EE 640 - 6 : Time Response & Stability

Note Title 29-06-2008

Review :

We have learnt how to implement a given differential equation on an ANALOG COMP and to write down the corr. state space eqns. We also learnt to translate the initial conditions of the diff eqns. to the initial conditions for the state space realization.

Hence we can simulate the diff. eqn. on the ANALOG COMP. and measure $x(t)$.

Q. Can we calculate $x(t)$ without running the simulation?

OR

Q. Is there an explicit formula for $x(t)$?

Time Response of a Realization

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(0^-) = x_0$$
$$y(t) = Cx(t)$$

Problem : Solve for $x(t)$.

Case I : The homogenous system OR $u(t) = 0$ for all t.

Then: $\dot{x}(t) = Ax(t) \qquad x(0^-) = x_0$
$\qquad\quad y(t) = Cx(t)$

Using Laplace transform,

$$sX(s) - X(0^-) = AX(s)$$

or $\quad X(s) = (sI - A)^{-1} x_0$

So $\quad x(t) = \mathcal{L}^{-1}\{X(s)\} = \{\mathcal{L}^{-1}(sI-A)^{-1}\} x_0$

$\qquad\qquad = e^{At} x_0 \quad$ (Refer to Section 4)

$$y = Ce^{At} x_0$$

Exercise: The matrix e^{At} is sometimes called the state-transition matrix. Why?

Before solving the general eqn, let us learn some properties of the exponential matrix.

Properties of e^{At}

1) $\frac{d}{dt} e^{At} = Ae^{At} = e^{At} A$

Proof: $\quad e^{At} = \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i$

$$\frac{d}{dt}(e^{At}) = \sum_{i=0}^{\infty} \frac{d}{dt}\left(\frac{t^i}{i!} A^i\right)$$

$$= \sum_{i=1}^{\infty} \frac{t^{i-1}}{(i-1)!} A^{i} = A \sum_{i=1}^{\infty} \frac{t^{i-1}}{(i-1)!} A^{(i-1)}$$

$$= A \sum_{j=0}^{\infty} \frac{t^{j}}{j!} A^{j} = A e^{At} = e^{At} A$$

(2) $e^{A(t_1+t_2)} = e^{At_1} e^{At_2}$

Proof: Exercise

NOTE: In general $e^{(A+B)t} \neq e^{At} e^{Bt}$.

Exercise: When does equality hold?

(3) e^{At} is always invertible

Proof: Use (2). Take $t_1 = t$, $t_2 = -t$

Then by (2),

$$I = e^{A(t-t)} = e^{At} \cdot e^{-At}$$

$$\Rightarrow e^{-At} = \left(e^{At}\right)^{-1}$$

(4) The effect of similarity Transform

$$e^{(T^{-1}AT)t} = T^{-1} e^{At} T$$

Proof: $e^{(T^{-1}AT)t} = \sum_{i=0}^{\infty} \frac{t^{i}}{i!} (T^{-1}AT)^{i}$

Consider $\left(T^{-1}AT\right)^{i} = \left(T^{-1}AT\right)\left(T^{-1}AT\right)\cdots\cdots\left(T^{-1}AT\right)$

$$= T^{-1}A^{i}T$$

Substituting:

$$e^{\left(T^{-1}AT\right)t} = \sum_{i=0}^{\infty} \frac{t^{i}}{i!} T^{-1}A^{i}T$$

$$= T^{-1}\left[\sum_{i=0}^{\infty} \frac{t^{i}}{i!} A^{i}\right]T = \underline{T^{-1}e^{At}T}$$

Solving the general equation

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \Big| \; x(0^{-}) = x_0$$
$$y(t) = Cx(t)$$

We have seen, (Section 4)

$$X(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1}BU(s)$$

Now, $\mathcal{L}^{-1}\left\{(sI-A)^{-1}BU(s)\right\}$

$$= \mathcal{L}^{-1}\left\{(sI-A)^{-1}B\right\} * \mathcal{L}^{-1}\left\{U(s)\right\}$$

$$= e^{At}B * u(t)$$

Convolution

So, $x(t) = e^{At}x_0 + e^{At}B * u(t)$

$$= e^{At}x_0 + \int_{0^{-}}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

OR

$$\boxed{\begin{aligned} x(t) &= e^{At}\left[x_0 + \int_{0^-}^{t} e^{-A\tau} B u(\tau)\, d\tau\right] \\ y(t) &= C x(t) \end{aligned}}$$ (*)

<u>Example</u>: $\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$$x(0^-) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Suppose $u(t) = e^t$. Find $x(t)$.

<u>Solⁿ</u>. First calculate e^{At}

$$(sI - A)^{-1} = \begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix}^{-1}$$

$$= \frac{1}{(s-1)^2} \begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix}$$

$\det(sI - A)$
$= (s-1)^2$

$\operatorname{adj}(sI - A)$
$= \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}^T$

Hence $e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$

According to (*) above,

$$x(t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-\tau} & -\tau e^{-\tau} \\ 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{\tau} d\tau \right)$$

$$= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^t \begin{pmatrix} -\tau e^{-\tau} \\ e^{-\tau} \end{pmatrix} e^{\tau} d\tau \right\}$$

$$= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^t \begin{pmatrix} -\tau \\ 1 \end{pmatrix} d\tau \right\}$$

$$= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 - t^2/2 \\ 1 + t \end{bmatrix}$$

$$= \begin{bmatrix} e^t - \frac{t^2}{2} e^t + te^t + t^2 e^t \\ e^t + te^t \end{bmatrix} = \begin{bmatrix} e^t + te^t + \frac{t^2}{2} e^t \\ e^t + te^t \end{bmatrix}$$

Exercise: Solve this problem using equation (**) instead of (*).

STABILITY OF LTI SYSTEMS

Asymptotic Stability / Internal Stability

Internal / Asymptotic stability refers to the stability of a realization of a system:

$$(*) \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$(*)$ is asymptotically stable if the solution of

$$(**) \quad \dot{x}(t) = Ax(t) \quad x(0) = x_0$$

satisfies $\|x(t)\|_2 \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

Where:

$$\|x(t)\|_2 = \sqrt{x_1^2(t) + x_2^2(t) + \cdots + x_n^2(t)}$$

Let us look at this requirement carefully. Some time back, we solved $(**)$ explicitly:

$$x(t) = e^{At} x_0$$

Now we require $\|e^{At} x_0\|$ to go to zero. So the relevant question is:

Q. What are the entries of e^{At}?

For simplicity assume all eigenvalues of A are real. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the distinct eigenvalues of matrix A. Then the characteristic polynomial of A

$$a(s) = (s-\lambda_1)^{n_1}(s-\lambda_2)^{n_2} \cdots\cdots (s-\lambda_k)^{n_k}$$

$n_i \geq 1$ $(i = 1, 2, \dots, k)$ are the multiplicity of λ_i in the characteristic polynomial

FACT: The minimal polynomial of A is

$$\mu(s) = (s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2} \cdots (s-\lambda_k)^{m_k}$$

where $1 \leq m_i \leq n_i$

and $m_i \equiv$ multiplicity of λ_i in the minimal polynomial.

Now recall the formula for $(sI-A)^{-1}$ given in section 4:

$$(sI-A)^{-1} = \frac{1}{\det(sI-A)} \operatorname{adj}(sI-A)$$

$$= \frac{1}{\mu(s)} \Gamma(s) \quad \text{After all possible cancellations.}$$

$$= \begin{bmatrix} \frac{\Gamma_{11}(s)}{\mu(s)} & \frac{\Gamma_{12}(s)}{\mu(s)} & \cdots & \frac{\Gamma_{1n}(s)}{\mu(s)} \\ \vdots & & & \\ \frac{\Gamma_{n1}(s)}{\mu(s)} & \cdots & & \frac{\Gamma_{nn}(s)}{\mu(s)} \end{bmatrix}$$

We know that $e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$

Hence each element of e^{At} is of the form

$$\mathcal{L}^{-1}\left\{\frac{\Gamma_{ij}(s)}{\mu(s)}\right\} \qquad \begin{array}{l} i = 1, \cdots, n \\ j = 1, \cdots, n \end{array}$$

which in turn consists of the terms

$e^{\lambda_i t}, te^{\lambda_i t}, \cdots\cdots, t^{(m_i-1)} e^{\lambda_i t}$

with $i = 1, 2, \cdots, k$.

$\left[\text{Recall : } \mathcal{L}^{-1}\left\{\frac{(k-1)!}{(s+a)^k}\right\} = t^{k-1} e^{-at}\right]$

Since $x(t) = e^{At} x_0$, when x_0 varies over the entire R^n space, $x(t)$ will have a linear combination of these terms as entries.

Hence, it is easy to see that

$\|x(t)\| \to 0$ <u>iff</u> $\lambda_i < 0 \quad \forall\, i = 1, 2, \cdots, n$

<u>Example</u>

$$\dot{x} = \underbrace{\left[\begin{array}{cc|c} -2 & 0 & 0 \\ 1 & -2 & 0 \\ \hline 0 & 0 & -2 \end{array}\right]}_{A} x$$

with $x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Characteristic polynomial of A is

$$a(s) = \det(sI - A) = (s+2)^3$$

$$(sI-A)^{-1} = \frac{1}{(s+2)^3}\begin{bmatrix} (s+2)^2 & 0 & 0 \\ (s+2) & (s+2)^2 & 0 \\ 0 & 0 & (s+2)^2 \end{bmatrix}$$

$$= \underbrace{\frac{1}{(s+2)^2}}_{\mu(s)} \underbrace{\begin{bmatrix} (s+2) & 0 & 0 \\ 1 & (s+2) & 0 \\ 0 & 0 & (s+2) \end{bmatrix}}_{\Gamma(s)}$$

Hence by our definition of minimal polynomial:

$$\mu(s) = (s+2)^2$$

So the multiplicity of the eigenvalue "2" in $a(s)$ is 3 and in $\mu(s)$ is 2.

If you calculate e^{At}:

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A^{-1})\right\} = \begin{bmatrix} e^{-2t} & 0 & 0 \\ te^{-2t} & e^{-2t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

For any arbitrary $x_0 = [x_{01} \; x_{02} \; x_{03}]^T \in \mathbb{R}^3$,

$$x(t) = \begin{bmatrix} e^{-2t} & 0 & 0 \\ te^{-2t} & e^{-2t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$$

$$\therefore \; x_1(t) = x_{01} e^{-2t}$$

$$x_2(t) = x_{01} . te^{-2t} + x_{02} e^{-2t}$$

$$x_3(t) = x_{03} e^{-2t}$$

It is now easy to verify that

$$\lim_{t \to \infty} x_1(t) = \lim_{t \to \infty} x_2(t) = \lim_{t \to \infty} x_3(t) = 0.$$

Hence the realization is asymptotically stable.

FOR GENERAL EIGENVALUES

In general the following FACT holds:

FACT: Let $x(t)$ be the solution to

$$\dot{x}(t) = Ax(t) \qquad x(0) = x_0 .$$

Then each coordinate $x_j(t)$ $(j = 1, \dots, n)$ of $x(t)$ is a linear combination of the following n functions:

a) the function $t^l e^{t\lambda}$ where λ runs through the distinct real eigenvalues of A, and l is an integer with $0 \leq l <$ multiplicity of λ.

AND

b) the functions

$$t^l e^{at} \cos bt \quad \text{and} \quad t^l e^{at} \sin bt$$

where $a+ib$ runs through the complex eigenvalues of A having $b>0$ and l is an integer in the range $0 \leq l <$ multiplicity of $a+ib$.

Hence it is easy to see:

FACT: The realization $\circledast$ is asymptotically stable if and only if $\mathrm{Re}(\lambda_i) < 0$ for all $i = 1, \dots, n$.

i.e. if and only if all eigenvalues are inside the open left half of the complex plane.

A slightly different (weaker) concept of stability is:

DEFINITION: The system is STABLE if $\|x(t)\|$ is bounded for all

$t \geq 0$, for all initial conditions.

From the FACT above, it is easy to characterize this form of stability also:

FACT: The system is stable if and only if $\operatorname{Re} \lambda_i \leq 0$ for all $i = 1, \cdots, n$ AND all eigenvalues with zero real part have multiplicity of 1 in the minimal polynomial of A.

The effect of the input / External Stability

Here we assume zero-initial conditions.

DEFN: A system is externally stable / Bounded input-bounded output stable (BIBO) stable if any bounded input $u(t) \leq M_1$ for $0 \leq t < \infty$ produces a bounded output, $y(t) \leq M_2$ for $0 \leq t < \infty$.

FACT: A system is BIBO stable if and only if the impulse response of the system $h(\cdot)$ (See Section 4) is absolutely summable i.e.

$$\int_0^\infty |h(t)| \, dt < M < \infty$$

Exercise: Prove this FACT.

Relation between BIBO Stability and Asymptotic Stability:

Claim: Asymptotic stability implies BIBO stability.

Proof: Consider the impulse response

$$h(t) = Ce^{At}B$$

$$= \text{Linear combination of } \{t^l e^{t\lambda}, t^l e^{ta}\cos bt, t^l e^{ta}\sin bt\}$$

For a asymptotically stable system, $\lambda < 0$ and $a < 0$. Hence each of these quantities are absolutely summable.
So $h(t)$ is also absolutely summable.

NOTE: In general the converse is not true. i.e.

$$\text{BIBO stability} \not\Rightarrow \text{Asymptotic stability}$$

In fact, BIBO stability is equivalent to asymptotic stability if the realization is minimal. (Section 4)

Example: Consider a realization:

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Eigenvalues of $A \equiv (1, -1)$.
Hence it is NOT asymptotically stable.

Now consider $h(t) = Ce^{At}B$

$$e^{At} = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e^{t} \\ e^{-t} \end{bmatrix}$$

$$= e^{-t}.$$

$$\int_0^\infty |h(t)|\, dt = \int_0^\infty |e^{-t}|\, dt = 1 < \infty$$

Hence the system is BIBO stable.

Exercise: 1) Check whether the realization is minimal?
2) Check whether the realization is observable?

Exercise: What do you think will happen if you try to implement this system on a ANALOG comp.?

An intermediate concept is Bounded Input Bounded State (BIBS) stability. Here also we assume zero initial conditions.

DEFN: A realization is BIBS stable if $\|x(t)\|$ is bounded for all $0 \le t < \infty$ whenever the input function $u(t)$ is bounded for all $0 \le t < \infty$.

FACT: A realization is BIBS stable if and only if it is asymptotically stable i.e. iff $Re(\lambda_i) < 0$ for all $i = 1, \cdots, n$.