2 x_2 + b_3 x_3$$

$$\dot{x} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

This is true in general:
Given any controller form realization

$$\dot{x} = A_c x + b_c u$$
$$y = c_c x$$

the corresponding transfer function is:

$$G(s) = \frac{c_{c_1} s^{n-1} + c_{c_2} s^{n-2} + \cdots + c_{c_n}}{a(s)}$$

Consequently we can write down a controller form realization from the transfer function by observation.

Q. Can we go from any realization to the controller form by a similarity transformation?

First lets see how the similarity transform affects the gain matrix k.

Effect of Similarity Transform on State Feedback

Given a realization $\dot{x} = Ax + bu$, $y = cx$
with state feedback k.

Apply a similarity transform
$$x = T\bar{x}$$

$$\dot{\bar{x}} = T^{-1}\dot{x} = T^{-1}\left[(A-bk)x+bu\right]$$
$$= T^{-1}(A-bk)T\bar{x} + T^{-1}bu$$
$$= (T^{-1}AT - \underbrace{T^{-1}b}\underbrace{kT})\bar{x} + T^{-1}bu$$
$$= (\bar{A} - \bar{b}\bar{k})\bar{x} + \bar{b}u$$

So $\bar{A} = T^{-1}AT$, $\bar{b} = T^{-1}b$, $\bar{c} = cT$

$$\boxed{\bar{k} = kT}$$

So for any two realization connected by a similarity transformation,

$$\{A, b, c\} \overset{T}{\longleftrightarrow} \{\bar{A}, \bar{b}, \bar{c}\}$$

it is enough to find the feedback gain of one of them; the feedback gain for the other one can be calculated directly using the above formula.

Let us now try to find a similarity transformation that will convert any controllable realization into the controller form i.e

$$\{A, B, c\} \xrightarrow{T} \{A_c, b_c, c_c\} \qquad \begin{aligned} A_c &= T^{-1}AT \\ b_c &= T^{-1}b \\ c_c &= cT \end{aligned}$$

Since both are controllable, we know that the connecting similarity transform is:

$$T = \mathcal{C}\,\mathcal{C}_c^{-1}$$

Now, it turns out that $\mathcal{C}_c^{-1} = \underline{a}^T$.

Verify this: Exercise (+HW6).

FACT: Every controllable realization can be brought into controller form by a similarity transform.

The effect of state feedback on the zeros of the transfer function

Consider a realization $\left.\begin{aligned} \dot{x} &= Ax+bu \\ y &= cx \end{aligned}\right\} G(s)$

Assume it is controllable. Let us bring our realization into controller form

$$\begin{cases} \dot{x} = A_c x + b_c u \\ y = c_c x \end{cases}$$

We know that the transfer function is invariant i.e. the it is the same $G(s)$.

Apply state feedback k_c in this realization

$$\alpha(s) = \det\left[sI - (A_c - b_c k_c)\right]$$
$$a(s) = \det(sI - A_c)$$

Since both are in controller form:

$$G(s) = \frac{c_{c_1} s^{n-1} + c_{c_2} s^{n-2} + \cdots + c_{c_n}}{a(s)}$$

$$G_{k_c}(s) = \frac{c_{c_1} s^{n-1} + c_{c_2} s^{n-2} + \cdots + c_{c_n}}{\alpha(s)}$$

So, state feedback does not directly affect the zeros of the transfer function.

However, if the roots of $\alpha(s)$ are chosen to be the roots of the eqn

$$c_{c_1} s^{n-1} + c_{c_2} s^{n-2} + \cdots + c_{c_n} = 0$$

then there are cancellations and this is the only way the numerator can change!

Q. Are there other methods for calculating the feedback gain?

A. Yes. Let us learn another.

<u>Ackermann's formula</u>

$$k = q_n \, \alpha(A)$$

where $\alpha(s)$ is the desired polynomial

$q_n = [0 \; \cdots \; 0 \; 1] \, \mathcal{C}^{-1} =$ last row of $\mathcal{C}^{-1}$

Let's assume we are already in controller form.

Then we know $k = \alpha - a$

So we have to show that

$$\alpha - a = q_n \, \alpha(A_c)$$

Now, $q_n \mathcal{C}_c^{-1} = q_n \, \underline{a}^T$

$$= [0 \cdots 0 \, 1] \begin{bmatrix} 1 & a_1 & a_2 & \cdots \\ & \ddots & \ddots & \ddots \\ & 0 & & \ddots \\ & & & 1 \end{bmatrix} = [0 - \; 0 \; 1]$$

Hence, we have to show that

$$\alpha - a = \text{last row of } \alpha(A_c)$$

$$\alpha(A_c) = A_c^n + \alpha_1 A_c^{n-1} + \cdots + \alpha_n I$$

But $A_c^n = -a_1 A_c^{n-1} - a_2 A_c^{n-2} - \cdots - a_n I$

So $\alpha(A_c) = (\alpha_1 - a_1) A_c^{n-1} + (\alpha_2 - a_2) A_c^{n-2} + \cdots + (\alpha_n - a_n) I$

Now, let $e_i = [0 \cdots 0 \, 1 \, 0 \cdots 0]$
$\uparrow$ i^{th} place

$$e_i A_c = e_{i-1} \quad \text{for } i = 2 \text{ to } n$$

$$e_1 A_c = [-a_1 \;\; -a_2 \;\; \cdots \;\; -a_n]$$

So
$$e_n A_c^{n-1} = e_1 = [1 \; 0 \cdots 0]$$
$$e_n A_c^{n-2} = e_2 = [0 \; 1 \cdots 0]$$
$$\vdots$$
$$e_n I = e_n = [0 \cdots 0 \; 1]$$

In other words,

$$q_n \quad \alpha(A_c) = e_n\left[(\alpha_1 - a_1)A_c^{n-1} + (\alpha_2 - a_2)A_c^{n-2} + \cdots + (\alpha_n - a_n)I\right]$$

$$= \begin{bmatrix} \alpha_1 - a_1 & \alpha_2 - a_2 & \cdots & \alpha_n - a_n \end{bmatrix}$$

$$= \alpha - a \,..$$