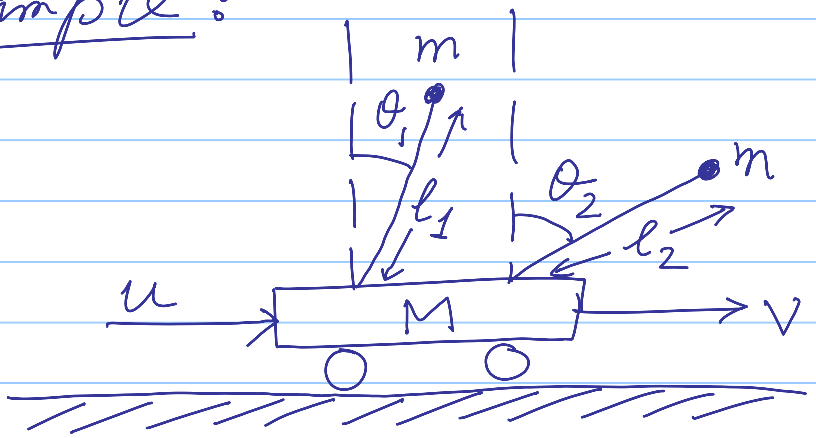
EE 640-8 Canonical Realization

Note Title 24-07-2008

Example:



Q. Can we bring both pendulums to vertical?

Q. Is it enough to look at only one of them?

The Equations of Motion

Assume $|\theta_1|, |\theta_2|$ are small

$\Rightarrow \sin\theta \approx \theta \quad \cos\theta \approx 1$

$v \equiv$ cart velocity

$u \equiv$ Force

Cart : $M\dot{v} = u - mg\theta_1 - mg\theta_2 \quad (1)$

Pendulum 1 : $m(\dot{v} + l_1\ddot{\theta}_1) = mg\theta_1 \quad (2)$

Pendulum 2 : $m(\dot{v} + l_2\ddot{\theta}_2) = mg\theta_2 \quad (3)$

But we are not interested in cart position; only in θ_1 & θ_2.

Let's eliminate v from the eqns:

$$\dot{v} = \frac{1}{M}\left[u - mg\theta_1 - mg\theta_2\right]$$

Substituting into (2), we get

$$\ddot{\theta}_1 = \frac{1}{l_1}\left[g\theta_1 - \dot{v}\right]$$

$$= \frac{g\theta_1}{l_1} - \frac{1}{l_1 M}\left[u - mg\theta_1 - mg\theta_2\right]$$

$$= \left[\frac{g}{l_1} + \frac{mg}{l_1 M}\right]\theta_1 + \frac{mg}{l_1 M}\theta_2 - \frac{1}{l_1 M}u$$

Similarly for (3),

$$\ddot{\theta}_2 = \frac{g\theta_2}{l_2} - \frac{1}{l_2 M}\left[u - mg\theta_1 - mg\theta_2\right]$$

$$= \frac{mg}{l_2 M}\theta_1 + \left[\frac{g}{l_2} + \frac{mg}{l_2 M}\right]\theta_2 - \frac{1}{l_2 M}u$$

We do not have any input derivative
So use Kelvin's method to get a realization.

Let $x_1 = \theta_1$; $x_3 = \dot{\theta}_1$
$x_2 = \theta_2$; $x_4 = \dot{\theta}_2$

Then, $\dot{x}_1 = \dot{\theta}_1 = x_3$
$\dot{x}_2 = \dot{\theta}_2 = x_4$

$$\dot{x}_3 = \ddot{\theta}_1 = \left[\frac{g}{\ell_1} + \frac{mg}{M\ell_1}\right] x_1 + \frac{mg}{\ell_1 M} x_2 - \frac{1}{\ell_1 M} u$$

$$\dot{x}_4 = \ddot{\theta}_2 = \frac{mg}{\ell_2 M} x_1 + \left[\frac{g}{\ell_2} + \frac{mg}{\ell_2 M}\right] x_2 - \frac{1}{\ell_2 M} u$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Let:

$$a_1 = \frac{g}{\ell_1} + \frac{mg}{M\ell_1}$$

$$a_4 = \frac{g}{\ell_2} + \frac{mg}{M\ell_2}$$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & \frac{mg}{\ell_1 M} & 0 & 0 \\ \frac{mg}{\ell_2 M} & a_4 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{\ell_1 M} \\ -1/\ell_2 M \end{bmatrix} u$$

The problem: Bring pendulums to vertical position and keep them there.

$\equiv \theta_1 = 0, \ \theta_2 = 0, \ \dot{\theta}_1 = 0, \ \dot{\theta}_2 = 0$

i.e. $x = 0$.

We have learnt that this is possible (starting from any initial condition) iff the realization is controllable.

$$\mathcal{C} = [B \quad AB \quad A^2B \quad A^3B]$$

It turns out that $\det(\mathcal{C}) = g^2(\ell_1 - \ell_2)^2$

So the realization is controllable

iff $l_1 \neq l_2$.

Q2) Is it sufficient to look at only one pendulum?

i.e. if $y = \theta_1 = [1\ 0\ 0\ 0]x$

Q. Can we calculate x from θ_1.

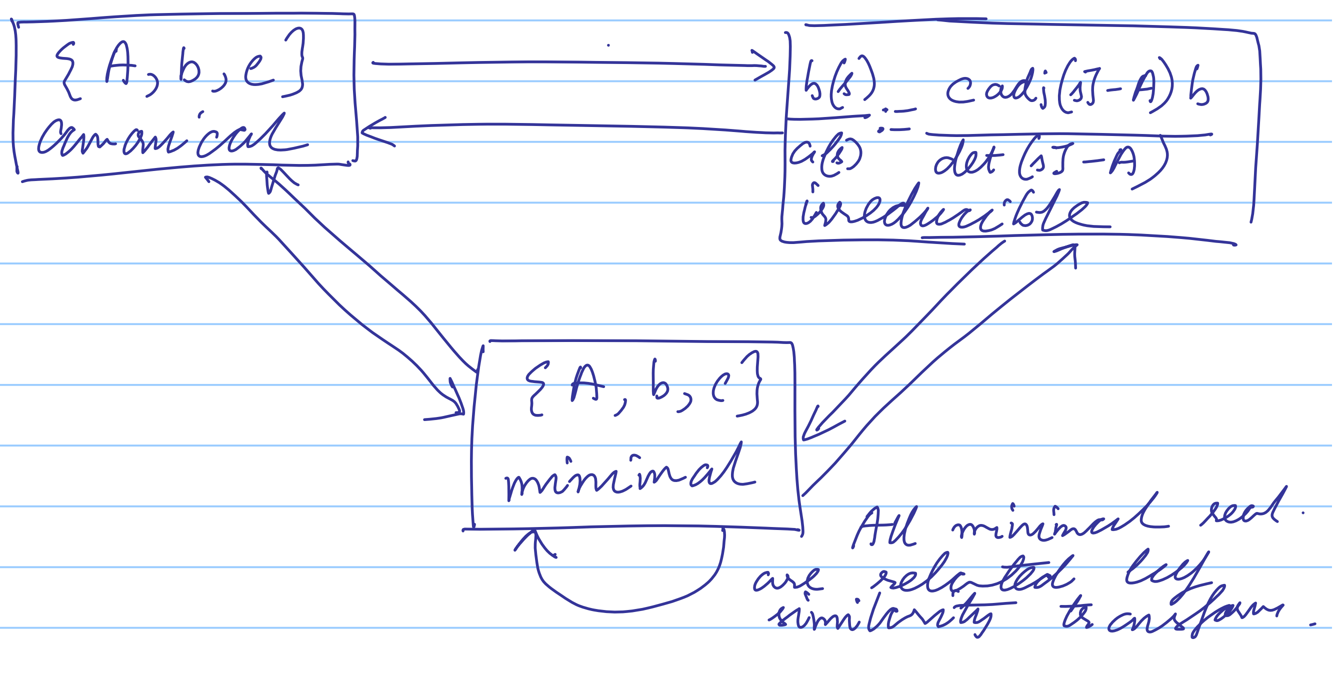
Ans: Yes, iff $\det(\mathcal{O}) \neq 0$.

$$\det(\mathcal{O}) = -\left(\frac{mg}{ml_1}\right)^2 \neq 0.$$

So the realization is observable and the answer Q2 is YES.

Realizations that are observable and controllable are called <u>canonical realizations</u>

Q. What are the properties of canonical realizations?



For proving these relations we need two intermediate results:

FACT 1: If a transfer function

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

has one controllable and observable n^{th} order realization, then all n^{th} order realization must be also controllable and observable.

Proof: We can easily check

$$\mathcal{O}(c, A)\, \mathcal{C}(A, b) = \begin{bmatrix} cb & cAb & \cdots & cA^{n-1}b \\ cAb & cA^2b & \cdots & cA^n b \\ \vdots & & & \vdots \\ cA^{n-1}b & \cdots & \cdots & cA^{2n-2}b \end{bmatrix}$$

Hankel Matrix $\nearrow$

But the elements are nothing but Markov parameters which we have seen, are uniquely determined by the transfer function.

Now, consider two n^{th} order realizations $\{A_1, b_1, c_1\}$ and $\{A_2, b_2, c_2\}$ of $H(s)$.

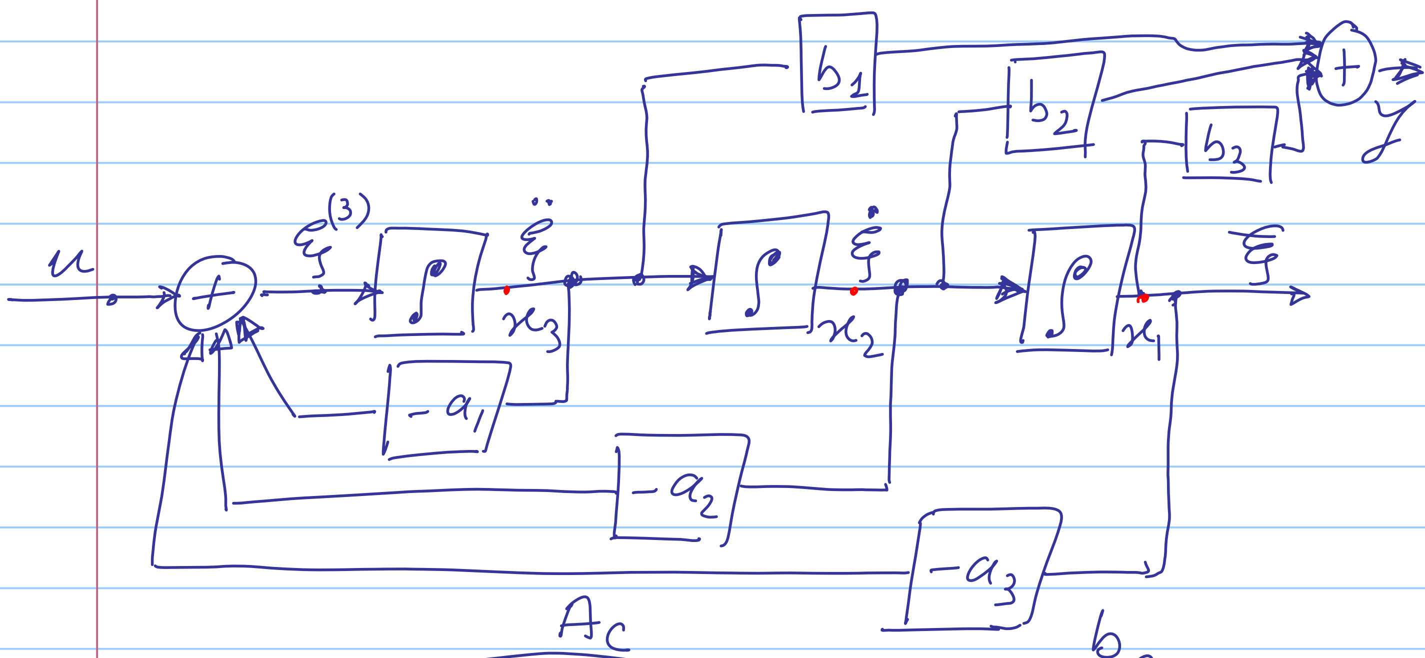
Then $\mathcal{O}(c_1, A_1)\, \mathcal{C}(A_1, b_1) = \mathcal{O}(c_2, A_2)\, \mathcal{C}(A_2, b_2)$

But by hypothesis, $\mathcal{O}(c_1, A_1)$ and $\mathcal{C}(A_1, b_1)$ are both non-singular, so is their product.

$\Rightarrow$ RHS is also non-singular

$\Rightarrow$ Each of $\mathcal{O}(c_2, A_2)$ and $\mathcal{C}(A_2, b_2)$ are non-singular for any n^{th} order realization $\{A_2, b_2, c_2\}$

A Special Form:



$$\dot{x} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}}^{A_c} x + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{b_c} u \qquad (\ast)$$

$$y = \underbrace{\begin{bmatrix} b_3 & b_2 & b_1 \end{bmatrix}}_{c_c} x$$

Check: Realization of this form is

always controllable.

$$\mathbb{C} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & * \\ 1 & -a_1 & * \end{bmatrix} \Rightarrow \det(\mathbb{C}) \neq 0.$$

FACT 2: The n^{th} order special $(*)$ form of $H(s) = \dfrac{b(s)}{a(s)}$, $n = \deg a(s)$, will be observable iff $b(s)$ and $a(s)$ are coprime, i.e. if $\dfrac{b(s)}{a(s)}$ is irreducible.

Proof: Let $b(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n$

and $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n$

We claim: $\mathcal{O}_c = [b_1 A_c^{n-1} + \cdots + b_n I]$

To see this, note:

$$\underbrace{\begin{bmatrix} 0 & \cdots & 0 & \underset{\uparrow \, i^{th} \text{ eleme}}{1} & 0 & \cdots & 0 \end{bmatrix}}_{e_i^T} \begin{bmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & \cdots & \\ & & & \ddots & \\ & & & & 1 \\ -a_n & -a_{n-1} & \cdots & & -a_1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & \cdots & 0 & \underset{\uparrow \, (i+1)^{th} \text{ element}}{1} & 0 & \cdots & 0 \end{bmatrix}}_{e_{i+1}^T} \qquad 1 \leq i \leq n-1$$

$$e_1^T A_c^{n-1} = e_2^T A_c^{n-2} = e_3^T A_c^{n-3} \cdots = e_{n-1}^T A_c = e_n^T$$

i.e $\quad e_i^T A_c = e_{i+1}^T \qquad 1 \leq i \leq n-1$

and $\quad e_n^T A_c = [-a_n \; -a_{n-1} \; \cdots \; -a_1]$

[Here $e_i \equiv i^{th}$ column of identity matrix]

Now, $e_1^T b(A_c) = e_1^T [b_1 A_c^{n-1} + \cdots + b_n I]$

$$= b_1 e_1^T A_c^{n-1} + b_2 e_1^T A_c^{n-2} + \cdots + b_n e_1^T I$$

$$= b_1 e_n^T + b_2 e_{n-1}^T + \cdots + b_n e_1^T$$

$$= [b_n \; b_{n-1} \; \cdots \; b_1] = c_c$$

Similarly, $e_2^T b(A_c) =$

$$= e_2^T [b_1 A_c^{n-1} + \cdots + b_n I]$$

$$= e_1^T A_c [b_1 A_c^{n-1} + \cdots + b_n I]$$

$$= e_1^T [b_1 A_c^{n-1} + \cdots + b_n I] A_c$$

$$= c_c A_c$$

Continuing like this,

$$O_c = \begin{bmatrix} c_c \\ c_c A_c \\ c_c A_c^2 \\ \vdots \\ c_c A_c^{n-1} \end{bmatrix} = \begin{bmatrix} e_1^T [b_1 A_c^{n-1} + \cdots + b_n I] \\ e_2^T [b_1 A_c^{n-1} + \cdots + b_n I] \\ \vdots \\ e_n^T [b_1 A_c^{n-1} + \cdots + b_n I] \end{bmatrix}$$

$$= \begin{bmatrix} e_1^T \\ e_2^T \\ \vdots \\ e_n^T \end{bmatrix} \left[b_1 A_c^{n-1} + \cdots + b_n I \right]$$

$$= \left[b_1 A_c^{n-1} + \cdots + b_n I \right]$$

Hence, $\det(\mathcal{O}_c) \neq 0 \iff \det[b(A_c)] \neq 0$.

FACT: If A_c has eigenvalues λ_i, $i = 1, 2, \ldots, n$ then $b(A_c)$ has eigenvalues $\{b(\lambda_i), i = 1, 2, \ldots, n\}$.

Proof: Exercise

Now, $\det[b(A_c)] = \prod_{i=1}^{n} b(\lambda_i)$

So $\det[b(A_c)] = 0$ iff one or more of the $\{b(\lambda_i)\}$ are zero. But by definition of the λ_i's,

$$\det(\lambda_i I - A_c) = 0.$$

Hence, $\det b(A_c)$ will be non-zero, and hence $\{A_c, b_c, c_c\}$ observable iff, $a(s)$ and $b(s)$ have no common roots.

Resulting FACT: A transfer function $H(s) = \dfrac{b(s)}{a(s)}$ is irreducible if and only if all n^{th} order realizations $n = \deg a(s)$, are controllable and observable (i.e. canonical)

Recall that we had defined the minimal realization as the one with an irreducible $c(sI-A)^{-1}B$. We had claimed that the minimal realization can be implemented using the minimum number of integrators.

FACT: A realization $\{A, b, c\}$ is of minimum order (ie. the smallest number of state variables OR the smallest number of integrators) iff $a(s) := \det(sI-A)$ and $b(s) := c \, \mathrm{adj}(sI-A) b$ are relatively prime.

Proof: Exercise.

Finally we note that any two minimal realizations are very closely related.

FACT: Any two minimal realizations can be connected by a unique similarity transform.

Proof: Let the minimal realizations be $\{A_1, b_1, c_1\}$ and $\{A_2, b_2, c_2\}$. Both are canonical, so, we define

$$T = \mathcal{O}^{-1}(c_1, A_1)\, \mathcal{O}(c_2, A_2)$$

We know,

$$\mathcal{O}(c_1, A_1)\,\mathcal{C}(A_1, b_1) = \mathcal{O}(c_2, A_2)\,\mathcal{C}(A_2, b_2)$$

So, $T = \mathcal{C}(A_1, b_1)\,\mathcal{C}^{-1}(A_2, b_2)$

We claim that $T^{-1} b_1 = b_2$

To prove we have to show

$$b_1 = \mathcal{C}_1 \mathcal{C}_2^{-1} b_2$$

i.e $\mathcal{O}_1 b_1 = \mathcal{O}_1 \mathcal{C}_1 \mathcal{C}_2^{-1} b_2$

i.e. $\mathcal{O}_1 b_1 = \mathcal{O}_2\, \mathcal{C}_2 \mathcal{C}_2^{-1} b_2$

i.e $\mathcal{O}_1 b_1 = \mathcal{O}_2 b_2$

But $\mathcal{O}_1 b_1 = \begin{bmatrix} c_1 \\ c_1 A_1 \\ \vdots \\ c_1 A_1^{n-1} \end{bmatrix} b_1 = \begin{bmatrix} c_1 b_1 \\ c_1 A_1 b_1 \\ \vdots \\ c_1 A_1^{n-1} b_1 \end{bmatrix}$

$$= \begin{bmatrix} c_2 b_2 \\ c_2 A_2 b_2 \\ \vdots \\ c_2 A_2^{n-1} b_2 \end{bmatrix} = \mathcal{O}_2 b_2$$

Similarly, $c_1 T = c_2$

Finally, we can easily verify

$$\mathcal{O}_1 A_1 \mathcal{C}_1 = \mathcal{O}_2 A_2 \mathcal{C}_2$$ [the elements are Markov parameters again)

Then,

$$A_2 = \mathcal{O}_2^{-1} \mathcal{O}_1 A_1 \mathcal{C}_1 \mathcal{C}_2^{-1}$$
$$= T^{-1} A_1 T$$

Hence T defines a similarity tran.

<u>Uniqueness</u>: Let $\tilde{T}$ be another similarity tr. relating $\{A_1, b_1, c_1\}$ and $\{A_2, b_2, c_2\}$

$$\mathcal{O}_2 = \begin{bmatrix} c_2 \\ c_2 A_2 \\ \vdots \\ c_2 A_2^{n-1} \end{bmatrix} = \begin{bmatrix} c_1 T \\ c_1 A_1 T \\ \vdots \\ c_1 A_1^{n-1} T \end{bmatrix} = \mathcal{O}_1 T$$

Similarly, $\mathcal{O}_2 = \mathcal{O}_1 \tilde{T}$

Hence, $\mathcal{O}_1 \left[T - \tilde{T} \right] = 0$

which implies that $T = \tilde{T}$ since $\mathcal{O}_1$ is non-singular.

<u>The Effect of Similarity Transformation on controllability and observability</u>

$$\{A, b, c\} \xrightarrow{T} \{\bar{A}, \bar{b}, \bar{c}\}$$

$\mathcal{O}, \mathcal{C}$ $\qquad\qquad\qquad\qquad$ $\mathcal{O}_T, \mathcal{C}_T$

What is the connection between the Controllability and observability matrix?

$$\mathcal{C}_T = [\bar{b} \quad \bar{A}\bar{b} \quad \cdots \quad \bar{A}^{n-1}\bar{b}]$$

$$= [T^{-1}b \quad T^{-1}ATT^{-1}b \quad \cdots\cdots]$$

$$= [T^{-1}b \quad T^{-1}Ab \quad \cdots \quad T^{-1}A^{n-1}b]$$

$$= T^{-1}\mathcal{C}$$

i.e $\boxed{\mathcal{C}_T = T^{-1}\mathcal{C}}$

Consequently, $\mathcal{C}$ is full rank iff $\mathcal{C}_T$ is full rank

i.e. A Similarity tr. does not affect controllability.

<u>Observability:</u> (shown in previous proof)

$$\boxed{\mathcal{O}_T = \mathcal{O}T}$$

The original realization is observable iff the tr. realization is observable

OR, A Sim. Tr. doesnot affect observability.

Corollary: Given two realizations known to be similar, we can find the similarity transform T that connects them as follows

1) If the realizations are controllable:

$$T = \mathcal{C}\,\mathcal{C}_T^{-1}$$

2) If the realizations are observable

$$T = \mathcal{O}^{-1}\mathcal{O}_T$$