Note Title 13-06-2008

EE 640 - State Variables from Differential Equations

Review: Solving differential equations by Laplace transforms.

We saw that transfer functions are inadequate tools to study the internal behaviour of systems.

ANALOG COMPUTER SIMULATIONS seemed to provide more insight. We will learn to implement differential equations on Analog Computers.

FACT: We cannot use a differentiator anywhere in the implementation.

Example: Suppose we have a very small signal

$$v(t) = 10^{-3} \sin 10^6 t$$

$$\frac{dv}{dt} \equiv \dot{v}(t) = 10^3 \cos 10^6 t$$

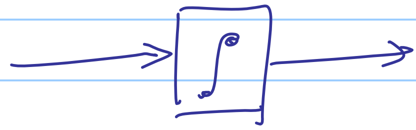
Consequently derivative amplifies disturbances noise, stray signals.

Whereas, integrating $v(t)$ gives

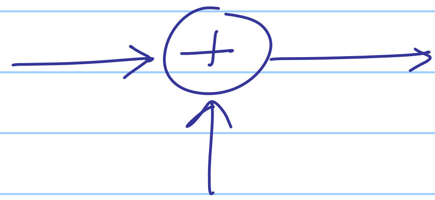
$$\int v(t) = -10^{-9} \cos 10^6 t$$

Hence, integration reduces high frequency disturbances.

Hence we use only the following components for implementing a differential equation:

1) Integrator 

2) Amplifier 

3) Adders 

<u>Methods for implementing a differential equation</u>

The Kelvin Method
- Valid for nonlinear systems also
- Restriction: No input derivative allowed.

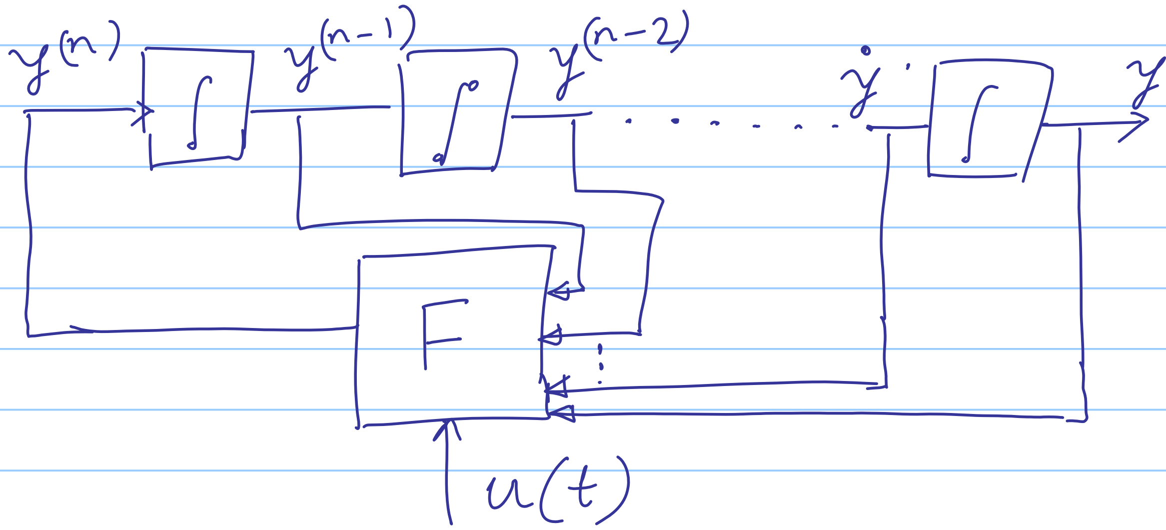
Given a differential Eqn relating
$y(t)$ = unknown (output)
$u(t)$ = given signal (input)

Express the highest derivative of $y(t)$ in terms of all other quantities

$$y^{(n)}(t) = F\left(y^{(n-1)}(t),\ y^{(n-1)}(t), \cdots, y(t), u(t)\right)$$

No input derivative allowed in this method

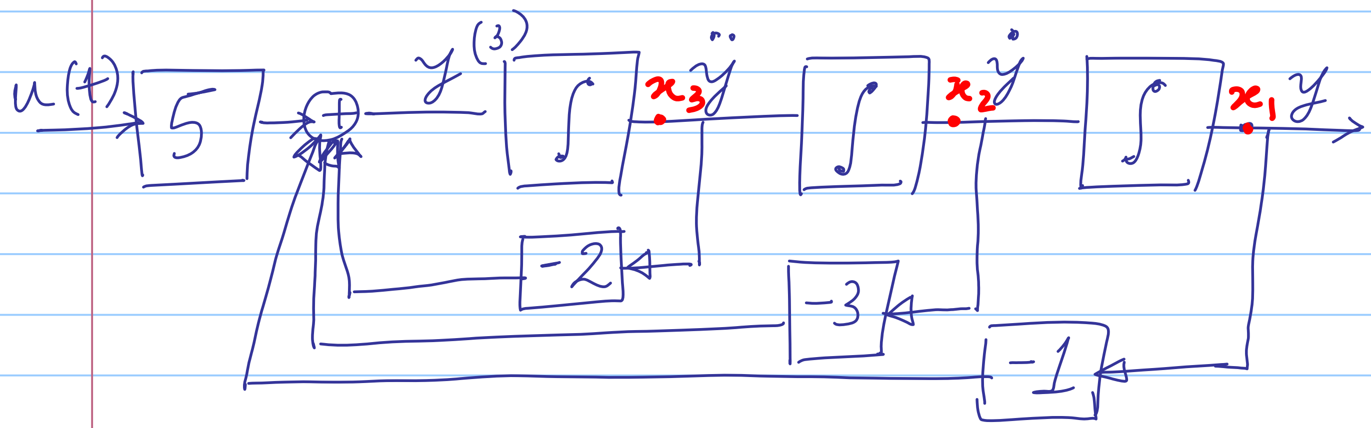
Start from $y^{(n)}(t)$ and integrate n times



Example: $y^{(3)}(t) + 2y^{(2)}(t) + 3\dot{y}(t) + y(t) = 5u(t)$ — (*)

By Kelvin's method:

$$y^{(3)}(t) = -2y^{(2)}(t) - 3\dot{y}(t) - y(t) + 5u(t)$$



Now observe that the output of the integrators have some special features: (Name them x_1, x_2, x_3)

1) The wiring diagram above can be completely & compactly described in terms of x_1, x_2, x_3.

2) (x_1, x_2, x_3) are related to $u(t)$ and $y(t)$ through a set of first order differential equations

■ How to write a system of first order differential equations from the wiring diagram.

Step 1: Give names to integrator output
Step 2: Write equations using only x's and $u(t)$'s.

$$\text{SE} \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -2x_3 - 3x_2 - x_1 + 5u \end{cases}$$

SE above are called the state equations and x_1, x_2, x_3 are the states of the system (*).

Step 3: How is y related to x's?

OE $\quad y = x_1$.

OE is called the output equation.

In matrix notation, SE and OE can be written compactly:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} =: \text{The state vector.}$$

(SE) can be written as:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u$$

(OE) is equivalent to:

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Summary: 1) We learnt to implement a differential equation (with no input derivative) on an analog computer.
2) We defined the output of the integrators as state variables.
3) State variables can compactly express the wiring diagram of the analog comp. simulation.

DISCLAIMER: (We will learn these later in the course)

1) State variables can be defined abstractly for a system, without any mention of analog comp.

2) State variables have many other uses other than being a compact notation of wiring diags.

3) State equations can be derived directly from the differential eqns. without drawing the wiring diagram.

Exercise:

1) We have already seen ~~one~~ two uses of state variable in an earlier lecture. What were they?

2) Try to figure out how to do Disclaimer (3) above for the differential equation $*$.

3) Can you derive the transfer function of $*$ from the state & output equations?

4) Where did all that talk about initial conditions disappear?

Another Method: Implementing diff. equations with input derivatives.

Consider a system described by the following differential equation

$$y^{(n)}(t) + a_1 y^{(n-1)} + \cdots + a_n y(t)$$
$$= b_0 u^{(m)}(t) + b_1 u^{(m-1)}(t) + \cdots + b_m u(t)$$

where $a_1, \ldots, a_n$ and $b_0, \ldots, b_m$ are constant coefficients.

FACT: For a causal system $m \leq n$.

Example: E1

$$\begin{cases} y^{(3)} + a_1 y^{(2)} + a_2 \dot{y} + a_3 y \\ = b_0 u^{(3)} + b_1 \ddot{u} + b_2 \dot{u} + b_3 u \end{cases}$$

Step 1: Express the highest $y(t)$ derivative in terms of all other quantities.

$$y^{(3)} = -a_1 \ddot{y} - a_2 \dot{y} - a_3 y + b_0 u^{(3)} + b_1 \ddot{u} + b_2 \dot{u} + b_3 u$$

Step 2: Integrate both sides $n (=3)$ times

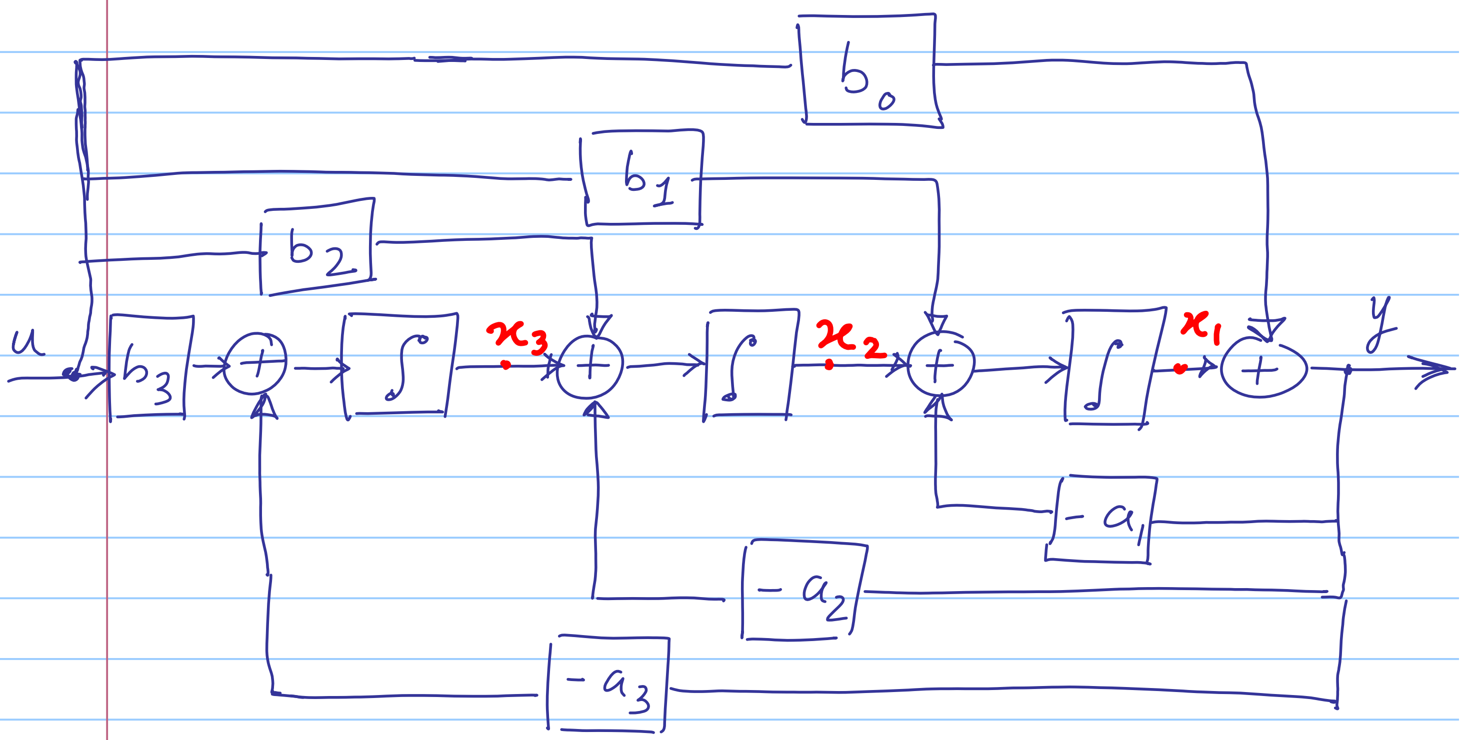
$$\iiint y^{(3)} dt = -a_1 \iiint \ddot{y} - a_2 \iiint \dot{y} - a_3 \iiint y$$
$$+ b_0 \iiint u^{(3)} + b_1 \iiint \ddot{u} + b_2 \iiint \dot{u} + b_3 \iiint u$$

Step 3: Cancel all possible derivatives

and integrals to leave only u and y.

$$y = -a_1\int y - a_2\iint y - a_3\iiint y + b_0 u$$
$$+ b_1\int u + b_2\iint u + b_3\iiint u \quad \ldots\ldots (*)$$

Step 4: Draw $n(=3)$ integrators. Put y at right end and u at left end



Step 5: Connect according to $(*)$.

Q) How to write "state" equations for the diagram?

1) Name the integrator outputs
2) Write equations for $\dot{x}$, using only x's and u's.

$\dot{x}_1 = -a_1 y + x_2 + b_1 u$ ------ (1)

[Note that $y = x_1 + b_0 u$ (which is incidentally the output eqn.)] (2)

Using (2) in (1),

$$\dot{x}_1 = -a_1 x_1 + x_2 + (b_1 - a_1 b_0) u$$

Similarly, $\dot{x}_2 = -a_2 y + x_3 + b_2 u$

$$= -a_2 (x_1 + b_0 u) + x_3 + b_2 u$$

$$= -a_2 x_1 + x_3 + (b_2 - a_2 b_0) u$$

Similarly, $\dot{x}_3 = -a_3 x_1 + (b_3 - a_3 b_0) u.$

In matrix form:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \qquad \dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}$$

(A)

$$\left[\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) + b_0 u(t) \end{aligned}\right.$$

Q2A) Is this the only way to simulate the differential equation (E1) on an ANALOG COMP.?

OR (same question)

Q2B) How many different implementations does a particular differential equation have?

Respectively the answers are:

NO and Infinitely many.

Let us try to build another implementation of equation (E1).

$$\left.\begin{aligned} &y^{(3)} + a_1 y^{(2)} + a_2 \dot{y} + a_3 y \\ &= b_0 u^{(3)} + b_1 \ddot{u} + b_2 \dot{u} + b_3 u \end{aligned}\right\} \quad (E1)$$

Consider the simpler system:

$$\xi^{(3)}(t) + a_1 \xi^{(2)}(t) + a_2 \dot{\xi}(t) + a_3 \xi(t) = u(t)$$

Define the linear form:

$$L(\xi) = \xi^{(3)} + a_1 \xi^{(2)} + a_2 \xi^{(1)} + a_3 \xi \quad (1)$$

Then $L(\xi) = u$ (1A)

So $\dot{u} = \frac{d}{dt} L(\xi) = L(\dot{\xi})$

Similarly $\ddot{u} = L(\ddot{\xi})$ and $u^{(3)} = L(\xi^{(3)})$

So, $b_0 u^{(3)} + b_1 u^{(2)} + b_2 u^{(1)} + b_3 u$

$= b_0 L(\xi^{(3)}) + b_1 L(\xi^{(2)}) + b_2 L(\dot{\xi}) + b_3 L(\xi)$

$= L(b_0 \xi^{(3)} + b_1 \xi^{(2)} + b_2 \dot{\xi} + b_3 \xi)$ ------- (2)

Now define, $w(t) = b_0 \xi^{(3)} + b_1 \xi^{(2)} + b_2 \dot{\xi} + b_3 \xi$

Then from (2) above,

$L(w(t)) = b_0 u^{(3)} + b_1 u^{(2)} + b_2 u^{(1)} + b_3 u$ ---- (3)

Recall that in (1) we defined

$L(w(t)) = w^{(3)}(t) + a_1 w^{(2)}(t) + a_2 w^{(1)}(t) + a_3 w(t)$

Hence by (3), $w(t)$ and $y(t)$ satisfy the same differential equation. Assuming, same initial conditions, the uniqueness of solutions, implies that

$w(t) = y(t)$ for all $t \geq 0$.

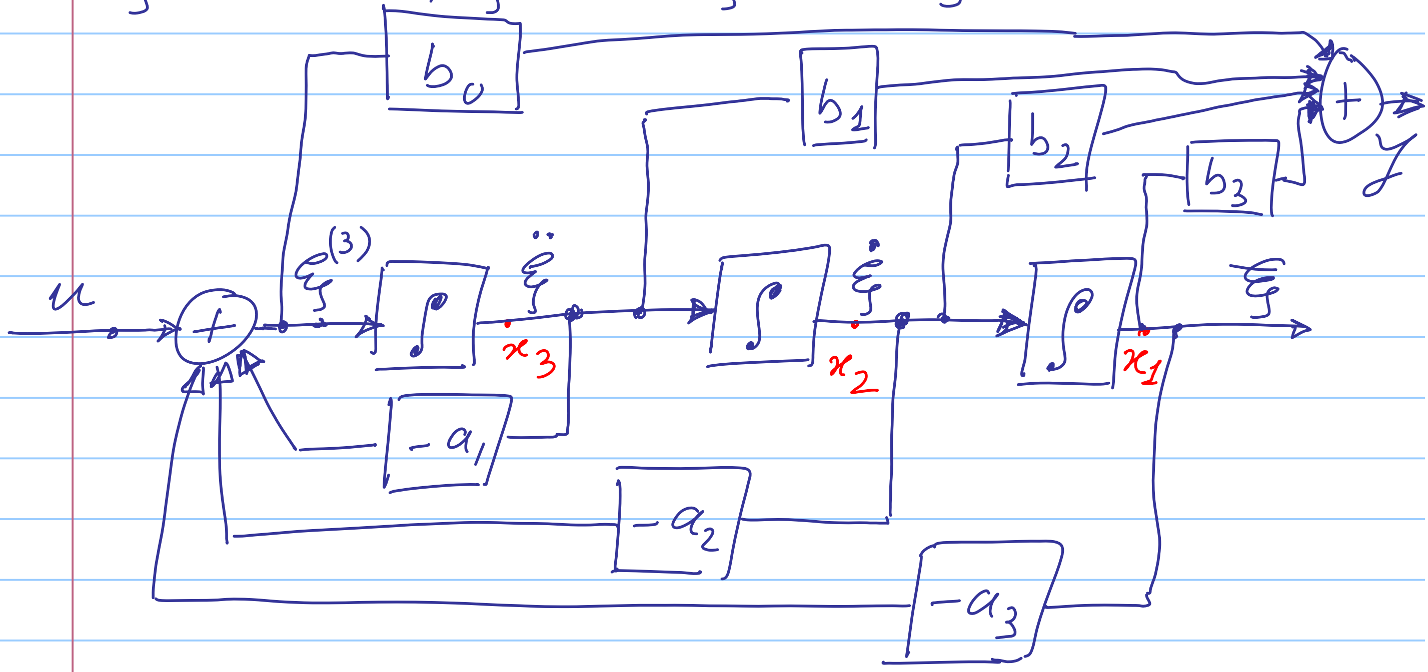
Hence we can use the following scheme for implementation.

1) Use Kelvin's method to implement ξ

2) Get y by adding.

By (1) and (1A), (and Kelvin's method)

$$\xi^{(3)} = -a_1 \ddot{\xi} - a_2 \dot{\xi} - a_3 \xi + u$$



State Equations:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -a_3 x_1 - a_2 x_2 - a_1 x_3 + u$$

Output Equations:

$$y = b_3 x_1 + b_2 x_2 + b_1 x_3 + b_0 \dot{x}_3$$
$$= b_3 x_1 + b_2 x_2 + b_1 x_3 + b_0 \left(-a_3 x_1 - a_2 x_2 - a_1 x_3 + u\right)$$
$$= (b_3 - b_0 a_3) x_1 + (b_2 - b_0 a_2) x_2 + (b_1 - b_0 a_1) x_3 + b_0 u$$

In matrix form,

$$
\text{(B)}\ \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\[2ex] y = \begin{bmatrix} (b_3 - b_0 a_3) & (b_2 - b_0 a_2) & (b_1 - b_0 a_1) \end{bmatrix} x \\ \qquad + b_0 u \end{cases}
$$

Clearly Equations (Implementations) (A) and (B) are different.

<u>Conclusion</u> : A differential equation <u>may have several different realizations / Implementations</u>.

(It can have infinitely many, as we will see later).

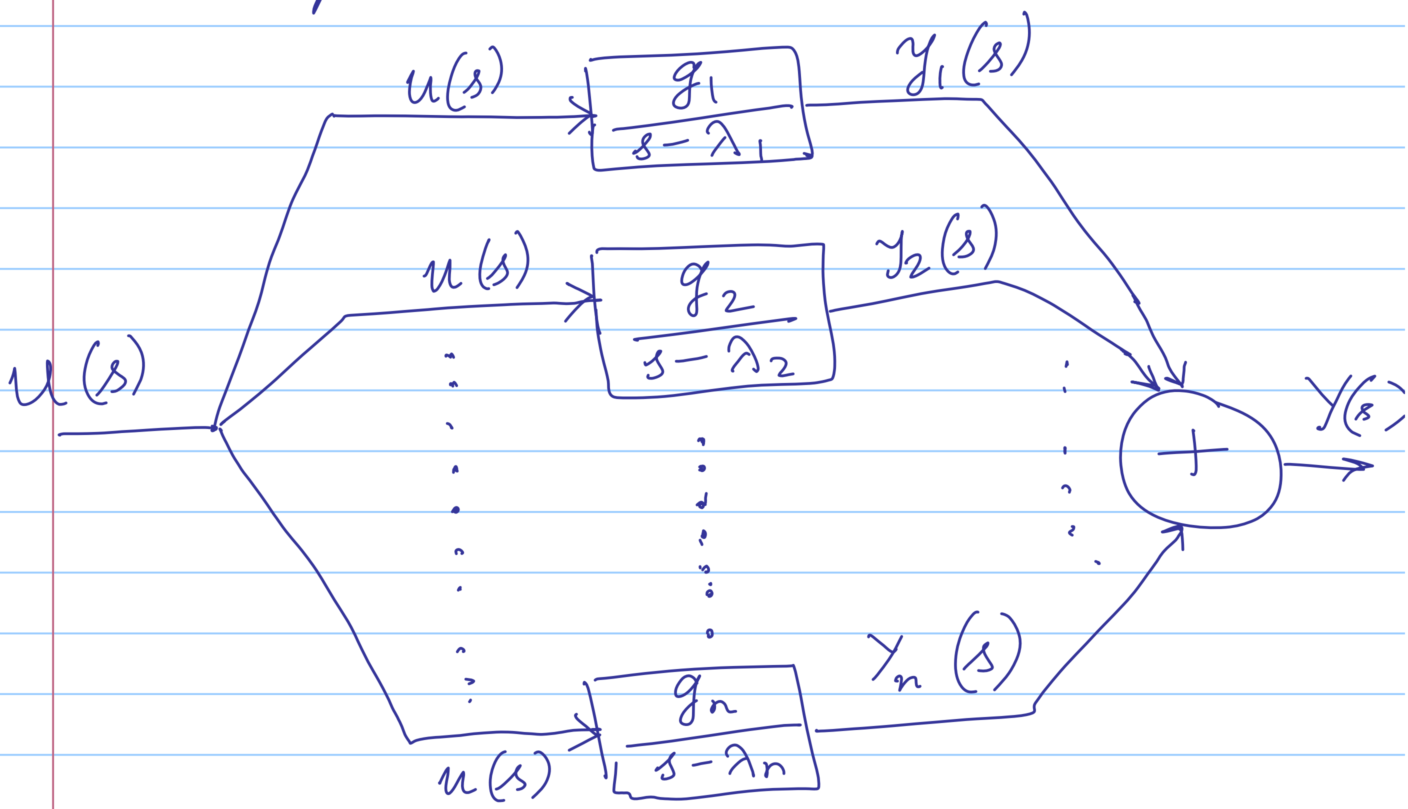
The Parallel Implementation

In contrast to the previous implementations we have discussed, the parallel implementation is possible only for systems whose transfer function is of the form:

$$(4) \quad \frac{Y(s)}{u(s)} = \frac{g_1}{s-\lambda_1} + \frac{g_2}{s-\lambda_2} + \cdots + \frac{g_n}{s-\lambda_n}$$

where $g_1, g_2, \ldots, g_n, \lambda_1, \ldots, \lambda_n$ are real constants and $\lambda_1, \ldots, \lambda_n$ are distinct.

Finding a realization:

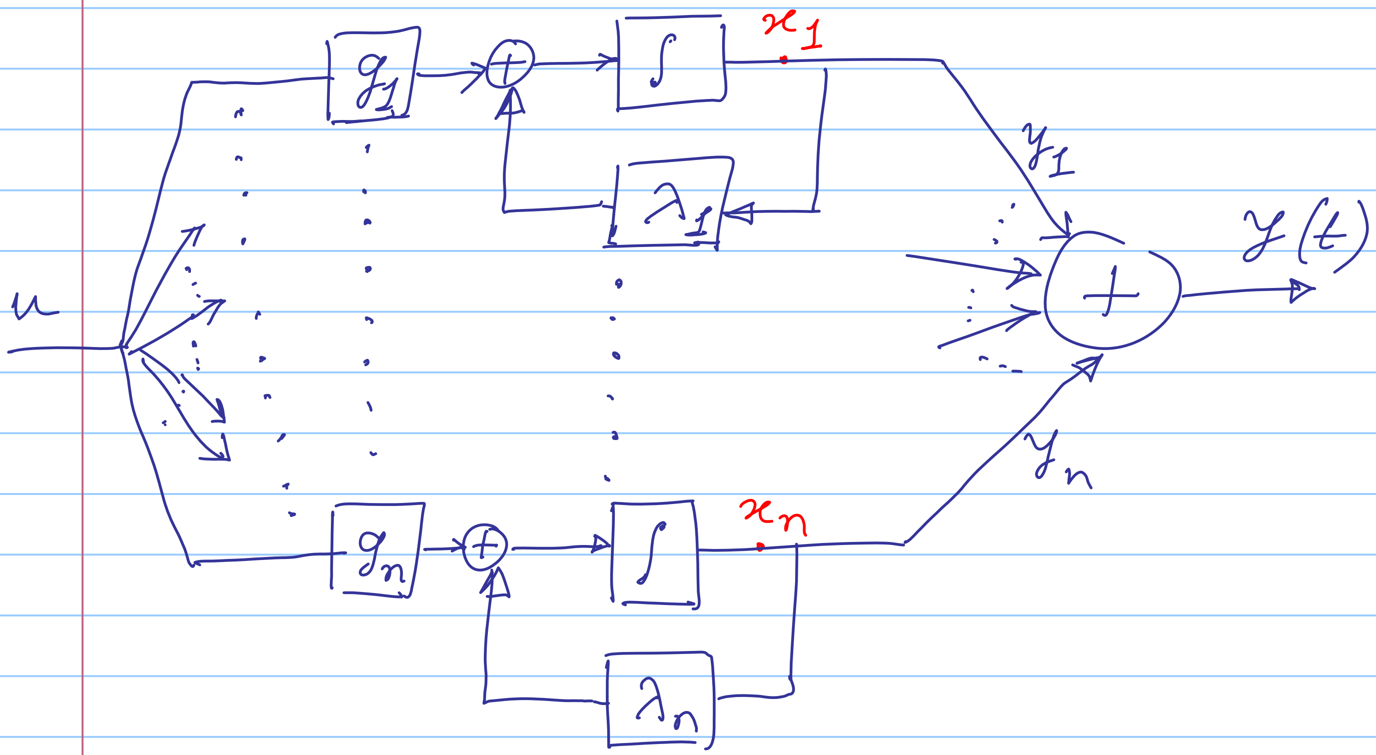


Consider the transfer function from

$$\frac{y_1(s)}{u(s)} = \frac{g_1}{s-\lambda_1}$$

$$\Rightarrow s\,y_1(s) - \lambda_1 y_1(s) = g_1 u(s)$$

$$\Rightarrow \dot{y}_1(t) = \lambda_1 y_1(t) + g_1 u(t)$$



State Equations :

$$\dot{x}_1 = \lambda_1 x_1 + g_1 u$$
$$\dot{x}_2 = \lambda_2 x_2 + g_2 u$$
$$\vdots \qquad \vdots$$
$$\dot{x}_n = \lambda_n x_n + g_n u$$

In matrix form:

$$
(C)\begin{cases}
\dot{x} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} u \\
y = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\end{cases}
$$

(C) is a DIAGONAL realization

The diagonal realization is easy to implement and analyse. However, as mentioned before, it is often not possible to get a diagonal implementation of a differential equation.

Conclusions:

* A differential equation of order n can be represented as
  * n first order differential equations.
  * $\equiv$ A wiring diagram for analog computer simulation
* A particular set of such first

order equations is called a realization.

* There may be different realizations of the same diff. equation

* A realization is of the form

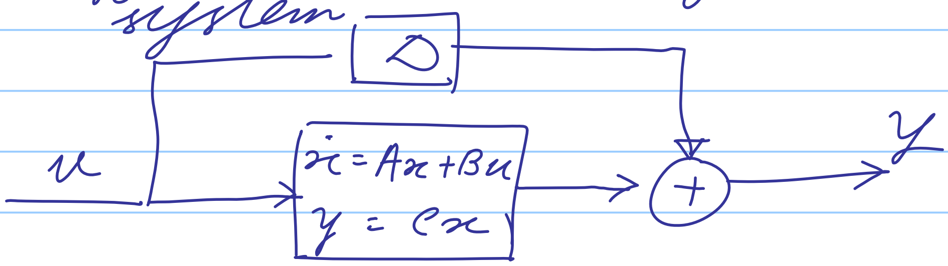
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

For a single-input single-output (SISO) system,

$A \in \mathbb{R}^{n \times n}$ $\quad$ $C \in \mathbb{R}^{1 \times n}$

$B \in \mathbb{R}^{n \times 1}$ $\quad$ $D \in \mathbb{R}^{1 \times 1}$

When $D \neq 0$, the term $Du(t)$ is called input coupling. This has no effect on the dynamics of the system



Hence we will usually assume $D=0$ and the state equations/realization of the form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$