A Dynamic Model for Lane-less (Indian) Traffic

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Popular perception is that Indian traffic is “chaotic”*

Nobody obeys lane discipline

The only rule is: “right of space” (as opposed to “right of way”)

The above rule frequently interpreted as “might is right”

We aim to build a car following model for Indian traffic
Car following model?

- Well developed microscopic model of traffic where each car is supposed to follow a leader.
- An early model as an example: \( \ddot{x}_{n+1} = c \frac{(\dot{x}_n - \dot{x}_{n+1})}{(x_n - x_{n+1})^2} \), where \( x_n \) is the position of the \( n \)-th vehicle.

**Figure**: Car Following Model (from Prof T. Mathew’s website)
Car following model—useful?

- Understanding vehicle level traffic/driver behaviour
- Used for traffic simulations
- For uniform velocity: such models can predict macro-level behaviour: capacity, density, interdependence between these quantities etc.
- Used to develop autonomous driverless cars
Does these models work for Indian Traffic?

- Several studies exist comparing simulations using available models calibrated with Indian traffic data
- A recent example (of course there are several more papers):

- Lane shift is usually incorporated extraneously to the car following simulations
- Numerical computation of macro-level data are accurate after complicated tuning of simulation parameters

- In our view, available models do not capture our lane-less driving behaviour.
Who follows whom?

- The right picture: laned traffic with clear validity of single leader following by each car
- Left picture: Typical Indian traffic, not clear who is following whom

Figure: Car Following? (picture from the G. Asaithambi et al. cited above)
Why?

- We do not follow a single car
  - We see every car in front, forever on the lookout for empty space
  - We see cars on both right and left, forever on ..........
- We move to occupy empty space
- We “lane-change” continuously in the process
- Obstacles are everywhere
- Roads are never straight

![Road Sign](image)
In this talk

- A model which better approximates Indian driving
- Validation (to some extent) of the model with real microscopic data
- Proven stability properties (will not be usable otherwise any way)
- Use of recently developed multi-agent formation results for these purposes
Experimental “Setup”

- Video of JVLR outside IITB

Figure: Original Image and corresponding google map image
Our Assumptions

Simplistic but some can be relaxed:

- All vehicles are identical double integrators in both directions
- Each driver uses identical driving laws
- Decoupled motion along longitudinal and lateral directions
- Cone of vision
- Road conditions create pseudo leader
- We sidestep mesh/string stability by assuming finite number of vehicles
Primary hypothesis
  - Each vehicle is influenced by all vehicles in the “layer” ahead of him

How to define layers?
  - All vehicles who influence a vehicle is in one layer ahead of him

Circular Logic - circumvented by the cone of vision argument
A **rooted directed tree** is a digraph such that there exists a node (called **root**) and a directed path from that node to all other nodes in the digraph.

A digraph $\vec{G}$ is said to contain a **directed spanning tree**, if there exists a rooted directed tree $\vec{G}_{\text{tree}} = (A_{\text{tree}}, E_{\text{tree}}, w)$ such that $A_{\text{tree}} = A$ and $E_{\text{tree}} \subseteq E$.

The Laplacian ($\mathcal{L}$) for a directed graph with weights $w_{ij}$ as follows: $\ell_{ij} := -w_{ij}$ if $(a_j, a_i) \in \vec{E}$, $\ell_{ij} := 0$ if $(a_j, a_i) \notin \vec{E}$, and $\ell_{ii} := \sum_{j=1}^{n} w_{ij}$ := cumulative weight of incoming edges.
Definitions

- For a directed graph $\vec{G} = (\mathcal{A}, \vec{E}, w)$, let $L = \{L_0, L_1, L_2, \ldots, L_k\}$, $k \geq 1$ be a partition of $\mathcal{A}$ such that if $(a_i, a_j) \in \vec{E}$ with $u_i \in L_p$ and $u_j \in L_q$, then $q < p$. Such $L$ is called a layering of $\vec{G}$ and $L_0, L_1, \ldots, L_k$ are referred to as layers.

- A digraph with layering is called a layered digraph. The index of a layer that contains a node $a_i$ is denoted by $l(a_i, L)$, where $l(a_i, L) = p$ if and only if $a_i \in L_p$. A layering $L$ is proper if all edges of $\vec{G}$ satisfy $s(e, L) = l(a_i, L) - l(a_j, L) = 1$. 
Graph Dynamics (Diagraph)

**Adjacency Matrix**

\[
A = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

or

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & w_{14} & 0 \\
0 & w_{24} & 0 & 0 & 0 & 0 \\
0 & 0 & w_{34} & 0 & 0 & 0 \\
0 & 0 & 0 & w_{45} & 0 & 0 \\
0 & 0 & 0 & 0 & w_{51} & 0 \\
\end{bmatrix}
\]

**Diagonal Matrix**

\[
D = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2 \\
\end{bmatrix}
\]

**Laplacian Matrix**

\[
L = D - A = \begin{bmatrix}
1 & 0 & 0 & -1 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
-1 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 \\
-1 & -1 & 0 & 0 & 2 \\
\end{bmatrix}
\]

Note that \((I + L)\) is row stochastic.

**Figure:** Laplacian of Directed Graph (from A. Das, Graph Consensus: Autonomous and Controlled, slideshare.net)
Figure: Algorithms for extracting layered graphs (from Ronald Kieft, Cross Minimization, slideshare.net)
Longitudinal Influence Graph

\[ a_{0} \]
\[ a_{1} \]
\[ a_{2} \]
\[ a_{3} \]
\[ a_{4} \]
\[ a_{5} \]

\[ a_{ov} \]
Longitudinal Influence Graph

\[ a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_4 \rightarrow a_5 \]

The graph shows the longitudinal influence of variables labeled as \( a_0, a_1, a_2, a_3, a_4, \) and \( a_5 \). The directed edges indicate the influence direction, with \( a_0 \) influencing \( a_1 \), \( a_1 \) influencing \( a_2 \), and so on. The path from \( a_0 \) to \( a_5 \) through \( a_1 \) and \( a_2 \) is indicated by the dashed line labeled \( a_{ov} \).
Longitudinal Influence Graph

\[ a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_4 \rightarrow a_5 \]

\[ a_{ov} \]
Longitudinal Influence Graph

$\text{ao}_v$

$\text{a}_0$

$\text{a}_1$

$\text{a}_2$

$\text{a}_3$

$\text{a}_4$

$\text{a}_5$
Longitudinal Influence Graph

\[ a_0 \leftrightarrow a_1 \leftrightarrow a_3 \leftrightarrow a_2 \leftrightarrow a_5 \]

\[ a_{ov} \]
Longitudinal Influence Graph

- $a_0$
- $a_1$
- $a_2$
- $a_3$
- $a_4$
- $a_5$

Directed edges indicate influence directions.

$a_{ov}$
Longitudinal Influence Graph

- \( w_{10} = 1 \)
- \( w_{20} = 1 \)
- \( w_{31} \)
- \( w_{32} \)
- \( w_{41} = 1 \)
- \( w_{52} = 1 \)
Proper Layered Graph

Algorithm 1: Extracting $\vec{G}^y$ from $\vec{G}_{cone}$

Assumption: $\vec{G}_{cone}$ contains a directed spanning tree rooted at $a_0^y$

1. $a_0^y$ is the leader node.
2. Number others vehicles as per their $Y$-coordinates, i.e. for two vehicles $a_i$ and $a_j$, $i < j$ if $y_i \geq y_j$. This implies $y_0 > y_1 \geq y_2 \geq \cdots \geq y_n > 0$.
3. For vehicle $a_k$, $k \in \{1, \ldots, n\}$
   1. Calculate maximum path length $l$ from $a_0^y$
   2. Assign level $L_l$ for $a_k$
4. Remove all long edges from $\vec{G}_{cone}$. 
Proper Layered Graph

Algorithm 1: Extracting $\vec{G}_y$ from $\vec{G}_{cone}$

Assumption: $\vec{G}_{cone}$ contains a directed spanning tree rooted at $a_0^y$

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3. For vehicle $a_k$, $k \in \{1, \ldots, n\}$
   1. Calculate maximum path length $l$ from $a_0^y$
   2. Assign level $L_l$ for $a_k$
4. Remove all long edges from $\vec{G}_{cone}$.

Theorem

If $\vec{G}_{cone}$ contains a directed spanning tree rooted at $a_0^y$, Algorithm 1 generates a proper layered graph $\vec{G}_y$ with layers $L_l$, $l = 0, 1, 2, \ldots, m$. 

Either we have six pair of eyes or we have additional cow vision.

Each driver aims to position (X) himself in the middle of the nearest perceived obstacles.

Edges of usable roads are pseudo-cars.

Figure: Cow Vision
Lateral Influence Graph

Boundaries of the road

$a_0$  $a_2$  $a_4$  $a_1$  $a_3$  $a_5$
Lateral Influence Graph

Boundaries of the road
Lateral Influence Graph

Boundaries of the road

\( a_0 \)

\( a_1 \)

\( a_2 \)

\( a_3 \)

\( a_4 \)

\( a_5 \)
Lateral Influence Graph

Boundaries of the road

Diagram showing the influence graph with nodes labeled as $a_0$, $a_1$, $a_2$, $a_3$, $a_4$, and $a_5$. The edges between the nodes represent the influence or causality between them, with some edges labeled as 'aov'.
Lateral Influence Graph

Boundaries of the road

$w_{10}$ $w_{20}$ $w_{23}$ $w_{32}$ $w_{31}$ $w_{35}$ $w_{42}$ $w_{45}$
Longitudinal Motion

Lane-less driving model for each car - Y direction

- Pseudo-Leader - models road, car or driver induced velocity limits and travels at constant velocity
  \[
  \dot{y}_0 = v_{yo} \\
  \dot{v}_{y0} = 0
  \]

- Influencing Neighbors following model for \( i = 1, ..., n \).
  \[
  \dot{y}_i = v_{yi} \\
  \dot{v}_{yi} = \sum_{j=\mathcal{N}_i} \left( b_y w_{ij} (v_{yj} - v_{yi}) + k_y \left( w_{ij} (y_j - y_i) - \frac{g_y}{|\mathcal{N}_i|} \right) \right)
  \]

- Compare with classical single car following models (\( j \) is the leader for \( i \)):
  \[
  \dot{v}_{yi} = c \frac{(v_{yj} - v_{yi})}{(y_j - y_i - g_y)^2} \text{ with } j = i - 1
  \]
Longitudinal Motion

Shorthand using the Laplacian of the influence graph

\[ \dot{y}_i = v_{yi} \]

\[ \dot{v}_{yi} = \sum_{j=\mathcal{N}_i} \left( b_y w_{ij} (v_{yj} - v_{yi}) + k_y \left( w_{ij} (y_j - y_i) - \frac{g_y}{|\mathcal{N}_i|} \right) \right) \]

\[
\begin{bmatrix}
\dot{y} \\
\dot{v}_y
\end{bmatrix} =
\begin{bmatrix}
0_{(n+1)\times(n+1)} & I_{(n+1)\times(n+1)} \\
-k_y \mathcal{L}^y & -b_y \mathcal{L}^y
\end{bmatrix}
\begin{bmatrix}
y \\
v_y
\end{bmatrix} - k_y g_y \begin{bmatrix}
0_{(n+2)\times1} \\
1_{n\times1}
\end{bmatrix}
\]

Theorem

For time invariant influence graphs containing a directed spanning tree rooted at the leader, as \( t \to \infty \), \( v_{yi} \to v_{y0} \) \( \forall i \in \{1,...,n\} \)

\[ |y_i(t) - y_j(t)\| \to 0 \] for all vehicles in same layer

\[ |y_i(t) - y_j(t)\| \to g_y \] for vehicles in consecutive layers
Longitudinal Motion

Shorthand using the Laplacian of the influence graph

\[
\begin{align*}
\dot{y}_i &= v_{yi} \\
\dot{v}_{yi} &= \sum_{j=\mathcal{N}_i} \left( b_y w_{ij} (v_{yj} - v_{yi}) + k_y \left( w_{ij} (y_j - y_i) - \frac{g_y}{|\mathcal{N}_i|} \right) \right)
\end{align*}
\]

\[
\begin{bmatrix}
\dot{y} \\
\dot{v}_y
\end{bmatrix} =
\begin{bmatrix}
0_{(n+1)\times(n+1)} & I_{(n+1)\times(n+1)} \\
-k_y \mathcal{L}^y & -b_y \mathcal{L}^y
\end{bmatrix}
\begin{bmatrix}
y \\
v_y
\end{bmatrix} - k_y g_y \begin{bmatrix} 0_{(n+2)\times1} \\ 1_{n\times1} \end{bmatrix}
\]

Theorem

For time invariant influence graphs containing a directed spanning tree rooted at the leader, as \( t \to \infty \)

\[ v_{yi} \to v_{y0} \ \forall \ i \in \{1, \ldots n\} \]

\[ |y_i(t) - y_j(t)| \to 0 \text{ for all vehicles in same layer} \]

\[ |y_i(t) - y_j(t)| \to g_y \text{ for vehicles in consecutive layers} \]
Pro and Cons

**Pros:**
- Proof uses the fact that the Laplacian is triangular
- Can be extended easily for influences beyond a single layer
- Works well for relatively dense traffic where influence graphs have no chance to change

**Cons:**
- Cannot handle mesh stability (on the to do list)
Lane-less driving model for each car - X direction

\[
\begin{align*}
\dot{x}_i &= v_{xi} \\
\dot{v}_{xi} &= \sum_{j \in \mathcal{N}_i} (b_{xw_{ij}} (v_{xj} - v_{xi}) + k_{wx_{ij}} (x_{j} - x_{i}))
\end{align*}
\]
Lane-less driving model for each car -X direction

\[
\begin{align*}
\dot{x}_i &= v_{xi} \\
\dot{v}_{xi} &= \sum_{j \in \mathcal{N}_i} (b_{xw}w_{ij}(v_{xj} - v_{xi}) + k_{xw}w_{ij}(x_j - x_i))
\end{align*}
\]

\[
\begin{bmatrix}
\dot{x} \\
\dot{v}_x
\end{bmatrix} =
\begin{bmatrix}
0_{(n+2) \times (n+2)} & I_{(n+2) \times (n+2)} \\
-k_x \mathcal{L}^x & -b_x \mathcal{L}^x
\end{bmatrix}
\begin{bmatrix}
x \\
 v_x
\end{bmatrix}
\]
Lateral Motion

Lane-less driving model for each car -X direction

\[
\begin{align*}
\dot{x}_i &= v_{xi} \\
\dot{v}_{xi} &= \sum_{j \in \mathcal{N}_i} (b_x w_{ij} (v_{xj} - v_{xi}) + k_x w_{ij} (x_j - x_i))
\end{align*}
\]

\[
\begin{bmatrix}
\dot{x} \\
\dot{v}_x
\end{bmatrix} = \begin{bmatrix}
0_{(n+2)\times(n+2)} & I_{(n+2)\times(n+2)} \\
-k_x \mathcal{L}^x & -b_x \mathcal{L}^x
\end{bmatrix} \begin{bmatrix}
x \\
v_x
\end{bmatrix}
\]

Theorem

For time invariant influence graph (with appropriate connectedness assumptions), as \( t \to \infty \)

\( v_{xi} \to 0 \quad \forall \ i \in \{1, \ldots, n\} \)

Position \( x_i \) of each vehicle \( a_i, \ i = 1, \ldots, n \) converge to the weighted average of the X-positions of its neighbours
Time varying graphs

- Dwell time: $\tau$

Longitudinal dynamics
$$\begin{bmatrix}
\dot{y} \\
\dot{v} \\
y
\end{bmatrix} =
\begin{bmatrix}
0 \\
(n+1) \times (n+1) I \\
-(n+1) \times (n+1) I
\end{bmatrix}
\begin{bmatrix}
y \\
v \\
y
\end{bmatrix}
- k y L_y \sigma - b y L_y \sigma$$

Lateral dynamics
$$\begin{bmatrix}
\dot{x} \\
\dot{v} \\
x
\end{bmatrix} =
\begin{bmatrix}
0 \\
(n+2) \times (n+2) I \\
-(n+2) \times (n+2) I
\end{bmatrix}
\begin{bmatrix}
x \\
v \\
x
\end{bmatrix}
- k x L_x \sigma - b x L_x \sigma$$

Theorem (Simplified) For each $\sigma$, assume that $\vec{G}_\sigma$ contains a spanning tree rooted at $a_0$. For any $\tau > 0$, the states of the vehicles are uniformly bounded.
Time varying graphs

- **Dwell time**: $\tau$

- **Longitudinal dynamics**

\[
\begin{bmatrix}
\dot{y} \\
\dot{v}_y
\end{bmatrix} =
\begin{bmatrix}
\mathbf{0}_{(n+1) \times (n+1)} & I_{(n+1) \times (n+1)} \\
-k_y \mathcal{L}^y_{\sigma} & -b_y \mathcal{L}^y_{\sigma}
\end{bmatrix}
\begin{bmatrix}
y \\
v_y
\end{bmatrix} - k_y g_y
\begin{bmatrix}
\mathbf{0}_{(n+2) \times 1} \\
\mathbf{1}_{n \times 1}
\end{bmatrix}
\]

- **Lateral dynamics**

\[
\begin{bmatrix}
\dot{x} \\
\dot{v}_x
\end{bmatrix} =
\begin{bmatrix}
\mathbf{0}_{(n+2) \times (n+2)} & I_{(n+2) \times (n+2)} \\
-k_x \mathcal{L}^x_{\sigma} & -b_x \mathcal{L}^x_{\sigma}
\end{bmatrix}
\begin{bmatrix}
x \\
v_x
\end{bmatrix}
\]
Time varying graphs

- **Dwell time**: $\tau$

- **Longitudinal dynamics**

\[
\begin{bmatrix}
\dot{y} \\
\dot{v}_y
\end{bmatrix} =
\begin{bmatrix}
0_{(n+1) \times (n+1)} & I_{(n+1) \times (n+1)} \\
-k_y \mathcal{L}^y_\sigma & -b_y \mathcal{L}^y_\sigma
\end{bmatrix}
\begin{bmatrix}
y \\
v_y
\end{bmatrix} - k_y g_y
\begin{bmatrix}
0_{(n+2) \times 1} \\
1_{n \times 1}
\end{bmatrix}
\]

- **Lateral dynamics**

\[
\begin{bmatrix}
\dot{x} \\
\dot{v}_x
\end{bmatrix} =
\begin{bmatrix}
0_{(n+2) \times (n+2)} & I_{(n+2) \times (n+2)} \\
-k_x \mathcal{L}^x_\sigma & -b_x \mathcal{L}^x_\sigma
\end{bmatrix}
\begin{bmatrix}
x \\
v_x
\end{bmatrix}
\]

**Theorem**

*(Simplified)* For each $\sigma$, assume that $\mathcal{G}_\sigma$ contain a spanning tree rooted at $a_0$. For any $\tau > 0$, the states of the vehicles are uniformly bounded.
Changing Graphs

Figure: The convoy of vehicles with the influence graphs

(a) Change in $\tilde{G}^x$
(b) Change in $\tilde{G}^y$ and $\tilde{G}^x$

Figure: Switching influence graphs
Other Complex Behaviours

Figure: Changes in influence graph due to obstacle $a_4$
Inter-vehicle spacing is usually based on two factors:
- **Constant time headway policy distance** i.e. the distance needed to decelerate to zero = \( k_i v_i = \tilde{k} v_i \).
- Desired constant spacing \( g_y > 0 \) so that, when \( v_0 = 0 \), the inter-vehicle spacing is not zero.

Modified dynamics:

\[
\begin{align*}
\dot{y}_i &= v_{yi} \\
\dot{v}_{yi} &= \sum_{j=\mathcal{N}_i} \left( b_y w_{ij} (v_{yj} - v_{yi}) + k_y \left( w_{ij} (y_j - y_i) + \frac{1}{|\mathcal{N}_i|} (g_y - \tilde{k} v_i) \right) \right)
\end{align*}
\]

**Theorem:** Under usual assumptions, if \( \tilde{k} > 0 \) and the velocity \( v_{y0} \) of \( a_0 \) is constant, then:

1. \( v_{yi} \to v_{y0} \) as \( t \to \infty \) \( \forall \ i \in \{1, \ldots, n\} \)
2. \( |y_i(t) - y_j(t)| \to 0 \) as \( t \to \infty \) for all \( a_i, a_j \in L_k, k = 1, 2, \ldots, m \).
3. For any two vehicles \( a_i \) and \( a_j \) such that \( a_i \in L_{k-1} \) and \( a_j \in L_k, k = 1, 2, \ldots, m \), as \( t \to \infty \), \( |y_i(t) - y_j(t)| \to g_y + \tilde{k} v_{y0} \).
Switch to ppt
Conclusions

- First attempt at lane-less traffic modeling
- Stability issues were solved
- Standard questions in transportation engineering e.g. flow, capacity etc. are still open.
- Heterogeneity not addressed
- Simulation based on proposed law should be validated with macro level data
Thank You!