Note Title 24-07-2008

EE 640 - Observers

Our design of state feedback was based on ALL the states being available for feedback.

Q. What do we do when we do not have access to the state?

Asymptotic Observer

Problem: * We do not have access to the state
* Calculating the state require derivatives of the input and the output.

We will show that for the purpose of state feedback:
* it is not necessary to calculate the exact state. It is enough to calculate an estimate $\hat{x}(t)$ of the state $x(t)$ that satisfies

$$\hat{x}(t) \longrightarrow x(t) \quad \text{as} \quad t \rightarrow \infty$$

$\hat{x}(t)$ is then called an asymptotic estimate of the state.

The system that produces $\hat{x}(t)$ from input/output data is called an Asymptotic Observer.

Building an Asymptotic Observer

Given is the realization

$$\dot{x}(t) = Ax(t) + bu(t)$$
$$y(t) = cx(t)$$
$$x(0^-) = x_0$$

Find $x(t)$, given $u(\theta), y(\theta)$ for $\theta \leq t$

Attempt 1: Use a dummy system

$$\dot{\hat{x}}(t) = A\hat{x}(t) + bu(t)$$
$$\hat{y}(t) = c\hat{x}(t)$$
$$\hat{x}(0^-) = x_0 - \varepsilon$$

There is an error of ε in the initial state. The observer error:

$$\tilde{x}(t) = x(t) - \hat{x}(t)$$

Lets try to find an equation for the error:

$$\dot{\tilde{x}}(t) = \dot{x}(t) - \dot{\hat{x}}(t)$$
$$= (Ax + bu) - (A\hat{x} + bu)$$
$$= A(x - \hat{x}) = A\tilde{x}$$

So $\dot{\tilde{x}}(t) = A\tilde{x}$; $\tilde{x}(0^-) = \varepsilon$

The initial condition: $\tilde{x}(0^-) = x(0^-) - \hat{x}(0^-) = \varepsilon$

If A has eigenvalues in the right side of the complex plane,

$$\tilde{x}(t) \to \infty \quad \text{as } t \to \infty$$

& our Attempt 1 will be unsuccessful in that case.

Attempt 2: To improve our attempt we can try to use the output error $y(t) - \hat{y}(t)$

Let us try to add this output error as a correction term to our dummy system.

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + bu(t) + l\left[y(t) - \hat{y}(t)\right] \\ \hat{y}(t) = c\,\hat{x}(t) \\ \hat{x}(0^-) = x_0 - \varepsilon \end{cases}$$

where $l = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix}$ is a constant vector.

The error is: $\tilde{x}(t) = x(t) - \hat{x}(t)$

To get an eqn.

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = [Ax + bu] - [A\hat{x} + bu + l(y - \hat{y})]$$

$$= [Ax + bu] - [A\hat{x} + bu + l(cx - c\hat{x})]$$

$$= (A - lc)(x - \hat{x}) = (A - lc)\tilde{x}$$

Error dynamics:

$$\dot{\tilde{x}} = (A - lc)\tilde{x}$$
$$\tilde{x}(0) = \varepsilon$$

Q. Can we choose l such that all eigenvalues of $(A-lc)$ are on the left side of the complex plane?

This question is similar to the state feedback eigenvalue assignment problem. To see the similarity consider:

$$\det[sI - (A-lc)] = \det[sI - A^T + c^T l^T]$$

So, $\det(sI - A^T + c^T l^T)$ ← observer
$\det(sI - A + bk)$ ← feedback

State feedback	Observer
A	A^T
b	c^T
k	l^T

We know, that for feedback there exists a k which can move arbitrary eigenvalues to the left half plane, whenever (A, b) is controllable.

Translation: For observer: A l^T

that moves all eigenvalues of $(A^T - c^T l^T)$ to the left side exists whenever (A^T, c^T) is controllable.

FACTS: (i) Eigenvalues of $(A^T - c^T l^T)$ are the same as those of $(A - lc)$ (since the transpose do not affect the characteristic polynomial)

(ii) We have seen that (A^T, c^T) is controllable iff (c, A) is observable

So we get: a vector l that moves all the eigenvalues of $(A - lc)$ to the left side exists if (c, A) is observable.

Calculating l:

Assume (c, A) is observable. Suppose we want $\lambda_1, \dots, \lambda_n$ to be the eigenvalues of the error system

$$\dot{\tilde{x}} = (A - lc)\tilde{x}$$

Calculate the polynomial

$$\alpha_0(s) = (s-\lambda_1)(s-\lambda_2) \cdots (s-\lambda_n)$$

$$\alpha_0 = (\alpha_1, \alpha_2, \dots, \alpha_n)$$

Let $a(s) = \det(sI - A) \div$ original characteristic polynomial

$$a = [a_1 \; a_2 \; \dots \; a_n]$$

Recall the feedback gain eq$^{\underline{n}}$:

$$k = (\alpha - a)(a_-^T)^{-1} \mathcal{C}^{-1}(A, b)$$

Using our translation table:

$$\ell^T = (\alpha_0 - a)(\underline{a}^T)^{-1} \mathcal{C}^{-1}(A^T, c^T)$$

So $\ell = \left[\mathcal{C}^T(A^T, c^T)\right]^{-1} \underline{a}^{-1} (\alpha_0 - a)^T$

<u>Recall</u>: $\mathcal{C}^T(A^T, c^T) = \mathcal{O}(c, A)$

$$\ell = \mathcal{O}^{-1}(c, A)\, \underline{a}^{-1} (\alpha_0 - a)^T$$

$$\boxed{\ell = \mathcal{O}^{-1} \underline{a}^{-1} (\alpha_0 - a)}$$

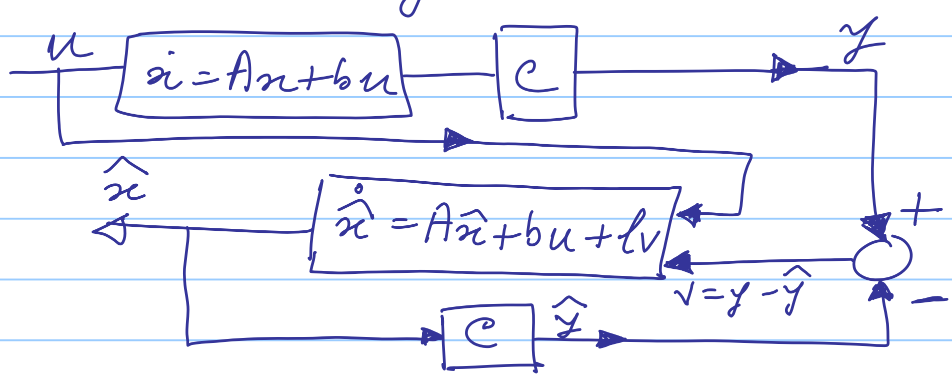
<u>The asymptotic observer</u>

$$\dot{\hat{x}} = A\hat{x} + bu + \ell(y - \hat{y})$$
$$\hat{y} = c\hat{x}$$
$$\hat{x}(0^-) = x_0 - \varepsilon$$

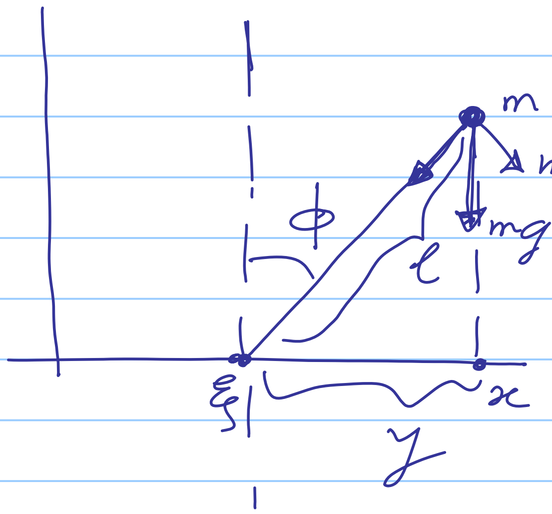
<u>FACT</u>: If the original realization

$\left\{\begin{array}{l} \dot{x} = Ax + bu \\ y = cx \end{array}\right\}$ is observable then there is an ℓ such that $\tilde{x}(t) = x(t) - \hat{x}(t) \to 0$ as $t \to \infty$. for any ε.

<u>Observer Diagram</u>



Example: Balancing a pointer



Acceleration of the base $= u$

The motion of the tip:

$$m\ddot{x} = mg\sin\phi = mg\phi$$

or

$$\ddot{x} = g\phi \qquad \text{——— (1)}$$

$$x = \xi + L\phi$$

$$\ddot{x} = \ddot{\xi} + L\ddot{\phi} \qquad \text{——— (2)}$$

From (1) & (2),

$$g\phi = L\ddot{\phi} + u$$

Let $z_1 = \phi$, $z_2 = \dot{\phi}$

$$\left.\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{g}{L} z_1 - \frac{1}{L} u \end{aligned}\right\} \quad \text{Take } L = 1,\ g = 9$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$$y = [1 \quad 0]\, z$$ {Suppose we measure ϕ only}

Q. Can we asymptotically estimate z_2

from y and u?

Is the realization observable?

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{observable}$$

So there is an asymptotic observer

Choose observer poles at

$$\lambda_1 = -10 + 10j$$
$$\lambda_2 = -10 - 10j$$

New characteristic polynomial:

$$\alpha_0(s) = s^2 + 20s + 200$$
$$\alpha_0 = [20 \quad 200]$$

The old characteristic polynomial:

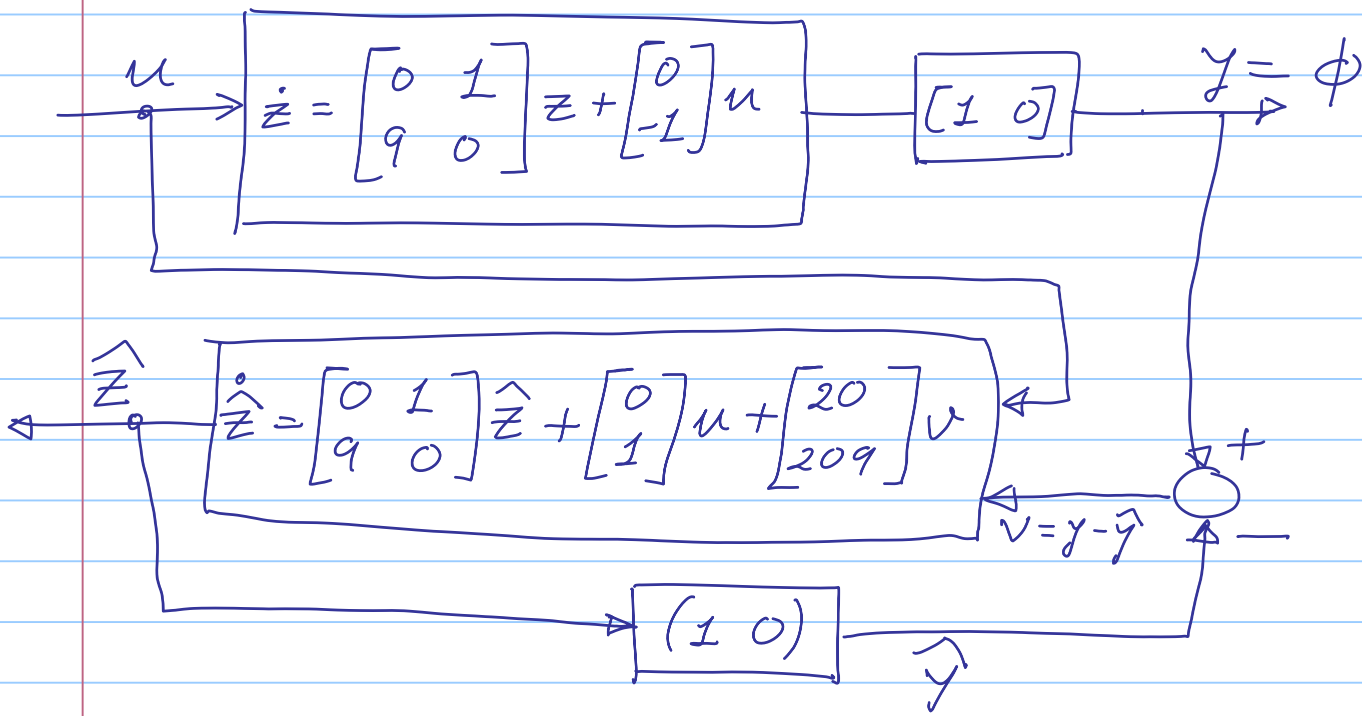
$$a(s) = \det(sI - A) = s^2 - 9$$
$$a = [0 \quad -9]$$

Then
$$l = \mathcal{O}^{-1} \underline{a}^{-1} (\alpha - a)^T$$
$$= \begin{bmatrix} 20 \\ 209 \end{bmatrix}$$

The observer:

$$\dot{\hat{z}} = A\hat{z} + bu + l(y - \hat{y})$$
$$= \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} \hat{z} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u + \begin{bmatrix} 20 \\ 209 \end{bmatrix} (y - \hat{y})$$

$$\hat{y} = [1 \quad 0] \hat{z}$$



Detectable Realizations

A realization $\begin{cases} \dot x = Ax + bu \\ y = cx \end{cases}$ is <u>detectable</u> if there is an asymptotic observer for x.

We already saw: An observable realization is detectable. So consider a realization which is unobservable.

Assume $\operatorname{rank}(\mathcal{O}) = r < n$

Then there is a similarity transformation T such that
$$\bar A = T^{-1} A T, \quad \bar b = T^{-1} b, \quad \bar c = cT$$
such that:
$$\bar A = \begin{array}{c} \left[\begin{array}{c|c} \bar A_o & 0 \\ \hline \bar A_{\bar o o} & \bar A_{\bar o} \end{array}\right] \begin{array}{l} \}r \\ \}n-r \end{array} \end{array} \qquad \bar c = [\bar c_o \;\; 0] \qquad \bar b = \begin{bmatrix} \bar b_o \\ \bar b_{\bar o} \end{bmatrix}$$

where $\{\bar{c}_o, \bar{A}_o\}$ is observable.

Lets try to build an observer for this realization:

$$\dot{\hat{\bar{x}}} = \bar{A}\hat{\bar{x}} + \bar{b}u + \bar{l}(y - \hat{y})$$
$$\hat{y} = c\hat{\bar{x}}$$

Error dynamics: $\tilde{\bar{x}} = \bar{x} - \hat{\bar{x}}$

$$\dot{\tilde{\bar{x}}} = (\bar{A} - \bar{l}\bar{c})\tilde{\bar{x}}$$

The characteristic polynomial of the error equation:

$$a_{\bar{l}}(s) = \det(sI - \bar{A} + \bar{l}\bar{c})$$

Write $\bar{l} = \begin{bmatrix} \bar{l}_o \\ \bar{l}_{\bar{o}} \end{bmatrix}$

Then: $\det(sI - \bar{A} + \bar{l}\bar{c})$

$$= \det\left[sI - \begin{pmatrix} \bar{A}_o & 0 \\ \bar{A}_{\bar{o}o} & \bar{A}_{\bar{o}} \end{pmatrix} + \begin{pmatrix} \bar{l}_o \\ \bar{l}_{\bar{o}} \end{pmatrix}\begin{pmatrix} \bar{c}_o & 0 \end{pmatrix}\right]$$

$$= \det\left[sI - \begin{pmatrix} \bar{A}_o & 0 \\ \bar{A}_{\bar{o}o} & \bar{A}_{\bar{o}} \end{pmatrix} + \begin{pmatrix} \bar{l}_o\bar{c}_o & 0 \\ \bar{l}_{\bar{o}}\bar{c}_o & 0 \end{pmatrix}\right]$$

$$= \det\begin{bmatrix} sI_r - \bar{A}_o + \bar{l}_o\bar{c}_o & 0 \\ -\bar{A}_{\bar{o}o} + \bar{l}_{\bar{o}}\bar{c}_o & sI_{n-r} - \bar{A}_{\bar{o}} \end{bmatrix}$$

$$= \underbrace{\det(sI_r - \bar{A}_o + \bar{l}_o\bar{c}_o)}_{a_1(s)}\underbrace{\det(sI_{n-r} - \bar{A}_{\bar{o}})}_{a_2(s)}$$

$a_2(s)$ is fixed (independent of $\bar{l}$)
Its roots are the unobservable modes of A.

$a_1(s) = \det(sI_r - \bar{A}_0 + \bar{l}_0 \bar{c}_0)$. We know we can change the roots arbitrarily since $\{\bar{c}_0, \bar{A}_0\}$ is observable.

<u>FACT</u>: The realization $\begin{cases} \dot{x} = Ax + bu \\ y = cx \end{cases}$ is detectable iff all unobservable modes of A are asymptotically stable.

<u>Designing the observer</u>

1) Check whether realization is detectable
2) Find T s.t. $\bar{A} = T^{-1}AT$, $\bar{b} = T^{-1}b$, $\bar{c} = cT$ is in the special form (as shown above)
3) Build an asymptotic observer for the r-dimensional observable realization $\{\bar{A}_0, \bar{b}_0, \bar{c}_0\}$.
Find $\bar{l}_0$ from the formula.
4) Augment $\bar{l}_0$ to $\bar{l} = \begin{bmatrix} \bar{l}_0 \\ \bar{l}_{\bar{o}} \end{bmatrix}$

where $\bar{l}_{\bar{o}}$ is arbitrary.
5) Transfer $\bar{l}$ back to the original coordinate system.

$$\boxed{l = T\bar{l}}$$

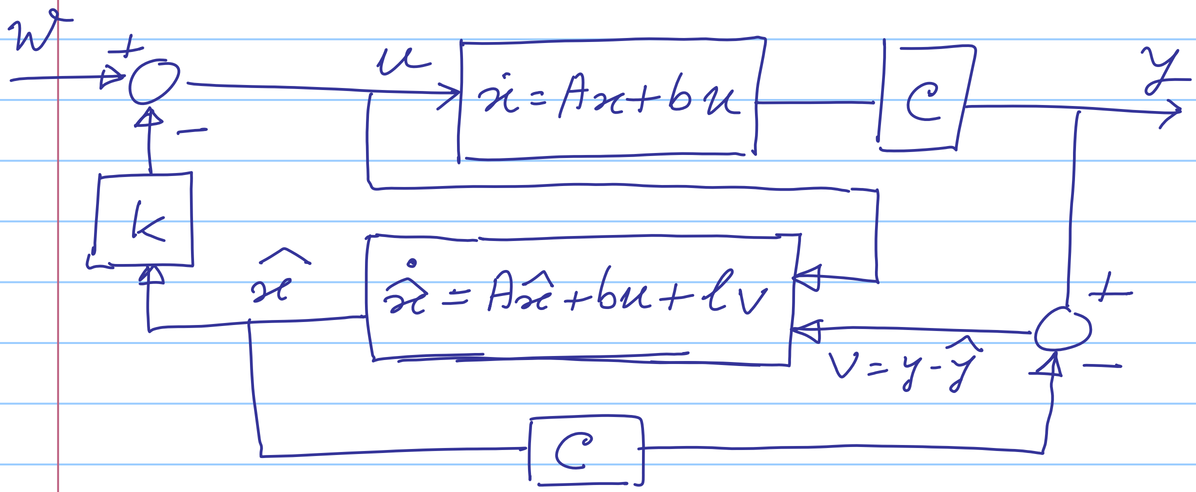
Exercise: Derive the above formula.

In the original coordinate system: the observer equations are:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + bu + l(y - \hat{y}) \\ \hat{y} = c\hat{x} \end{cases}$$

Combining Asymptotic Observer with State Feedback

Consider the realization $\{A, b, c\}$



Observer - Controller Configuration

Q. What are the condition for the existence of l, k such that the configuration is asymptotically stable.

States of the configuration $z = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$

$$\dot{x} = Ax + bu = Ax + b(w - k\hat{x})$$
$$= Ax - bk\hat{x} + bw$$

$y = cx$; $\hat{y} = c\hat{x}$; $u = w - k\hat{x}$

$$\dot{\hat{x}} = A\hat{x} + bu + l(y - \hat{y})$$
$$= A\hat{x} + b(w - k\hat{x}) + l(cx - c\hat{x})$$
$$= lcx + (A - bk - lc)\hat{x} + bw$$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -bk \\ lc & A-bk-lc \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} b \\ b \end{bmatrix} w$$

$$y = \begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

Q) Are there l & k such that all eigenvalues of $\begin{bmatrix} A & -bk \\ lc & A-bk-lc \end{bmatrix}$ are in the stable region?

Consider the characteristic polynomial:

$$\det\left[sI_{2n} - \begin{pmatrix} A & -bk \\ lc & A-lc-bk \end{pmatrix} \right]$$

$$= \det \begin{bmatrix} sI_n - A & bk \\ -lc & sI_n - A + lc + bk \end{bmatrix}$$

Add the first n-columns to the last n-columns:

$$= \det \begin{bmatrix} sI_n - A & sI_n - A + bk \\ -lc & sI_n - A + bk \end{bmatrix}$$

Subtract the last n rows from the

top n-rows.

$$= \det \begin{bmatrix} sI_n - A + lc & 0 \\ -lc & sI_n - A + bk \end{bmatrix}$$

$$= \underbrace{\det(sI_n - A + lc)}_{a_o(s) \equiv \text{char. polynomial of observer}} \underbrace{\det(sI_n - A + bk)}_{a_c(s) \equiv \text{char polynomial of true state feedback}}$$

So the char. polynomial of the observer - controller configuration factors into two parts

$a_o(s) \leftarrow$ depends only on the observer (l)
$a_c(s) \leftarrow$ depends only on the state feedback (k)

"The SEPARATION PRINCIPLE"

<u>Hence</u>: The observer - controller configuration can be stabilized iff
(1) $a_o(s)$ has all roots in the open left half of the complex plane
(2) $a_c(s)$ has all root in the open left half of the complex plane

<u>FACT</u>: The observer - controller configuration can be stabilized iff the given realization is detectable and stabilizable.

* FACT : If a realization cannot be stabilized by the observer-controller configuration, then it cannot be stabilized at all.

The transfer function of the observer-controller configuration

$$G_{oc}(s) = \begin{bmatrix} c & 0 \end{bmatrix} \left[sI - \begin{pmatrix} A & -bk \\ lc & A-lc-bk \end{pmatrix} \right]^{-1} \begin{bmatrix} b \\ b \end{bmatrix}$$

Let us do a similarity transformation
This has no effect on $G_{oc}(s)$

Recall the observer error $\dot{\tilde{x}} = (A-lc)\tilde{x}$
So lets use $\begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$ instead of $\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$

$$\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = T \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} \qquad \tilde{x} = x - \hat{x}$$
$$\text{or } \hat{x} = x - \tilde{x}$$

So $T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$

$$\bar{A} = T^{-1} \begin{bmatrix} A & -bk \\ lc & A-lc-bk \end{bmatrix} T = \begin{bmatrix} A-bk & bk \\ 0 & A-lc \end{bmatrix}$$

$$\bar{b} = T^{-1} \begin{bmatrix} b \\ b \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$\bar{c} = \begin{bmatrix} c & 0 \end{bmatrix} T = \begin{bmatrix} c & 0 \end{bmatrix}$$

So

$$(*)\quad \begin{cases} \begin{bmatrix}\dot{x}\\ \dot{\tilde{x}}\end{bmatrix} = \begin{bmatrix} A-bk & bk \\ 0 & A-lc\end{bmatrix}\begin{bmatrix}x\\ \tilde{x}\end{bmatrix} + \begin{bmatrix} b\\ 0\end{bmatrix} w \\ y = \begin{bmatrix} c & 0\end{bmatrix}\begin{bmatrix}x\\ \tilde{x}\end{bmatrix} \end{cases}$$

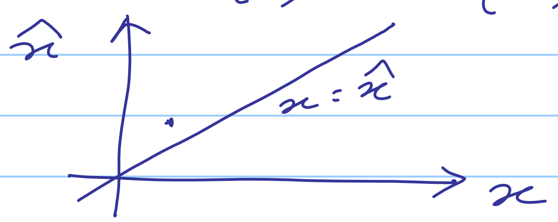
But we know, (e.g. HW1) that $(*)$ & the realization:
$\{A-bk,\ b,\ c\}$ have the same transfer function.

$$\therefore \quad G_{oc}(s) = c\,(sI - A + bk)^{-1}\, b$$

which is the same transfer function we got from true state feedback without observer.

This implies the observer states are not controllable.
— Why? Exercise.

E.g. From zero initial condition
$\hat{x}(t) = x(t) \quad \forall t$



Example: Overall Design

Given a system with the transfer function:

$$G(s) = \frac{s^2 - 5s + 6}{s^3 - 5s^2 + 4s}$$

Stabilize the system.

$G(s)$ is irreducible, so any realization is controllable + observable

Step 1 : Build a realization. Lets use a controller form realization:

$\dot{x}_c = A_c x_c + b_c u$
$y = c_c x_c$

$$A_c = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad b_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$c_c = \begin{bmatrix} 1 & -5 & 6 \end{bmatrix}$$

Step 2 : Choose eigenvalues of the state feedback system

$\lambda_1 = -5$; $\lambda_2 = -1-j$, $\lambda_3 = -1+j$

Then $\alpha(s) = (s+5)(s+1-j)(s+1+j)$
$= s^2 + 7s^2 + 12s + 10$

$\alpha = [7 \quad 12 \quad 10]$

Step 3 Design k_c :

Original char. poly:
$a(s) = s^2 - 5s + 4$
$a = [-5 \quad 4 \quad 0]$

$k_c = \alpha - a = [7 \; 12 \; 10] - [-5 \; 4 \; 0]$

$$= [12 \quad 8 \quad 10]$$

Step 4: Building the observer:

$$\dot{\hat{x}}_c = A_c \hat{x}_c + b_c u + l_c (y - \hat{y})$$
$$\hat{y} = c_c x_c$$

Select the observer error poles::
choose $\lambda_1^o = -6$, $\lambda_2^o = -6-6j$, $\lambda_3^o = -6+6j$

Then $\alpha_o(s) = (s+6)(s+6+6j)(s+6-6j)$
$$= s^3 + 18s^2 + 144s + 432$$

or $\alpha_o = [18 \quad 144 \quad 432]$

$$l_c = \mathcal{O}^{-1} a_-^{-1} (\alpha - a)^T$$

$$a_- = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 4 & -5 & 1 \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 6 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$l_c = \begin{bmatrix} 807.5 \\ 127.5 \\ -24.5 \end{bmatrix}$$

Q. We see that some of the gains of l_c are too high. Can we reduce it in some way?

Let us try a similarity transformation T.

$$x = T\bar{x}$$
$$\bar{A} = T^{-1} A_c T, \quad \bar{b} = T^{-1} b_c \quad \bar{c} = c_c T$$

The new observer gain: $\bar{l} = T^{-1} l_c$ — (1)

The feedback: to keep the same feedback:

$$\bar{K} = K_c T \quad \text{— (2)}$$

So, from (1) & (2), we can see that T can be used to "move" some gain from l to K & vice versa.

In the example above:

$$K_c = [12 \ \ 8 \ \ 10]$$

$$l_c = \begin{bmatrix} 807.5 \\ 125.5 \\ -24.5 \end{bmatrix}$$

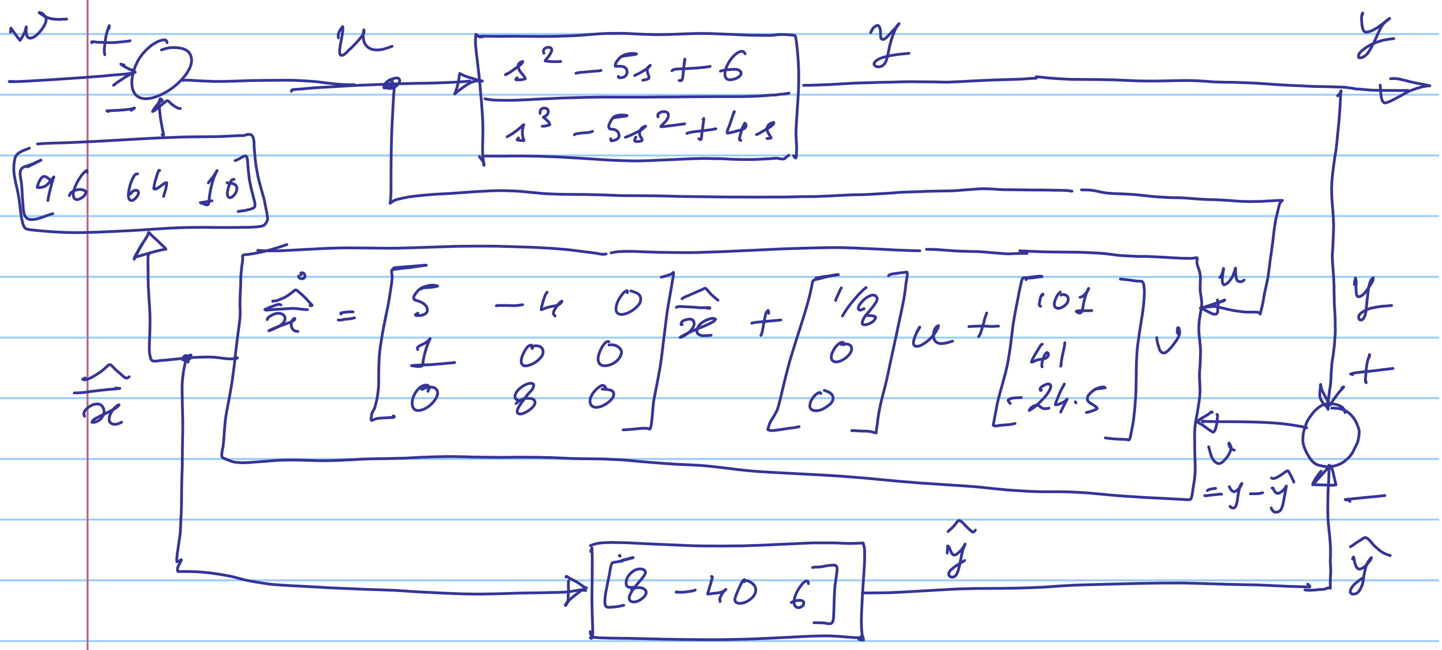
Let's choose $T = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\bar{l} = T^{-1} l_c = \begin{bmatrix} 1/8 & 0 & 0 \\ 0 & 1/8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 807.5 \\ 125.5 \\ -24.5 \end{bmatrix} = \begin{bmatrix} 101 \\ 41 \\ -24.5 \end{bmatrix}$$

$$\bar{K} = K_c T = [12 \ \ 8 \ \ 10] \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [96 \ \ 64 \ \ 10]$$

$$\bar{A} = T^{-1} A_c T = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 0 \\ 0 & 8 & 0 \end{bmatrix} \qquad b = T^{-1} b_c = \begin{bmatrix} 1/8 \\ 0 \\ 0 \end{bmatrix}$$

$$c = c_c T = [8 \ \ -40 \ \ 6]$$



w
+
−
u
$\frac{s^2-5s+6}{s^3-5s^2+4s}$
y
y
[96 64 10]
$\dot{\hat{x}} = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 0 \\ 0 & 8 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 1/8 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 101 \\ 41 \\ -24.5 \end{bmatrix} v$
u
y
+
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$v = y - \hat{y}$
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[8 −40 6]
$\hat{y}$