On characteristic cones of discrete autonomous nD systems: theory and algorithm

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Abstract

In this paper, we provide a complete answer to the question of characteristic cones for a general discrete autonomous nD system described by partial difference equations with real constant coefficients. A characteristic cone is a special subset (having the structure of a cone) of the domain, here $\mathbb{Z}^n$, such that the knowledge of the trajectories on this set uniquely determines them over the whole domain. The study of characteristic sets is relevant in many system-theoretic aspects. This importance of characteristic sets stems from the fact that they help quantify the 'information' required to solve a system of partial difference equations. In spite of its importance, the issue of characteristic sets for multidimensional systems have not been explored in its full generality except for Valcher's seminal work for the special case of 2D systems in 2000. This apparent lack of progress in the last fifteen years is perhaps due to inapplicability of a crucial intermediate result by Valcher to cases with $n > 2$. We illustrate this inapplicability of the above-mentioned result in Section 3 with the help of an example. We then provide an answer to this open problem of characterizing characteristic cones for general n by proving a necessary and sufficient condition for a cone to be a characteristic cone for a given system of partial difference equations. In the second

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part of the paper we convert this necessary and sufficient condition to another equivalent algebraic condition, which is more suited from an algorithmic perspective. Using this result, we provide algorithms, based on Gröbner bases theory, that are implementable using standard computer algebra packages for testing whether a given cone is a characteristic cone for a given system of nD partial difference equations.

1 Introduction

Research on multidimensional systems theory has been steadily progressing in the past few decades benefiting from the interaction with modern algebraic and analytic geometry. A large number of highly diverse applications such as, image and signal processing, seismic data processing, repetitive processes, delay-differential systems, distributed systems etc. use the theory developed for multidimensional systems. (See [1] for some case studies.)

In this paper, by nD systems we mean systems that are described by linear partial difference equations with real constant coefficients. One of the fundamental problems in nD systems theory is concerning their *characteristic sets*. By a characteristic set we mean a subset of the domain (here, $\mathbb{Z}^n$), with the defining property that, for every trajectory in the system, the knowledge of its values on the characteristic set uniquely identifies the trajectory over the whole domain. The question of characteristic sets is irrelevant for systems having inputs/free variables. This is because free variables can take arbitrary values over the entire domain $\mathbb{Z}^n$ and therefore no proper subset of $\mathbb{Z}^n$ can be a characteristic set. Systems having no free variables are called *autonomous*; these are the systems that admit proper subsets of the domain as characteristic sets [2], [3]. In this paper, we focus on the question of finding *characteristic cones* (characteristic sets that have the structure of a cone) for the above-mentioned type of nD systems, i.e., autonomous systems described by simultaneous linear partial difference equations with constant real coefficients. We provide a complete algebraic characterization of characteristic cones for general autonomous nD systems, with $n \geqslant 2$.

The necessity of studying characteristic sets, and their properties, arise because of their applicability in studying a number of system-theoretic properties such as stability [3], [4], Markovian-ness [5], finite dimensionality [3], [2], time/space-relevance [6], [7], Lyapunov theory of nD systems [8], [9] and many more. Characteristic sets also play a central role in the canonical Cauchy problem [10]. For 1D discrete autonomous systems, it is known that a characteristic set is always a finite collection of points (See [11]). However, for multidimensional systems with $n \geqslant 2$, characteristic sets are often infinite. Moreover, for $n \geqslant 2$, characteristic sets may come in numerous different shapes and sizes. For example, it was shown in [12] that every 2D autonomous system admits a finite union of parallel lines as characteristic sets.

As far as characteristic sets of nD systems are concerned, *cones* stand out among various types of subsets of the domain. This special status of cones arises from their significance in stability analysis of nD systems. Indeed, the literature on stability of nD systems has been predominantly concerned with stability with respect to cones [4], [3]. A possible reason for this predominance is perhaps the natural ability of cones to provide a dichotomy of the domain into 'past' and 'future' [4], [3]. A cone is a collection of half-lines called *rays*, and stability with respect to a cone has the natural meaning that every trajectory of the system must die down to zero along every such ray as infinity is approached. However, in nD systems, often such half-lines turn out to be *free* (see [13]). Meaning, the values that an arbitrary trajectory takes on the half-line are freely assignable. Note that the question of stability then becomes relevant only for those half-lines that are not-free. Interestingly, the set of non-free half-lines often forms a thin set [13, Theorem 28]. In this scenario, the question of conic stability becomes irrelevant for most of the situations. However, a remedy to this conundrum can be obtained in the following manner (see [3], [14]): stability along a cone should be asked of those trajectories whose 'initial conditions' are well-behaved. In this sense, it was proposed in [3] that in order to answer the question of stability with respect to a cone $\mathcal{C}$ the negative cone $-\mathcal{C}$ must be a characteristic cone so that the values that a trajectory takes on $-\mathcal{C}$ may serve as initial conditions. It is this point of view that we take our inspiration from for this paper: given a cone in the domain and the describing difference equations of an nD system, check whether the cone is a characteristic set for the system or not. Note that, in this paper, we do not delve into the question of conic stability at all; settling the issue of conic stability using the ideas and algorithms developed in this paper will be the focus of our future work.

The problem of determining if a given cone is a characteristic cone for a discrete nD system, with $n = 2$, was studied in meticulous detail by Valcher in [3]. Valcher's method of determining if a given cone is a characteristic cone for a 2D system relies on the fact that every 2D autonomous system can be decomposed into a finite dimensional subsystem and a square autonomous subsystem [3, Proposition 4.1]. However, such a decomposition does not always exist for $n \geqslant 3$ (see Example 2 in Section 3). Thus, Valcher's method becomes inapplicable for higher dimensional systems, with $n \geqslant 3$. Interestingly, the question of characterizing characteristic cones for nD systems with $n \geqslant 3$ has remained open since Valcher's seminal contribution in the $n = 2$ case. Perhaps, the above-mentioned inapplicability of Valcher's crucial reduction step [3, Proposition 4.1] for the $n \geqslant 3$ case made it impossible to have any straight-forward extension of Valcher's methods to $n \geqslant 3$ case. This further could have caused the apparent lack of progress in the study of characteristic sets for $n \geqslant 3$ case. In this paper, we provide a complete solution to this problem that has remained open for the past fifteen years.

A crucial observation that helps us in circumventing the problem associated with the above-mentioned decomposition ([3, Proposition 4.1]) (of autonomous nD systems with $n \geqslant 3$) is that cones (under a mild assumption of rationality) in higher dimensions have a rich algebraic structure: they are *affine semigroups* (See [15, Chapter 7] for more details). We show in this paper how this structure can be exploited to give a necessary and sufficient algebraic condition for a given cone to be a characteristic set for a general autonomous nD system (Theorem 4). We further provide an algorithm for checking the same using Gröbner basis. We also delineate a crucial reduction process that converts the algebraic condition given by Theorem 4 to an equivalent statement in terms of modules over polynomial rings so as to ensure standard application of the Gröbner basis techniques.

The paper is organized as follows: Notation and preliminaries required for this paper are stated in Section 2. Section 3 explains why extension of Valcher's result for $n \geqslant 3$ is not possible. The relation between polyhedral cones, affine semigroups and the algebra generated by them is discussed in Section 4. One of the main results, the algebraic characterization, is presented in Section 5. The second main result, the algorithm, is stated in Section 6. We conclude the paper with some examples in Section 7.

2 Notation and Preliminaries

2.1 Notation

The notation used is standard. We use the symbols $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ to denote, respectively, the ring of integers, the field of real numbers and the field of complex numbers. The set of n tuples of integers, real numbers and complex numbers are denoted by $\mathbb{Z}^n$, $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively. The set of non-negative real numbers is denoted by $\mathbb{R}_{\geqslant 0}$ and the set of all non-negative n-tuples of integers is denoted by $\mathbb{Z}^n_{\geqslant 0}$. We use $\mathbb{R}[\boldsymbol{\xi}]$ to denote the ring of polynomials in n variables $\xi_1, \xi_2, \ldots, \xi_n$ having real coefficients. For Laurent polynomial rings in n variables with real coefficients $\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}] =: \mathcal{A}$ is used. A monomial in $\mathcal{A}$ is of the form $\boldsymbol{\xi}^{\boldsymbol{\nu}} = \xi_1^{\nu_1} \ldots \xi_n^{\nu_n}$ where $\boldsymbol{\nu} \in \mathbb{Z}^n$. We use the symbol $\mathcal{W}$ to denote the set of all scalar trajectories, that is, $\mathcal{W} := \{w : \mathbb{Z}^n \to \mathbb{R}\}$. We use $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{-1}$ to denote the n-tuples of shift operators and inverse shift operators respectively, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $\boldsymbol{\sigma}^{-1} = (\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_n^{-1})$. The action of the i-th shift operator, σ_i, on a scalar trajectory $w \in \mathcal{W}$ is defined in the following manner:

$$(\sigma_i w)(\boldsymbol{k}) = w(k_1, k_2, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_n). \tag{1}$$

The symbol $\bullet$ is used for denoting a quantity which is unspecified. For example, $R(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A}^{\bullet \times q}$ means R is a matrix having entries from $\mathcal{A}$ with q columns and an unspecified number of rows. For a set Γ, $|\Gamma|$ denotes the cardinality of Γ.

2.2 Discrete nD systems

A discrete nD system is described by a set of partial difference equations having n independent variables. The difference equations are succinctly written in terms of n shift operators $\sigma_1, \sigma_2, \ldots, \sigma_n$. These shift operators act on trajectories w_i's which are real valued multi-indexed sequences with the n dimensional integer grid, $\mathbb{Z}^n$, as the indexing set (also called the *domain*). In other words, for an integer n-tuple $\boldsymbol{k} = (k_1, k_2, \ldots, k_n)$ and a trajectory $w_i \in \mathcal{W}$, we have $w_i(k_1, k_2, \ldots, k_n) \in \mathbb{R}$, that is, $w_i : \mathbb{Z}^n \to \mathbb{R}$.

The action of a Laurent monomial, $\boldsymbol{\xi}^{\boldsymbol{\nu}} = \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_n^{\nu_n}$, on a scalar trajectory $w_i \in \mathcal{W}$ is defined in the following manner:

$$(\boldsymbol{\sigma}^{\boldsymbol{\nu}})\, w_i(\boldsymbol{k}) = w_i(\boldsymbol{k}+\boldsymbol{\nu}) = \left(\boldsymbol{\sigma}^{\boldsymbol{k}+\boldsymbol{\nu}}\right) w_i(\boldsymbol{0}) = w_i(k_1+\nu_1,\dots,k_n+\nu_n). \quad (2)$$

A Laurent polynomial, $f \in \mathcal{A}$, is a finite linear combination of Laurent monomials; that is,

$$f(\boldsymbol{\xi},\boldsymbol{\xi}^{-1}) \in \mathcal{A} \Rightarrow f = \sum_{\boldsymbol{\nu}\in\Gamma} \alpha_{\boldsymbol{\nu}}\boldsymbol{\xi}^{\boldsymbol{\nu}} \quad (3)$$

where, $\Gamma \subseteq \mathbb{Z}^n$ is finite and $\alpha_{\boldsymbol{\nu}} \in \mathbb{R}$. The action of a Laurent polynomial on a scalar trajectory is defined as

$$f(\boldsymbol{\sigma},\boldsymbol{\sigma}^{-1})w_i := \sum_{\boldsymbol{\nu}\in\Gamma} \alpha_{\boldsymbol{\nu}}\boldsymbol{\sigma}^{\boldsymbol{\nu}} w_i. \quad (4)$$

Thus $f : \mathcal{W} \to \mathcal{W}$.

Using this definition, a row of Laurent polynomials

$$\boldsymbol{r}(\boldsymbol{\xi},\boldsymbol{\xi}^{-1}) := \begin{bmatrix} r_1(\boldsymbol{\xi},\boldsymbol{\xi}^{-1}) & r_2(\boldsymbol{\xi},\boldsymbol{\xi}^{-1}) & \dots & r_q(\boldsymbol{\xi},\boldsymbol{\xi}^{-1}) \end{bmatrix} \in \mathcal{A}^{1\times q}$$

can be made to act on a column of trajectories $w := \mathrm{col}(w_1, w_2, \dots, w_q) \in \mathcal{W}^q$ in the following manner:

$$\boldsymbol{r}(\boldsymbol{\sigma},\boldsymbol{\sigma}^{-1})w := \sum_{i=1}^{q} r_i(\boldsymbol{\sigma},\boldsymbol{\sigma}^{-1})w_i. \quad (5)$$

We call $w \in \mathcal{W}^q$ a vector-valued trajectory and view it as a column vector.

Thus a discrete nD system described by a system of partial difference equations with real constant coefficients can be represented as

$$R(\boldsymbol{\sigma},\boldsymbol{\sigma}^{-1})w = 0 \quad (6)$$

where, $R(\boldsymbol{\xi},\boldsymbol{\xi}^{-1}) \in \mathcal{A}^{\bullet\times q}$. In this article, we always consider that a q-tuple of polynomials in the shift operators act on the vector valued trajectory w as defined in equation (5). It is important to note that the rows of $R(\boldsymbol{\xi},\boldsymbol{\xi}^{-1})$ form a *submodule* of the free module $\mathcal{A}^{1\times q}$. We elaborate more on this in the following section.

2.3 Kernel representation

Given a discrete nD system described by a system of partial difference equations, the collection of all trajectories that satisfy the system of equations (6) is known as the *behavior* of the system and is denoted by $\mathfrak{B}$. That is,

$$\mathfrak{B} := \left\{ w \in \mathcal{W}^q \mid R(\boldsymbol{\sigma},\boldsymbol{\sigma}^{-1})w = 0 \right\} = \ker R(\boldsymbol{\sigma},\boldsymbol{\sigma}^{-1}).$$

This is known as a *kernel representation* of the behavior $\mathfrak{B}$ and $R(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})$ is called a *kernel representation matrix.*

It is known that kernel representations are not unique. However, if two distinct kernel representations have the same *row-span* over $\mathcal{A}$ then they give rise to the same behavior. Indeed, let $R(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A}^{\bullet \times q}$ and $\mathcal{R} :=$ rowspan$_{\mathcal{A}} R(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})$, then the behavior $\mathfrak{B} = \ker R(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})$ is equivalently given by

$$\mathfrak{B}(\mathcal{R}) = \left\{ w \in \mathcal{W}^q | \boldsymbol{f}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w = 0 \;\forall\; \boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{R} \right\}. \tag{7}$$

The submodule $\mathcal{R}$ of the free module $\mathcal{A}^{1\times q}$ generated by the rows of a kernel representation matrix $R(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})$ is known as the *equation module* of the behavior $\mathfrak{B}$. It was shown in [16] that submodules of $\mathcal{A}^{1\times q}$ and discrete nD behaviors having q dependent variables are in one-to-one inclusion reversing correspondence with each other.

A behavior $\mathfrak{B}$ defined by a kernel representation, or, equivalently by an equation module, is closed under addition and under multiplication by scalars in $\mathbb{R}$. Thus $\mathfrak{B}$ has the structure of an $\mathbb{R}$-vector space. Further, $\mathfrak{B}$ is also closed under multiplication by scalars from $\mathcal{A}$, where scalar multiplication by an $f \in \mathcal{A}$ to a trajectory $w \in \mathfrak{B}$ is defined as the component-wise action $f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w$ given by equation (4). For $w \in \mathfrak{B}$ we have $f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w \in \mathfrak{B}$ for all $f \in \mathcal{A}$. Thus $\mathfrak{B}$ also has the structure of a module over $\mathcal{A}$.

2.4 Quotient Module

We now elaborate on the algebraic notion of a *quotient module* which will be of crucial importance in the sequel. Given an equation module $\mathcal{R}$, the quotient module $\mathcal{M} := \mathcal{A}^{1\times q}/\mathcal{R}$ is the set of all equivalence classes originating from the equivalence relation on $\mathcal{A}^{1\times q}$ defined as: two elements $\boldsymbol{f_1}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}), \boldsymbol{f_2}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A}^{1\times q}$ are related if $\boldsymbol{f_1} - \boldsymbol{f_2} \in \mathcal{R}$. For an element $\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A}^{1\times q}$, its equivalence class is denoted by $\overline{\boldsymbol{f}}$. This gives us the $\mathcal{A}$-module homomorphism $\mathcal{A}^{1\times q} \twoheadrightarrow \mathcal{M}$, called the *canonical surjection*, where, every element in $\mathcal{A}^{1\times q}$ is mapped to its equivalence class in $\mathcal{M}$.

For an element $\boldsymbol{r}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A}^{1\times q}$ let $\overline{\boldsymbol{r}} = m \in \mathcal{M}$ be the image of $\boldsymbol{r}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})$ under the canonical surjection. We call $\boldsymbol{r}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})$ a *lift* of m. The action of elements from the quotient module on trajectories is defined in the following manner: suppose $m \in \mathcal{M}$ and $w \in \mathfrak{B}$,

$$m(w) := \big(\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w\big) = \sum_{i=1}^{q} r_i(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w_i \tag{8}$$

where, $\boldsymbol{r}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A}^{1\times q}$ is a lift of m. It is important to note that m may have several distinct lifts in $\mathcal{A}^{1\times q}$, but all of them have the same action on $w \in \mathfrak{B}$. This can be seen from the following argument: let $\boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathcal{A}$ be two distinct lifts of the same $m \in \mathcal{M}$. It then follows from the definition of $\mathcal{M}$ that $\boldsymbol{r}_1 - \boldsymbol{r}_2 \in \mathcal{R}$. However, since $r(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w = 0$ for all $\boldsymbol{r} \in \mathcal{R}$ and $w \in \mathfrak{B}$, we get that $(\boldsymbol{r}_1(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) - \boldsymbol{r}_2(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}))\, w = \boldsymbol{r}_1(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w - \boldsymbol{r}_2(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w = 0$. Therefore, $\boldsymbol{r}_1(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w = \boldsymbol{r}_2(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w$. Thus, the definition of action of $m \in \mathcal{M}$ on $\mathfrak{B}$ is independent of the choice made in getting the lift of m. In other words, the action of $\mathcal{M}$ on $\mathfrak{B}$ is well-defined. Note that $\mathcal{M}$ also has the structure of an $\mathcal{A}$-module and an $\mathbb{R}$-vector space.

2.5 Autonomous systems

An nD system is called autonomous if it does not have any free variables. Several equivalent conditions for nD autonomous systems can be found; in [17] it was shown that 2D autonomous systems have a full column rank kernel representation matrix. This is equivalent to the condition that 2D autonomous systems have proper subsets of $\mathbb{Z}^2$ as characteristic cones as shown in [3]. In this paper, we follow the definition of autonomy given in [4]; this definition is equivalent to the above-mentioned ones [3], [17]. In order to state this definition of autonomy, we need the following algebraic objects associated with an nD behavior: *characteristic ideal*, *characteristic variety* and *annihilator ideal*.

Let the behavior $\mathfrak{B}$ of a discrete nD system be given by a kernel representation $\mathfrak{B} = \ker\ (R(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}))$ with $R(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A}^{\bullet\times q}$. The characteristic ideal of $\mathfrak{B}$, denoted by $\mathcal{I}(\mathfrak{B})$, is defined as the ideal of $\mathcal{A}$ generated by the $(q \times q)$ minors of $R(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})$. Associated with the characteristic ideal is the characteristic variety which is defined as the set

$$\mathbb{V}(\mathfrak{B}) := \{\boldsymbol{\zeta} \in \mathbb{C}^n \setminus \{\mathbf{0}\} \mid f(\boldsymbol{\zeta}) = 0 \text{ for all } f \in \mathcal{I}(\mathfrak{B})\}\,.$$

If the number of rows in $R(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})$ is less than the number of columns q, then $\mathcal{I}(\mathfrak{B})$ is defined to be the zero ideal and $\mathbb{V}(\mathfrak{B})$ is all of $\mathbb{C}^n \setminus \{\mathbf{0}\}$. An element m of a module $\mathcal{M}$ over a ring $\mathcal{A}$ is called a *torsion element* if there exists an element $f \in \mathcal{A}$ such that $fm = 0 \in \mathcal{M}$. A module $\mathcal{M}$ is called a *torsion module* if all its elements are torsion elements. Let $\mathcal{M}$ be a torsion module over the Laurent polynomial ring $\mathcal{A}$. The collection of all polynomials $f \in \mathcal{A}$, whose actions on all elements from $\mathcal{M}$ produce zero, has the structure of an ideal; this ideal is called the *annihilator ideal.* That is,

$$\operatorname{ann} \mathcal{M} := \left\{ f(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A} \,\middle|\, f(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})m = 0 \ \forall \ m \in \mathcal{M} \right\}.$$

A discrete nD system is said to be autonomous if the characteristic ideal $\mathcal{I}(\mathfrak{B})$ is nonzero. Equivalently, a behavior $\mathfrak{B}$ is autonomous if and only if the quotient module $\mathcal{M}$ is a torsion module. This, in turn, is equivalent to the annihilator ideal, ann $\mathcal{M}$ being non-zero. Further, an autonomous behavior is said to be *strongly autonomous* if the quotient ring $\mathcal{A}/\mathcal{I}(\mathfrak{B})$ is a finite dimensional vector space over $\mathbb{R}$. In other words, $\mathfrak{B}$ is strongly autonomous if and only if its characteristic variety $\mathbb{V}(\mathfrak{B})$ is a finite set [4].

2.6 Characteristic sets

We first define what we mean by restriction of a trajectory to a subset of the domain. Given a trajectory $w : \mathbb{Z}^n \to \mathbb{R}^q$ and a subset $\mathcal{C} \subseteq \mathbb{Z}^n$, the restriction of w to $\mathcal{C}$, denoted by $w|_{\mathcal{C}}$, is defined as

$$\begin{aligned} w|_{\mathcal{C}} &: \mathcal{C} \to \mathbb{R}^q \\ w|_{\mathcal{C}}(\boldsymbol{k}) &= w(\boldsymbol{k}) \ \forall \ \boldsymbol{k} \in \mathcal{C}. \end{aligned} \tag{9}$$

For a discrete autonomous nD system with behavior $\mathfrak{B}$, a characteristic set is a special subset of the domain (here $\mathbb{Z}^n$) that has the property that every trajectory $w \in \mathfrak{B}$ can be uniquely extended to the entire domain with the knowledge of w restricted to this set. The formal definition of a characteristic set is adopted from [3].

Definition 1. *Given a behavior $\mathfrak{B}$, a subset $\mathcal{C}$ of $\mathbb{Z}^n$ is said to be a* characteristic set *for $\mathfrak{B}$ if for every trajectory w in $\mathfrak{B}$, the restriction of w to the set $\mathcal{C}$, allows to uniquely determine the remaining portion of w, i.e., $w|_{\mathbb{Z}^n \setminus \mathcal{C}}$ can be uniquely determined if $w|_{\mathcal{C}}$ is known.*

Throughout this paper, we consider proper cones in $\mathbb{Z}^n$ as characteristic sets and call them *characteristic cones.* Valcher gives a complete description of characteristic cones for autonomous 2D behaviors in [3]. She also proposes a method to check if a given cone is a characteristic cone for a 2D autonomous behavior. However, Valcher's method becomes inapplicable for higher dimensional systems with $n \geqslant 3$. We elaborate on this in the next section.

3 Why Valcher's results do not extend to $n \geqslant 3$?

Valcher's crucial observation was that every discrete 2D autonomous behavior can be decomposed as a sum of two special type of autonomous behaviors. These two special subclasses of autonomous behaviors are: *finite dimensional behaviors* and *square behaviors.* A finite dimensional behavior is nothing but a strongly autonomous behavior. On the other hand, square autonomous behaviors are defined as kernels of nonsingular square Laurent polynomial matrices. Valcher's method of determining whether a given cone is a characteristic cone for a 2D autonomous behavior, heavily uses this decomposition.

More elaborately, given a 2D autonomous behavior $\mathfrak{B} = \ker R$ where, $R \in \mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}]^{g \times q}$, it was shown in [3, Proposition 4.1] that $\mathfrak{B}$ can be decomposed as $\mathfrak{B} = \mathfrak{B}_{\text{fd}} + \mathfrak{B}_{\text{sq}}$, where $\mathfrak{B}_{\text{fd}}$ is a finite dimensional behavior and $\mathfrak{B}_{\text{sq}}$ is a square behavior. This decomposition is done in the following manner. A kernel representation matrix $R \in \mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}]^{g \times q}$ for $\mathfrak{B}$ can always be factorized as $R = \widetilde{R}\Delta$ where, $\widetilde{R} \in \mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}]^{g \times q}$ is *right-factor-prime*[1] and $\Delta \in \mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}]^{q \times q}$ is square and non-singular. It then follows that by defining $\mathfrak{B}_{\text{sq}} := \ker \Delta$ there exists a right-factor-prime matrix R_{fd} such that $\mathfrak{B}_{\text{fd}} := \ker R_{\text{fd}}$ is a finite dimensional behavior and $\mathfrak{B}$ can be written as $\mathfrak{B} = \mathfrak{B}_{\text{fd}} + \mathfrak{B}_{\text{sq}}$. The construction of R_{fd} can be found in the proof of [3, Proposition 4.1]. While it is clear why $\ker \Delta$ is square, the fact that $\ker R_{\text{fd}}$ is finite dimensional (strongly autonomous) follows from R_{fd} being right-factor-prime [19].

Using this decomposition, it was shown in [3, Proposition 2.6] that a *proper* cone[2] is a characteristic cone for $\mathfrak{B}$ if and only if it is a characteristic

[1] A matrix $P(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})$ is called left-factor-prime if any decomposition $P = EP_1$, where E is square, implies E is unimodular. A matrix $R(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})$ is said to be right-factor-prime if $R^T(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})$ is left-factor-prime. See [18] for more details.

[2] A closed, pointed, solid, convex cone is called a *proper cone*; we elaborate more on

cone for $\mathfrak{B}_{\mathrm{sq}}$. It was further shown that a proper cone is a characteristic set for the square behavior $\mathfrak{B}_{\mathrm{sq}}$ if and only if it is a characteristic set for the scalar behavior $\mathfrak{B}_\delta$, where $\mathfrak{B}_\delta := \ker(\det \Delta)$. Thus the problem of determining if a given proper cone is a characteristic cone for a 2D behavior was reduced to checking if the cone is a characteristic cone for such a scalar behavior, which is the kernel of a single polynomial. For a scalar behavior then the verification of whether a proper cone is a characteristic cone was done by a neat graphical method [3, Proposition 2.8].

Obviously, this analysis holds if the above-mentioned decomposition exists. Thus, in order to extend Valcher's graphical method to nD systems, with $n \geqslant 3$, an extension of the decomposition result becomes mandatory. Unfortunately, the decomposition does not extend for $n \geqslant 3$ as we show in Example 2 below.

Example 2. Consider the 3D discrete autonomous system $\mathfrak{B} = \ker R$, where $R = \begin{bmatrix} 1+\sigma_1 \\ 1+\sigma_2 \end{bmatrix} \in \mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \sigma_3^{\pm 1}]$. Note that R is already right-factor-prime. So, as per the above-mentioned decomposition of 2D behaviors, the square part of $\mathfrak{B}$ here is just $\{0\}$. In this scenario, for the decomposition to work, the square part $\mathfrak{B}_{\mathrm{sq}}$ needs to be zero, which forces $\mathfrak{B}$ to be equal to $\mathfrak{B}_{\mathrm{fd}}$. However, note that this is *impossible*, for $\mathfrak{B}$ is not finite dimensional, although R is right-factor-prime. (Indeed, $\mathfrak{B} = \ker R$ cannot be finite dimensional because the characteristic variety is not a finite collection of points.)

Thus extension of Valcher's method for higher dimensional systems with $n \geqslant 3$ is impossible. However, the question of characterizing characteristic cones for nD systems with $n \geqslant 3$ still remains relevant and interesting. Here, it is important to note that for 2D a proper cone has a relatively simple structure: every proper cone in $\mathbb{R}^2$ is the collection of points that are *non-negative* linear combination of two independent vectors in $\mathbb{R}^2$. However, this is not the case for cones in higher dimensions. For example, in $\mathbb{R}^3$ a cone can be given by the intersection of four half-spaces thus forming a cone with a quadrilateral base, that is, 4 generating vectors. Hence a cone in $\mathbb{R}^n$ can have a generating set whose cardinality is more than n. Therefore, the first step in solving the problem of characteristic cones would be to

this in Section 4.

understand the structure of cones in higher dimensions. Interestingly, cones in $\mathbb{Z}^n$ have a rich algebraic structure – that of an affine semigroup [15]. In the following section we discuss in short this structure of polyhedral cones as affine semigroups. For details readers are referred to [15]. This lays the groundwork on which the algebraic analysis of characteristic cones would be based in the sequel.

4 Polyhedral cones, affine semigroups and semigroup-algebras

A set $\mathcal{C} \subseteq \mathbb{R}^n$ is called a *cone* if $\lambda\mathcal{C} \subseteq \mathcal{C}$ for all $\lambda \in \mathbb{R}_{\geqslant 0}$. If there exist vectors $\boldsymbol{c_1}, \dots, \boldsymbol{c_d} \in \mathbb{R}^n$ such that

$$\mathcal{C} = \{\lambda_1 \boldsymbol{c_1} + \dots + \lambda_d \boldsymbol{c_d} \mid \lambda_1, \dots, \lambda_d \in \mathbb{R}_{\geqslant 0}\},$$

then the cone $\mathcal{C}$ is said to be finitely generated by $\boldsymbol{c_1}, \dots, \boldsymbol{c_d}$ and is called a *polyhedral cone*. Further, $\mathcal{C}$ is called a *rational* cone if the generating vectors $\boldsymbol{c_1}, \dots, \boldsymbol{c_d}$ are vectors of rational numbers.

A cone $\mathcal{C}$ is said to be *convex* if the line segment joining any two points in the cone also belongs to the cone. A convex cone is *solid* if it contains an open ball of $\mathbb{R}^n$ and it is *pointed* if $\mathcal{C} \cap -\mathcal{C} = \{0\}$. A closed, pointed, solid, convex cone is a *proper* cone.

A *semigroup* is a subset of a group which is closed under the group operation and follows associativity. A semigroup is an *affine semigroup* if it is isomorphic to a subsemigroup of $\mathbb{Z}^d$ for some d. According to Gordan's Lemma ([15, Theorem 7.16]), for every rational proper cone $\mathcal{C} \subseteq \mathbb{R}^n$, the intersection $\mathcal{C} \cap \mathbb{Z}^n$ is an affine sub-semigroup of the Abelian group $\mathbb{Z}^n$ (under addition as the group operation). It further follows from [15, Proposition 7.15, Theorem 7.16] that such a cone $\mathcal{C} \cap \mathbb{Z}^n$ admits a representation

$$\mathcal{C} \cap \mathbb{Z}^n = \left\{\lambda_1 \boldsymbol{c_1} + \dots + \lambda_r \boldsymbol{c_r} \mid \lambda_1, \dots, \lambda_r \in \mathbb{Z}_{\geqslant 0}\right\}, \tag{10}$$

where $\boldsymbol{c_1}, \dots, \boldsymbol{c_r} \in \mathbb{Z}^n$. In this paper, by a *cone in* $\mathbb{Z}^n$ we mean the intersection of $\mathcal{C} \subseteq \mathbb{R}^n$ with $\mathbb{Z}^n$, where, $\mathcal{C}$ is a proper rational polyhedral cone. From now on, we do a slight abuse of notation by using $\mathcal{C}$ to mean a cone in $\mathbb{Z}^n$ which, as mentioned above, actually is the intersection of a rational proper cone in $\mathbb{R}^n$ with $\mathbb{Z}^n$.

Let $\mathcal{C} \subseteq \mathbb{Z}^n$ be a cone (that is, the intersection of a proper rational cone in $\mathbb{R}^n$ with $\mathbb{Z}^n$). As mentioned above, $\mathcal{C}$ has the structure of an affine sub-semigroup of $\mathbb{Z}^n$. Therefore, $\mathcal{C}$ is closed under addition, and $\mathcal{C}$ admits a representation given by equation (10). The *semigroup algebra*, denoted by $\mathbb{R}[\mathcal{C}]$, is the $\mathbb{R}$-vector space of finite linear combinations of monomials having their exponent tuples in $\mathcal{C}$. That is,

$$\mathbb{R}[\mathcal{C}] := \left\{ \sum_{\boldsymbol{\nu} \in \mathcal{S}} \alpha_{\boldsymbol{\nu}} \boldsymbol{\xi}^{\boldsymbol{\nu}} \;\middle|\; \mathcal{S} \subseteq \mathcal{C},\ |\mathcal{S}| < \infty,\ \alpha_{\boldsymbol{\nu}} \in \mathbb{R} \right\}. \tag{11}$$

Note that, $\mathbb{R}[\mathcal{C}]$ is closed under addition. Further, $\mathcal{C}$ being closed under addition (because of the semigroup structure) implies $\mathbb{R}[\mathcal{C}]$ is closed under multiplication. Thus $\mathbb{R}[\mathcal{C}]$ is a subring (or, equivalently, a subalgebra over $\mathbb{R}$) of $\mathcal{A}$.

Next we consider the set $\mathbb{R}[\mathcal{C}]^{1\times q}$. Note that $\mathbb{R}[\mathcal{C}]^{1\times q}$ is the free module (of rank q) over the ring $\mathbb{R}[\mathcal{C}]$. Since $\mathbb{R}[\mathcal{C}]$ is a subalgebra of $\mathcal{A}$, it follows that $\mathbb{R}[\mathcal{C}]^{1\times q}$ sits inside $\mathcal{A}^{1\times q}$, that is, $\mathbb{R}[\mathcal{C}]^{1\times q} \hookrightarrow \mathcal{A}^{1\times q}$, where the inclusion map can be viewed as a morphism of $\mathbb{R}$-algebra modules via the $\mathbb{R}$-algebra homomorphism $\mathbb{R}[\mathcal{C}] \hookrightarrow \mathcal{A}$. In particular, the inclusion map $\mathbb{R}[\mathcal{C}]^{1\times q} \hookrightarrow \mathcal{A}^{1\times q}$ is $\mathbb{R}$-linear.

5 Algebraic characterization of characteristic cones

Given a cone $\mathcal{C}$ in $\mathbb{Z}^n$, let $\mathbb{R}[\mathcal{C}]$ be the algebra defined by the cone (see equation (11) above). Recall that the equation module $\mathcal{R}$ is a submodule of the free module $\mathcal{A}^{1\times q}$. We also have the free $\mathbb{R}[\mathcal{C}]$-module $\mathbb{R}[\mathcal{C}]^{1\times q}$ constructed by taking q copies of $\mathbb{R}[\mathcal{C}]$. Let us define, $\widetilde{\Psi}$, as

$$\widetilde{\Psi} : \mathbb{R}[\mathcal{C}]^{1\times q} \hookrightarrow \mathcal{A}^{1\times q}. \tag{12}$$

As mentioned earlier, $\widetilde{\Psi}$ is a morphism of $\mathbb{R}$-algebra modules via the $\mathbb{R}$-algebra homomorphism $\mathbb{R}[\mathcal{C}] \hookrightarrow \mathcal{A}$. Clearly, $\widetilde{\Psi}^{-1}(\mathcal{R}) = \mathcal{R} \cap \mathbb{R}[\mathcal{C}]^{1\times q}$ is a submodule of $\mathbb{R}[\mathcal{C}]^{1\times q}$. We denote by $\mathcal{Q}$ the quotient module $\frac{\mathbb{R}[\mathcal{C}]^{1\times q}}{\mathcal{R}\cap\mathbb{R}[\mathcal{C}]^{1\times q}}$, that is,

$$\mathcal{Q} := \frac{\mathbb{R}[\mathcal{C}]^{1\times q}}{\mathcal{R} \cap \mathbb{R}[\mathcal{C}]^{1\times q}}.$$

Note that $\mathcal{Q}$ also has the structure of a (possibly infinite dimensional) vector space over $\mathbb{R}$. We define the homomorphism of $\mathbb{R}$-algebra modules (via the

$\mathbb{R}$-algebra homomorphism $\mathbb{R}[\mathcal{C}] \hookrightarrow \mathcal{A}$)

$$\Psi : \mathcal{Q} \to \mathcal{M} \tag{13}$$

in the following way: for $\boldsymbol{p} \in \mathcal{Q}$, let $\widehat{\boldsymbol{p}}$ be a lift of $\boldsymbol{p}$ in $\mathbb{R}[\mathcal{C}]^{1\times q}$. By the natural inclusion map $\widetilde{\Psi}$, $\widehat{\boldsymbol{p}} \in \mathcal{A}^{1\times q}$. Let $\overline{\widehat{\boldsymbol{p}}}$ be the image of $\widehat{\boldsymbol{p}}$ under the canonical surjection $\mathcal{A}^{1\times q} \twoheadrightarrow \mathcal{M}$. Then Ψ is defined as

$$\Psi : \boldsymbol{p} \mapsto \overline{\widehat{\boldsymbol{p}}}. \tag{14}$$

To show Ψ is well defined, suppose $\boldsymbol{p}$ has two distinct lifts $\widehat{\boldsymbol{p}}_1$ and $\widehat{\boldsymbol{p}}_2$ in $\mathbb{R}[\mathcal{C}]^{1\times q}$ satisfying $\widehat{\boldsymbol{p}}_1 - \widehat{\boldsymbol{p}}_2 \in \mathcal{R} \cap \mathbb{R}[\mathcal{C}]^{1\times q}$. By the natural inclusion $\widetilde{\Psi}$, $\widehat{\boldsymbol{p}}_1 \neq \widehat{\boldsymbol{p}}_2$ in $\mathcal{A}^{1\times q}$. However, under the surjection $\mathcal{A}^{1\times q} \twoheadrightarrow \mathcal{M}$, $\overline{\widehat{\boldsymbol{p}}_1} = \overline{\widehat{\boldsymbol{p}}_2}$ because $\widehat{\boldsymbol{p}}_1$ and $\widehat{\boldsymbol{p}}_2$ are equivalent modulo $\mathcal{R}$. Thus Ψ is well defined. The definition of Ψ is illustrated by the commutative diagram (Figure 1) below.

$$\begin{tikzcd}
\mathbb{R}[\mathcal{C}]^{1\times q} \arrow[r, "\widetilde{\Psi}"] \arrow[d] & \mathcal{A}^{1\times q} \arrow[d] \\
\mathcal{Q} \arrow[r, "\Psi"] & \mathcal{M}
\end{tikzcd}$$

Figure 1: Commutative diagram showing Ψ.

Recall that Ψ is a morphism of $\mathbb{R}$-algebra modules via an $\mathbb{R}$-algebra homomorphism. Therefore, Ψ is $\mathbb{R}$-linear. The following result follows immediately from the definitions.

Lemma 3. *The homomorphism of $\mathbb{R}$-algebra modules $\Psi : \mathcal{Q} \to \mathcal{M}$ is injective.*

Proof: Let $\boldsymbol{p}_1, \boldsymbol{p}_2 \in \mathcal{Q}$ be such that $\Psi(\boldsymbol{p}_1) = \Psi(\boldsymbol{p}_2)$. It follows from the definition of Ψ that $\overline{\widehat{\boldsymbol{p}}_1} = \overline{\widehat{\boldsymbol{p}}}_2$, where $\widehat{\boldsymbol{p}}_1, \widehat{\boldsymbol{p}}_2 \in \mathbb{R}[\mathcal{C}]^{1\times q}$ are lifts of $\boldsymbol{p}_1, \boldsymbol{p}_2$, respectively. However, $\overline{\widehat{\boldsymbol{p}}}_1 = \overline{\widehat{\boldsymbol{p}}}_2$ implies that $\widehat{\boldsymbol{p}}_1 - \widehat{\boldsymbol{p}}_2 \in \mathcal{R}$. Also, $\widehat{\boldsymbol{p}}_1 - \widehat{\boldsymbol{p}}_2 \in \mathbb{R}[\mathcal{C}]^{1\times q}$. It then follows that $\widehat{\boldsymbol{p}}_1 - \widehat{\boldsymbol{p}}_2 \in \mathcal{R} \cap \mathbb{R}[\mathcal{C}]^{1\times q}$. Hence $\boldsymbol{p}_1 - \boldsymbol{p}_2 = 0 \in \mathcal{Q}$. □

We now state one of the main results of this paper, Theorem 4. While the map Ψ is always injective – as shown in Lemma 3 above – Theorem 4 shows that in order for a cone $\mathcal{C}$ to be a characteristic cone, it is necessary and sufficient that the $\mathbb{R}$-algebra homomorphism Ψ be surjective as well.

Theorem 4. *Let $\mathfrak{B}$ be an nD discrete autonomous behavior with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$. Then a cone (or, equivalently, an affine semigroup) $\mathcal{C} \subseteq \mathbb{Z}^n$ is a characteristic cone for the behavior $\mathfrak{B}$ if and only if the homomorphism of $\mathbb{R}$-algebra modules (via the $\mathbb{R}$-algebra homomorphism $\mathbb{R}[\mathcal{C}] \hookrightarrow \mathcal{A}$) $\Psi : \mathcal{Q} \to \mathcal{M}$, as defined in equation (14), is surjective.*

The proof of Theorem 4 requires some more machinery which is developed in the following subsections.

5.1 Duality of behaviors and $\mathcal{A}$-modules

Suppose $\mathfrak{B}$ is a behavior with equation module $\mathcal{R}$. Recall that the quotient module $\mathcal{M}$ has the structure of an $\mathbb{R}$-vector space and an $\mathcal{A}$-module. Also note that the solution space $\mathcal{W}^q$ has the structure of an $\mathcal{A}$-module. The well-known Malgrange's Theorem [16] states that the set of $\mathcal{A}$-module morphisms from $\mathcal{M}$ to $\mathcal{W}^q$ is isomorphic to the behavior $\mathfrak{B}$ as an $\mathcal{A}$-module, that is, $\mathfrak{B} \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{W}^q)$.

We define, $\mathcal{M}^* := \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}, \mathbb{R})$, the *algebraic dual* of $\mathcal{M}$ as a vector space over $\mathbb{R}$. In other words, $\mathcal{M}^*$ is the set of all $\mathbb{R}$-linear functionals on $\mathcal{M}$. The following result, Proposition 5, is easy to prove.

Proposition 5. *$\mathcal{M}^*$ has the structure of an $\mathcal{A}$-module, where multiplication by scalars from $\mathcal{A}$ is defined as follows: for $\varphi \in \mathcal{M}^*$,*

$$(f\varphi)(m) := \varphi(fm) \text{ for all } f \in \mathcal{A}.$$

We prove the following result – a variant of Malgrange's Theorem – that the behavior $\mathfrak{B}$ and $\mathcal{M}^*$, the algebraic dual of $\mathcal{M}$, are also isomorphic as $\mathcal{A}$-modules. This result is not new; it can be found in various earlier works, see for example [16]. However, we give a proof of this result for the sake of completeness and easy referencing in the sequel.

Proposition 6. *Let $\mathfrak{B}$ be a discrete autonomous nD behavior with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$. Let $\mathcal{M}$ be the quotient module $\mathcal{A}^{1\times q}/\mathcal{R}$ and $\mathcal{M}^*$ its algebraic dual. Recall the definition of action of $\mathcal{M}$ on $\mathfrak{B}$ as defined in equation (8). Define the $\mathcal{A}$-module morphism $\Gamma : \mathfrak{B} \to \mathcal{M}^*$ in the following manner: for $w \in \mathfrak{B}$ and $m \in \mathcal{M}$,*

$$(\Gamma(w))\,(m) := (m(w))(\mathbf{0}).$$

Then Γ is an isomorphism of $\mathcal{A}$-modules.

Proof: First note that Γ is an $\mathcal{A}$-module homomorphism. That is, for $w_1, w_2 \in \mathfrak{B}$, $m \in \mathcal{M}$ and $r(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A}$, we have

$$\begin{aligned}\left(\Gamma(w_1 + w_2)\right)(m) &= \left(m(w_1 + w_2)\right)(\mathbf{0})\\ &= \left(m(w_1)\right)(\mathbf{0}) + \left(m(w_2)\right)(\mathbf{0})\\ &= \Gamma(w_1)(m) + \Gamma(w_2)(m),\end{aligned}$$

and,

$$\begin{aligned}\left(\Gamma\left(r(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w_1\right)\right)(m) &= \left(m\left(r(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w_1\right)\right)(\mathbf{0})\\ &= r(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})\left(m(w_1)\right)(\mathbf{0})\\ &= r(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})\left(\Gamma(w_1)\right)(m)\end{aligned}$$

To show Γ is an isomorphism we need to show that Γ is injective and surjective.

(Injectivity) Suppose, for a $w \in \mathfrak{B}$ we have $\Gamma(w) = 0 \in \mathcal{M}^*$, that is $\left(\Gamma(w)\right)(m) = 0$ for all $m \in \mathcal{M}$. We want to show that this means $w \equiv 0$, that is, $w_i(\boldsymbol{k}) = 0$ for all $i = 1, \ldots q$ and for all $\boldsymbol{k} \in \mathbb{Z}^n$. Let $i \in \{1, 2, \ldots, q\}$ and $\boldsymbol{k} \in \mathbb{Z}^n$ both be arbitrary. Then for any $w \in \mathfrak{B}$, we have

$$\begin{aligned}w_i(\boldsymbol{k}) &= \left(\begin{bmatrix}0 & \cdots & 0 & \boldsymbol{\sigma}^{\boldsymbol{k}} & 0 & \cdots & 0\end{bmatrix} w\right)(\mathbf{0})\\ &= \left(\overline{\begin{bmatrix}0 & \cdots & 0 & \boldsymbol{\sigma}^{\boldsymbol{k}} & 0 & \cdots & 0\end{bmatrix}} w\right)(\mathbf{0}),\end{aligned}$$

where $\boldsymbol{\sigma}^{\boldsymbol{k}}$ appears at the i^{th} position. Using the definition of Γ, and assuming that $\Gamma(w) = 0 \in \mathcal{M}^*$, we get

$$w_i(\boldsymbol{k}) = \left(\Gamma(w)\right)\left(\overline{\begin{bmatrix}0 & \cdots & 0 & \boldsymbol{\sigma}^{\boldsymbol{k}} & 0 & \cdots & 0\end{bmatrix}}\right) = 0.$$

Since this is true for any arbitrary $i \in \{1, 2, \ldots, q\}$ and $\boldsymbol{k} \in \mathbb{Z}^n$, we have $w \equiv 0$ thus proving Γ is injective.

(Surjectivity) Suppose $\varphi \in \mathcal{M}^*$, we want to show that there exists $w \in \mathfrak{B}$ such that $\Gamma(w) = \varphi$ on $\mathcal{M}$. We do this by constructing such a w. For $\boldsymbol{k} \in \mathbb{Z}^n$ and $i \in \{1, 2, \ldots, q\}$, define

$$w_i(\boldsymbol{k}) := \varphi\left(\overline{\begin{bmatrix}0 & \cdots & 0 & \boldsymbol{\sigma}^{\boldsymbol{k}} & 0 & \cdots & 0\end{bmatrix}}\right) = \varphi\left(\overline{\boldsymbol{\sigma}^{\boldsymbol{k}}\boldsymbol{e}_{\boldsymbol{i}}^T}\right)$$

where $\boldsymbol{e}_{\boldsymbol{i}}$ is the standard basis column vector in $\mathbb{R}^q$. Using this definition,

$$\begin{aligned}\left(\boldsymbol{\sigma}^{\boldsymbol{k}} w\right)(\mathbf{0}) &= w(\boldsymbol{k}) = w_1(\boldsymbol{k})\boldsymbol{e}_{\mathbf{1}} + \cdots + w_q(\boldsymbol{k})\boldsymbol{e}_{\boldsymbol{q}}\\ &= \varphi\left(\overline{\boldsymbol{\sigma}^{\boldsymbol{k}}\boldsymbol{e}_{\mathbf{1}}^T}\right)\boldsymbol{e}_{\mathbf{1}} + \cdots + \varphi\left(\overline{\boldsymbol{\sigma}^{\boldsymbol{k}}\boldsymbol{e}_{\boldsymbol{q}}^T}\right)\boldsymbol{e}_{\boldsymbol{q}}.\end{aligned}$$

We first claim that $w \in \mathfrak{B}$. Note that $w \in \mathcal{W}^q$. Let $f(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A}$ be an arbitrary Laurent polynomial given by

$$f(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) = \sum_{\boldsymbol{\nu} \in \mathcal{S}} \alpha_{\boldsymbol{\nu}} \boldsymbol{\xi}^{\boldsymbol{\nu}},$$

where $\mathcal{S} \subseteq \mathbb{Z}^n$ is finite and $\alpha_{\boldsymbol{\nu}} \in \mathbb{R}$. It then follows that, for the $f \in \mathcal{A}$ defined above, we must have

$$\begin{aligned}
\left(f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w_i\right)(\mathbf{0}) &= \sum_{\boldsymbol{\nu} \in \mathcal{S}} \alpha_{\boldsymbol{\nu}} \left(\left(\boldsymbol{\sigma}^{\boldsymbol{\nu}} w_i\right)(\mathbf{0})\right) \\
&= \sum_{\boldsymbol{\nu} \in \mathcal{S}} \alpha_{\boldsymbol{\nu}} \left(\varphi\left(\overline{\boldsymbol{\sigma}^{\boldsymbol{\nu}} \boldsymbol{e}_{\boldsymbol{i}}^T}\right)\right) \quad \text{(from the definition of } w\text{)} \\
&= \varphi\left(\sum_{\boldsymbol{\nu} \in \mathcal{S}} \alpha_{\boldsymbol{\nu}} \overline{\boldsymbol{\sigma}^{\boldsymbol{\nu}} \boldsymbol{e}_{\boldsymbol{i}}^T}\right) \quad \text{(since } \varphi \text{ is } \mathbb{R}\text{-linear)} \\
&= \varphi\left(\overline{f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) \boldsymbol{e}_{\boldsymbol{i}}^T}\right). \qquad (15)
\end{aligned}$$

Now, suppose $\boldsymbol{r}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A}^{1 \times q}$, then

$$\boldsymbol{r}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) = \begin{bmatrix} r_1(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) & r_2(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) & \ldots & r_q(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \end{bmatrix}.$$

The action of $\boldsymbol{r}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})$ on a trajectory $w \in \mathcal{W}^q$ is given by

$$\left(\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\mathbf{0}) = \sum_{i=1}^{q} \left(r_i(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w_i\right)(\mathbf{0}).$$

Using equation (15) we have

$$\begin{aligned}
\left(\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\mathbf{0}) &= \sum_{i=1}^{q} \varphi\left(\overline{r_i(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) \boldsymbol{e}_{\boldsymbol{i}}^T}\right) \\
&= \varphi\left(\sum_{i=1}^{q} \overline{r_i(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) \boldsymbol{e}_{\boldsymbol{i}}^T}\right) \\
&= \varphi\left(\overline{\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})}\right). \qquad (16)
\end{aligned}$$

Now $\boldsymbol{r} \in \mathcal{R} \subseteq \mathcal{A}^{1 \times q}$ implies that $\overline{\boldsymbol{r}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})} = 0$. Thus, for all $\boldsymbol{r} \in \mathcal{R}$ we have

$$\left(\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\mathbf{0}) = \varphi\left(\overline{\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})}\right) = \varphi(0) = 0$$

since φ is $\mathbb{R}$-linear. Given $\boldsymbol{r} \in \mathcal{R}$, observe that $\boldsymbol{\xi}^{\boldsymbol{k}} \boldsymbol{r} \in \mathcal{R}$ for all $\boldsymbol{k} \in \mathbb{Z}^n$. Therefore, it follows that

$$\left(\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\boldsymbol{k}) = \left(\boldsymbol{\sigma}^{\boldsymbol{k}} \boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\mathbf{0}) = \varphi\left(\overline{\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})}\right) = 0.$$

Thus, for all $\boldsymbol{r} \in \mathcal{R}$, we have $\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w \equiv 0$, which means $w \in \mathfrak{B}$.

Next, we claim that for this w we must have $\Gamma(w) = \varphi$ on $\mathcal{M}$. Let $m \in \mathcal{M}$ be arbitrary and suppose $\boldsymbol{r} \in \mathcal{A}^{1\times q}$ is a lift of m. Then from the definition of Γ we have

$$(\Gamma(w))\,(m) = (m(w))\,(\mathbf{0}).$$

However, by equation (8), we have $(m(w))\,(\mathbf{0}) = (\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w)\,(\mathbf{0})$. It follows from equation (16) that $(\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w)\,(\mathbf{0}) = \varphi\left(\overline{\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})}\right) = \varphi(m)$. Since $m \in \mathcal{M}$ was chosen arbitrarily, we have $\Gamma(w) = \varphi$ on $\mathcal{M}$. This proves that Γ is surjective. $\square$

Proposition 6 provides the desired duality between behaviors and the algebraic dual of the quotient module. This duality enables us to devise an algorithm for obtaining trajectories in a behavior, given the equation module $\mathcal{R}$. We elaborate on this algorithm in Lemma 7 below which will be crucial in proving the main result Theorem 4. Similar methods have been presented in various earlier works; see for example, [16].

Lemma 7. *Let $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ be an equation module with behavior $\mathfrak{B}$. Further, let $\mathcal{E} = \{m_1, m_2, m_3, \ldots\} \subseteq \mathcal{M}$ be a (Hamel) basis*[3] *of $\mathcal{M}$ as a vector space over $\mathbb{R}$. Let $\varphi \in \mathcal{M}^*$ be given. Define $w_i : \mathbb{Z}^n \to \mathbb{R}$ in the following manner: for $\boldsymbol{k} \in \mathbb{Z}^n$, suppose $\overline{\boldsymbol{\sigma}^{\boldsymbol{k}} \boldsymbol{e}_{\boldsymbol{i}}^T} = \sum_j \beta_j m_j$, then define*

$$w_i(\boldsymbol{k}) := \sum_j \beta_j \varphi(m_j). \tag{17}$$

Since $\mathcal{E}$ is a basis of $\mathcal{M}$, the above-mentioned sums are finite. Define $w \in \mathcal{W}^q$ as $w := \operatorname{col}(w_1, w_2, \ldots, w_q)$, each w_i is as defined in equation (17). Then w thus defined is a trajectory in $\mathfrak{B}$.

Proof: First we note the following fact, which follows from $\mathbb{R}$-linearity of φ: suppose $m \in \mathcal{M}$ can be written in the Hamel basis $\mathcal{E}$ as $m = \sum_j \alpha_j m_j$,

[3] A Hamel basis of a possibly infinite dimensional vector space $\mathcal{V}$ over a field $\mathbb{K}$ is a subset $\mathcal{E}$ of $\mathcal{V}$ that satisfies:

1. elements in $\mathcal{E}$ are *linearly independent* over $\mathbb{K}$, that is, no finite non-zero linear combination of elements in $\mathcal{E}$ equals zero, and
2. every element of $\mathcal{V}$ can be written as a finite linear combination of elements from $\mathcal{E}$.

See [20, Section 2]. Note also that $\mathcal{M}$ admits a countable Hamel basis. This justifies writing the basis as a list.

where all but finitely many α_js are zero, then we must have

$$\varphi(m) = \sum_j \alpha_j \varphi(m_j). \tag{18}$$

In particular, combining equations (17) and (18) we get that

$$w_i(\boldsymbol{k}) = \varphi\left(\overline{\boldsymbol{\sigma}^{\boldsymbol{k}} \boldsymbol{e}_{\boldsymbol{i}}^T}\right). \tag{19}$$

Now suppose that $\boldsymbol{r} \in \mathcal{R}$. Note that $\boldsymbol{r}$ admits an expansion of the following form:

$$\boldsymbol{r}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) = \sum_{i=1}^{q} \sum_{\boldsymbol{k} \in \mathbb{Z}^n} \alpha_{i,\boldsymbol{k}} \boldsymbol{\xi}^{\boldsymbol{k}} e_i^T,$$

where all but finitely many $\alpha_{i,\boldsymbol{k}}$s are zero. Making this $\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})$ act on $w = \mathrm{col}(w_1, w_2, \dots, w_q) \in \mathcal{W}^q$, as defined in the statement of the lemma, we get that

$$\begin{aligned}
\left(\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\boldsymbol{0}) &= \sum_{i=1}^{q} \sum_{\boldsymbol{k} \in \mathbb{Z}^n} \left(\alpha_{i,\boldsymbol{k}} \boldsymbol{\sigma}^{\boldsymbol{k}} e_i^T w\right)(\boldsymbol{0}) \\
&= \sum_{i=1}^{q} \sum_{\boldsymbol{k} \in \mathbb{Z}^n} \alpha_{i,\boldsymbol{k}} w_i(\boldsymbol{k}) = \sum_{i=1}^{q} \sum_{\boldsymbol{k} \in \mathbb{Z}^n} \alpha_{i,\boldsymbol{k}} \varphi\left(\overline{\boldsymbol{\sigma}^{\boldsymbol{k}} \boldsymbol{e}_{\boldsymbol{i}}^T}\right),
\end{aligned}$$

where the last equality follows from equation (19). Applying the $\mathbb{R}$-linearity property of the map φ we get

$$\begin{aligned}
\left(\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\boldsymbol{0}) &= \varphi\left(\sum_{i=1}^{q} \sum_{\boldsymbol{k} \in \mathbb{Z}^n} \alpha_{i,\boldsymbol{k}} \overline{\boldsymbol{\sigma}^{\boldsymbol{k}} \boldsymbol{e}_{\boldsymbol{i}}^T}\right) \\
&= \varphi\left(\overline{\sum_{i=1}^{q} \sum_{\boldsymbol{k} \in \mathbb{Z}^n} \alpha_{i,\boldsymbol{k}} \boldsymbol{\sigma}^{\boldsymbol{k}} \boldsymbol{e}_{\boldsymbol{i}}^T}\right) \\
&= \varphi\left(\overline{\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})}\right) = 0,
\end{aligned}$$

where the last equality comes from noting that $\boldsymbol{r} \in \mathcal{R}$ implies $\overline{\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})} = 0 \in \mathcal{M}$, and linearity of φ forces $\varphi(0) = 0$. Since $\boldsymbol{r} \in \mathcal{R}$ was chosen arbitrarily, we come to the conclusion that $\left(\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\boldsymbol{0}) = 0$ for all $\boldsymbol{r} \in \mathcal{R}$. Therefore, $w \in \mathfrak{B}$. $\square$

Remark 8. *It is a noteworthy fact that equation* (19) *implies that* $\Gamma(w) = \varphi$*, where* Γ *is as defined in Proposition 6. This follows from the 'surjectivity' part of the proof of Proposition 6. Saying alternatively,* $w = \Gamma^{-1}$*. Since* Γ *is an isomorphism between* $\mathfrak{B}$ *and* $\mathcal{M}^*$*, Lemma 7 can be used to provide all trajectories in* $\mathfrak{B}$ *by varying* φ *over* $\mathcal{M}^*$*. Thus Lemma 7 gives a parametrization of* $\mathfrak{B}$ *by* $\varphi \in \mathcal{M}^*$*.*

5.2 Proof of Theorem 4

With Proposition 6 and Lemma 7 in place, we are now in a position to prove one of our main results, Theorem 4.

Proof of Theorem 4: (If) Suppose Ψ is surjective, we have to show that $\mathcal{C}$ is a characteristic cone for $\mathfrak{B}$. It is enough to show that for all $w \in \mathfrak{B}$ we have, $w|_{\mathcal{C}} = 0$ implies $w \equiv 0$. That is, for $w \in \mathfrak{B}$,

$$w(\boldsymbol{k}) = 0 \text{ for all } \boldsymbol{k} \in \mathcal{C} \;\Rightarrow\; w(\boldsymbol{k}) = 0 \text{ for all } \boldsymbol{k} \in \mathbb{Z}^n.$$

In order to show this let $w \in \mathfrak{B}$ be such that $w|_{\mathcal{C}} = 0$, and let $\boldsymbol{k} \in \mathbb{Z}^n$ be arbitrary. Now, since Ψ is surjective, it follows from the definition of Ψ that for all $i \in \{1, 2, \dots, q\}$ there exists $\boldsymbol{f_i} \in \mathbb{R}[\mathcal{C}]^{1\times q}$ such that

$$\boldsymbol{\xi^k e_i^T} - \boldsymbol{f_i} \in \mathcal{R}.$$

It then follows that $\left(\boldsymbol{\sigma^k e_i^T} - \boldsymbol{f_i}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})\right)(w) \equiv 0$. Therefore,

$$w_i(\boldsymbol{k}) = \left(\boldsymbol{\sigma^k} w_i\right)(\boldsymbol{0}) = \left(\boldsymbol{f_i}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w\right)(\boldsymbol{0}). \tag{20}$$

Now, $w|_{\mathcal{C}} = 0$ implies that for all $\boldsymbol{f_i} \in \mathbb{R}[\mathcal{C}]^{1\times q}$ we must have $(\boldsymbol{f_i}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w)(\boldsymbol{0}) = 0$. Hence from equation (20)

$$w_i(\boldsymbol{k}) = \left(\boldsymbol{\sigma^k} w_i\right)(\boldsymbol{0}) = (\boldsymbol{f_i}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w)(\boldsymbol{0}) = 0.$$

Since $\boldsymbol{k} \in \mathbb{Z}^n$ was arbitrary, it follows that $w \equiv 0$.

(Only if) We want to show that if Ψ is *not* surjective then $\mathcal{C}$ is *not* a characteristic cone for $\mathfrak{B}$. Recall that $\mathcal{Q}$ denotes the $\mathbb{R}[\mathcal{C}]$-module $\frac{\mathbb{R}[\mathcal{C}]^{1\times q}}{\mathcal{R}\cap\mathbb{R}[\mathcal{C}]^{1\times q}}$. Let $\mathcal{E} = \{\epsilon_1, \epsilon_2, \dots\} \subseteq \mathcal{Q}$ be a (Hamel) basis of $\mathcal{Q}$ as a vector space over $\mathbb{R}$. Define

$$\widetilde{\mathcal{E}} := \{\widetilde{\epsilon_1}, \widetilde{\epsilon_2}, \dots\},$$

where $\widetilde{\mathcal{E}} = \Psi(\mathcal{E})$. That is, $\widetilde{\mathcal{E}} = \{\Psi(\epsilon_1), \Psi(\epsilon_2), \dots\}$. Since Ψ is injective (see Lemma 3), $\widetilde{\mathcal{E}}$ is a linearly independent set in $\mathcal{M}$. It then follows that $\mathcal{M}$ admits a (Hamel) basis $\mathcal{E}'$ such that $\widetilde{\mathcal{E}} \subseteq \mathcal{E}'$ (see [20, Corollary 2.2]). Note that we must have $\widetilde{\mathcal{E}} \subsetneq \mathcal{E}'$ because we have assumed that Ψ is not surjective.

Recall the definition of the map under consideration $\Psi : \mathcal{Q} \to \mathcal{M}$ (see equation (14)). It follows that we have for all $i \in \{1, 2, \dots, q\}$

$$\Psi(\epsilon_i) = \overline{\widehat{\boldsymbol{\epsilon_i}}},$$

where $\widehat{\boldsymbol{\epsilon_i}} \in \mathbb{R}[\mathcal{C}]^{1\times q}$ is a lift of ϵ_i. Due to this, we get that for all $\boldsymbol{k} \in \mathcal{C}$ and $i \in \{1, 2, \ldots, q\}$ we must have

$$\overline{\boldsymbol{\xi^k e_i^T}} \in \text{ span } \widetilde{\mathcal{E}}. \tag{21}$$

Furthermore, Ψ being not surjective also implies that there exists $\boldsymbol{k}^* \in \mathbb{Z}^n \setminus \mathcal{C}$ and $j \in \{1, 2, \ldots, q\}$ such that

$$\overline{\boldsymbol{\xi^{k^*} e_j^T}} \notin \text{span } \widetilde{\mathcal{E}}.$$

In other words, there exists $c \in \mathcal{E}' \setminus \widetilde{\mathcal{E}}$ such that

$$\overline{\boldsymbol{\xi^{k^*} e_j^T}} = \alpha c + \sum_{m_i \in \mathcal{E}'} \beta_i m_i, \tag{22}$$

where the sum is finite and $\alpha \neq 0$.

Now, we shall define a $\varphi \in \mathcal{M}^*$ in the following manner. Since φ is $\mathbb{R}$-linear and $\mathcal{E}'$ is a basis of $\mathcal{M}$ as a vector space over $\mathbb{R}$, in order to define φ, it is enough to define its action on the elements of $\mathcal{E}'$. Moreover, this action of φ on the elements of $\mathcal{E}'$ can be defined independently because elements in $\mathcal{E}'$ are linearly independent. Therefore, we can define $\varphi \in \mathcal{M}^*$ to be such that

$$\begin{aligned} \varphi(c) &= 1 \\ \varphi|_{\mathcal{E}' \setminus \{c\}} &= 0. \end{aligned}$$

Then we construct a trajectory $w : \mathbb{Z}^n \to \mathbb{R}^q$,

$$w(\boldsymbol{k}) = \varphi(\overline{\boldsymbol{\sigma^k e_1^T}})\boldsymbol{e_1} + \varphi(\overline{\boldsymbol{\sigma^k e_2^T}})\boldsymbol{e_2} + \cdots + \varphi(\overline{\boldsymbol{\sigma^k e_q^T}})\boldsymbol{e_q} \tag{23}$$

following equation (19) in Lemma 7. By Lemma 7, this $w \in \mathfrak{B}$. Now, for all $\boldsymbol{k} \in \mathcal{C}$, $\overline{\boldsymbol{\sigma^k e_i^T}} \in \text{span } \widetilde{\mathcal{E}}$. Therefore, from the construction of φ it follows that

$$w(\boldsymbol{k}) = 0 \text{ for all } \boldsymbol{k} \in \mathcal{C}.$$

In other words, $w|_{\mathcal{C}} = 0$. However, $w \not\equiv 0$, because $w_j(\boldsymbol{k}^*) = \alpha \neq 0$. This shows that $\mathcal{C}$ cannot be a characteristic set for $\mathfrak{B}$. □

6 Algorithm to verify whether a cone is a characteristic cone

In the last section (Section 5) we have given an algebraic characterization for a given cone to be a characteristic cone for a discrete autonomous nD behavior in terms of a homomorphism between two $\mathbb{R}$-algebra modules. In fact, a given cone is a characteristic cone if and only if the homomorphism is surjective. Checking surjectivity of module morphisms is a standard problem in computational commutative algebra and the theory of Gröbner bases is one of the approaches used for solving this problem. In this section we reduce the condition of Theorem 4 to another equivalent algebraic condition which is more suitable for checking using Gröbner bases techniques.

In the first part of this section we briefly state some preliminary results on Gröbner bases theory. Details can be found in textbooks like [21], [22]. Subsection 6.2 discusses various reductions required to transform Theorem 4 to another equivalent condition given by Proposition 13. Using this proposition and Lemma 14 we state Theorem 15 which is an equivalent condition based on Gröbner basis to check when a given cone is a characteristic cone for a behavior $\mathfrak{B}$. We finally provide an algorithm for doing this test.

6.1 Gröbner basis preliminaries

Let $\mathbb{R}[\boldsymbol{\xi}]$ be the polynomial ring in n variables. Ordering of monomials play an important role in multivariable polynomial rings. Such an ordering in $\mathbb{R}[\boldsymbol{\xi}]$, equivalently on $\mathbb{Z}^n_{\geqslant 0}$, is called a *term ordering*. We use the symbol $\succ$ to denote monomial term ordering. By a term ordering we mean a total ordering on $\mathbb{Z}^n_{\geqslant 0}$, such that, if $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\beta} \in \mathbb{Z}^n_{\geqslant 0}$ and $\boldsymbol{\nu}_1 \succ \boldsymbol{\nu}_2$, then $\boldsymbol{\nu}_1 + \boldsymbol{\beta} \succ \boldsymbol{\nu}_2 + \boldsymbol{\beta}$. Also $\succ$ is a well-ordering on $\mathbb{Z}^n_{\geqslant 0}$. (See [21, Chapter 2, Section 2] for more details). Some common monomial orderings are lexicographic ordering, graded reverse lexicographic ordering etc. (See [21, Chapter 2, Section 2] for more details). Under a given term ordering $\succ$, the terms of a polynomial are ordered uniquely.

Let $f = \sum \alpha_{\boldsymbol{\nu}} \boldsymbol{\xi}^{\boldsymbol{\nu}}$ be a polynomial in $\mathbb{R}[\boldsymbol{\xi}]$ and $\succ$ be a term order. The *multidegree* of f is defined as

$$\mathrm{md}(f) := \max(\boldsymbol{\nu} \in \mathbb{Z}^n_{\geqslant 0} | \alpha_{\boldsymbol{\nu}} \neq 0).$$

The *leading term* of f is the term corresponding to the multidegree of f with respect to a given monomial order, that is,

$$\mathrm{LT}(f) := \alpha_{\mathrm{md}(f)} \boldsymbol{\xi}^{\mathrm{md}(f)}. \tag{24}$$

The idea of term ordering can be generalized to polynomial modules (see [22], [23]). To define a module term ordering let $\succ$ be a term ordering on

$\mathbb{Z}^n_{\geqslant 0}$ and let $\boldsymbol{\xi}^{\boldsymbol{\nu}'}$ and $\boldsymbol{\xi}^{\boldsymbol{\nu}''}$ be monomials in $\mathbb{R}[\boldsymbol{\xi}]$. Then for $\boldsymbol{\xi}^{\boldsymbol{\nu}'}\boldsymbol{e}_i \in \mathbb{R}[\boldsymbol{\xi}]^q$ and $\boldsymbol{\xi}^{\boldsymbol{\nu}''}\boldsymbol{e}_j \in \mathbb{R}[\boldsymbol{\xi}]^q$ we have,

$$\begin{aligned} &\boldsymbol{\xi}^{\boldsymbol{\nu}'}\boldsymbol{e}_i \succ_{\text{TOP}} \boldsymbol{\xi}^{\boldsymbol{\nu}''}\boldsymbol{e}_j \\ \Longleftrightarrow &\left(\boldsymbol{\xi}^{\boldsymbol{\nu}'} \succ \boldsymbol{\xi}^{\boldsymbol{\nu}''}\right) \text{ or } (\boldsymbol{\xi}^{\boldsymbol{\nu}'} = \boldsymbol{\xi}^{\boldsymbol{\nu}''} \text{ and } i > j). \end{aligned}$$

Such an ordering is called "term over position" (TOP) ordering. Another term ordering is the "position over term" (POT) ordering. We consider TOP module ordering here. Once the term ordering is fixed, the leading term of an element $\boldsymbol{f} \in \mathbb{R}[\boldsymbol{\xi}]^q$ can be defined in the following manner (See [22, Chapter 3, Section 5]). For a non-zero $\boldsymbol{f} \in \mathbb{R}[\boldsymbol{\xi}]^q$,

$$\begin{aligned} \boldsymbol{f} &= f_1\boldsymbol{e}_1 + f_2\boldsymbol{e}_2 + \cdots + f_q\boldsymbol{e}_q \\ &= \left(\sum_{\boldsymbol{\nu}\in\mathcal{S}_1} \alpha_{\boldsymbol{\nu}}\boldsymbol{\xi}^{\boldsymbol{\nu}}\right)\boldsymbol{e}_1 + \left(\sum_{\boldsymbol{\nu}\in\mathcal{S}_2} \alpha_{\boldsymbol{\nu}}\boldsymbol{\xi}^{\boldsymbol{\nu}}\right)\boldsymbol{e}_2 + \cdots + \left(\sum_{\boldsymbol{\nu}\in\mathcal{S}_q} \alpha_{\boldsymbol{\nu}}\boldsymbol{\xi}^{\boldsymbol{\nu}}\right)\boldsymbol{e}_q \end{aligned}$$

where, $\mathcal{S}_i \subseteq \mathbb{Z}^n$ is finite and $\alpha_{\boldsymbol{\nu}} \in \mathbb{R}$. Since $\mathcal{S}_i$'s are finite, using the TOP term ordering we can write

$$\boldsymbol{f} = \alpha_1\boldsymbol{X}_1 + \alpha_2\boldsymbol{X}_2 + \cdots + \alpha_r\boldsymbol{X}_r$$

where, $\boldsymbol{X}_i := \boldsymbol{\xi}^{\boldsymbol{\nu}}\boldsymbol{e}_j$, $j \in \{0, 1, \ldots, q\}$ are monomial tuples such that $\boldsymbol{X}_1 \succ \boldsymbol{X}_2 \succ \cdots \succ \boldsymbol{X}_r$. Now, the *leading monomial* of $\boldsymbol{f}$ is $\boldsymbol{X}_1$ and the *leading term* of $\boldsymbol{f}$ is $\alpha_1\boldsymbol{X}_1$.

For a submodule $D \subseteq \mathbb{R}[\boldsymbol{\xi}]^q$, the leading term module of D is defined as

$$\langle \mathrm{LT}(D)\rangle = \langle \mathrm{LT}(\boldsymbol{d}) \mid \boldsymbol{d} \in D\rangle \subseteq \mathbb{R}[\boldsymbol{\xi}]^q.$$

A finite subset $\mathcal{G} = \{\boldsymbol{g_1}, \ldots, \boldsymbol{g_t}\} \subseteq D$ with $\boldsymbol{g}_i \neq \boldsymbol{0}$ is said to be a Gröbner basis of D if the module generated by the leading terms of $\mathcal{G}$ and the module generated by the leading terms of elements in D are equal. That is,

$$\langle \mathrm{LT}(\mathcal{G})\rangle = \langle \mathrm{LT}(D)\rangle.$$

We state some important properties of Gröbner bases, without proof, which are used later in this paper. Proofs can be found in any standard textbook on Gröbner bases, for instance, [22].

Proposition 9. *[22, Chapter 3, Section 5, Theorem 14] Let* $\mathcal{G} = \{\boldsymbol{g_1}, \ldots, \boldsymbol{g_t}\}$ *be a Gröbner basis for a submodule* $D \subseteq \mathbb{R}[\boldsymbol{\xi}]^q$ *and let* $\boldsymbol{f} \in \mathbb{R}[\boldsymbol{\xi}]^q$*. Then*

1. *There exists* $h_1, \dots, h_t \in \mathbb{R}[\boldsymbol{\xi}]$ *and a* unique *remainder* $\boldsymbol{r} \in \mathbb{R}[\boldsymbol{\xi}]^q$ *such that* $\boldsymbol{f} = h_1\boldsymbol{g_1} + \cdots + h_t\boldsymbol{g_t} + \boldsymbol{r}$.

2. *Let* $\overline{\boldsymbol{f}}^{\mathcal{G}}$ *denote the remainder obtained after dividing*[4] $\boldsymbol{f}$ *by elements in* $\mathcal{G}$. *Then,* $\boldsymbol{f} \in D$ *if and only if* $\overline{\boldsymbol{f}}^{\mathcal{G}} = 0$.

Eliminating variables from a system of equations will be of crucial importance in the sequel. For this we next state the *elimination theorem* [22, Chapter 3, Section 6, Theorem 6].

Proposition 10. *Consider a module* $D \subseteq \mathbb{R}[\boldsymbol{\xi}]^q$. *Let* $\succ$ *be an elimination term order in* $\mathbb{R}[\boldsymbol{\xi}]$ *with* $\xi_1 \succ \cdots \succ \xi_n$. *Let* $\mathcal{G}$ *be a Gröbner basis for* D *with respect to TOP module ordering* $\succ_{\text{TOP}}$ *on* $\mathbb{R}[\boldsymbol{\xi}]^q$. *Then* $\mathcal{G}_r := \mathcal{G} \cap \mathbb{R}[\xi_k, \xi_{k+1}, \dots, \xi_n]^q$ *is a Gröbner basis for* $D \cap \mathbb{R}[\xi_k, \xi_{k+1}, \dots, \xi_n]^q$.

The theory of Gröbner bases is well suited for polynomial rings in several indeterminates. However, they are not suited for Laurent polynomial rings [10] [24]. To apply Gröbner bases results for Laurent polynomial rings in n variables we define the $2n$ variable polynomial ring $\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}]$ and the $\mathbb{R}$-algebra map as

$$\begin{aligned} \pi : \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}] &\twoheadrightarrow \mathcal{A} \\ \xi_i &\mapsto \xi_i \\ \eta_i &\mapsto \xi_i^{-1}. \end{aligned} \tag{25}$$

It follows from First Isomorphism Theorem [25] that, $\mathcal{A} \cong \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}]/\ker \pi$, where, $\ker \pi$ is the ideal of relations between variables $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ given by $\langle \xi_1\eta_1 - 1, \dots, \xi_n\eta_n - 1\rangle$.

Remark: Another way to apply Gröbner bases results to Laurent polynomial rings is to introduce one additional variable and form the $(n+1)$ variable polynomial ring and define a similar $\mathbb{R}$-algebra homomorphism from this new ring to $\mathcal{A}$. We, however, do not follow this approach, in spite of its advantages like, being computationally more efficient. This is mainly because, here, we do not intend to delve into computational aspects but rather provide algorithms that can be implemented. Discussions pertaining to computational issues of the algorithms presented here will be dealt with in future works.

[4] It is important to note that division here refers to division in $\mathbb{R}[\boldsymbol{\xi}]^q$. An algorithm for this can be found in [22, Algorithm 3.5.1]

Consider the $2n$ variable polynomial ring $\mathbb{R}[\boldsymbol{\xi},\boldsymbol{\eta}]$ and the set $\mathbb{R}[\boldsymbol{\xi},\boldsymbol{\eta}]^{1\times q}$. Now, $\mathbb{R}[\boldsymbol{\xi},\boldsymbol{\eta}]^{1\times q}$ is the free module of rank q over the ring $\mathbb{R}[\boldsymbol{\xi},\boldsymbol{\eta}]$. Also $\pi : \mathbb{R}[\boldsymbol{\xi},\boldsymbol{\eta}] \twoheadrightarrow \mathcal{A}$ is a surjection given in equation (25). We construct the homomorphism of $\mathbb{R}$-algebra modules induced by the $\mathbb{R}$-algebra homomorphism π in the following manner.

$$\begin{aligned} \Pi : \mathbb{R}[\boldsymbol{\xi},\boldsymbol{\eta}]^{1\times q} &\twoheadrightarrow \mathcal{A}^{1\times q} \\ \xi_i \boldsymbol{e_j}^T &\mapsto \xi_i \boldsymbol{e_j}^T \ \forall i=1,\dots,n, \forall j=1,\dots,q, \\ \eta_i \boldsymbol{e_j}^T &\mapsto \xi_i^{-1} \boldsymbol{e_j}^T \ \forall i=1,\dots,n, \forall j=1,\dots,q. \end{aligned} \tag{26}$$

The kernel of Π is a submodule of $\mathbb{R}[\boldsymbol{\xi},\boldsymbol{\eta}]^{1\times q}$ and is given by the rowspan of the following matrix,

$$P = \operatorname{diag}\left(\underbrace{\begin{bmatrix}\xi_1\eta_1 - 1\\ \vdots \\ \xi_n\eta_n - 1\end{bmatrix}, \begin{bmatrix}\xi_1\eta_1 - 1\\ \vdots \\ \xi_n\eta_n - 1\end{bmatrix}, \dots, \begin{bmatrix}\xi_1\eta_1 - 1\\ \vdots \\ \xi_n\eta_n - 1\end{bmatrix}}_{q \text{ entries}}\right).$$

In other words,

$$\ker \Pi = \operatorname{rowspan}_{\mathbb{R}[\boldsymbol{\xi},\boldsymbol{\eta}]} P.$$

6.2 Cones and semigroup rings

We now define the algebra of the cone in terms of a ring homomorphism. Recall that a cone $\mathcal{C}$ is an affine sub-semigroup of $\mathbb{Z}^n$ generated by $\{\boldsymbol{c}_1,\dots,\boldsymbol{c}_r\} \subseteq \mathbb{Z}^n$. Therefore, any element $\boldsymbol{c} \in \mathcal{C}$ can be written as a non-negative linear combination of elements in $\mathcal{C}$. That is, $\boldsymbol{c} = \alpha_1\boldsymbol{c}_1 + \alpha_2\boldsymbol{c}_2 + \cdots + \alpha_r\boldsymbol{c}_r$, where $\alpha_i \in \mathbb{Z}_{\geqslant}0$. Also the generators $\boldsymbol{c}_1,\dots,\boldsymbol{c}_r$ are associated to monomials $\boldsymbol{\xi}^{\boldsymbol{c}_1},\dots,\boldsymbol{\xi}^{\boldsymbol{c}_r}$ respectively. Thus, the monomial corresponding to $\boldsymbol{c}$, that is, $\boldsymbol{\xi}^{\boldsymbol{c}} \in \mathbb{R}[\mathcal{C}]$ can be written as

$$\begin{aligned} \boldsymbol{\xi}^{\boldsymbol{c}} &= \boldsymbol{\xi}^{\alpha_1\boldsymbol{c}_1+\alpha_2\boldsymbol{c}_2+\cdots+\alpha_r\boldsymbol{c}_r} = \boldsymbol{\xi}^{\alpha_1\boldsymbol{c}_1}\boldsymbol{\xi}^{\alpha_2\boldsymbol{c}_2}\dots\boldsymbol{\xi}^{\alpha_r\boldsymbol{c}_r} \\ &= (\boldsymbol{\xi}^{\boldsymbol{c}_1})^{\alpha_1}(\boldsymbol{\xi}^{\boldsymbol{c}_2})^{\alpha_2}\dots(\boldsymbol{\xi}^{\boldsymbol{c}_r})^{\alpha_r}. \end{aligned} \tag{27}$$

Since every element in $\mathbb{R}[\mathcal{C}]$ is a finite $\mathbb{R}$-linear combination of monomials of the form $\boldsymbol{\xi}^{\boldsymbol{c}}$, where $\boldsymbol{c} \in \mathcal{C}$, it follows from equation (27) above that every element in $\mathbb{R}[\mathcal{C}]$ is a polynomial in $\boldsymbol{\xi}^{\boldsymbol{c}_1},\dots,\boldsymbol{\xi}^{\boldsymbol{c}_r}$. In other words, $\mathbb{R}[\mathcal{C}]$ is generated by the monomials $\boldsymbol{\xi}^{\boldsymbol{c}_1},\dots,\boldsymbol{\xi}^{\boldsymbol{c}_r}$ as an $\mathbb{R}$-algebra. That is $\mathbb{R}[\mathcal{C}] =$

$\mathbb{R}[\boldsymbol{\xi}^{\boldsymbol{c}_1}, \ldots, \boldsymbol{\xi}^{\boldsymbol{c}_r}]$. Therefore, for every element $f(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathbb{R}[\mathcal{C}]$ there exists an r-variable polynomial z such that f can be written as

$$f(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) = z(\boldsymbol{\xi}^{\boldsymbol{c}})$$

where, $z(\boldsymbol{\xi}^{\boldsymbol{c}}) \in \mathbb{R}[\boldsymbol{\xi}^{\boldsymbol{c}_1}, \ldots, \boldsymbol{\xi}^{\boldsymbol{c}_r}]$. This is the content of Lemma 11 below. We follow this convention frequently henceforth. Since there are r generators of the cone we define the r-variable polynomial ring $\mathbb{R}[\boldsymbol{\delta}]$ and the $\mathbb{R}$-algebra map

$$\begin{aligned} \Phi : \mathbb{R}[\boldsymbol{\delta}] &\longrightarrow \mathcal{A} \\ \delta_1 &\mapsto \boldsymbol{\xi}^{\boldsymbol{c}_1}, \\ \delta_2 &\mapsto \boldsymbol{\xi}^{\boldsymbol{c}_2}, \\ &\vdots \\ \delta_r &\mapsto \boldsymbol{\xi}^{\boldsymbol{c}_r}. \end{aligned} \tag{28}$$

Lemma 11. *Let $\mathcal{C} \subseteq \mathbb{Z}^n$ be a cone and $\mathbb{R}[\mathcal{C}]$ be the algebra generated by the cone as defined in equation* (11)*. Then $\mathbb{R}[\mathcal{C}] = \mathrm{im}\ \Phi$, where, $\Phi : \mathbb{R}[\boldsymbol{\delta}] \to \mathcal{A}$ is as defined in equation* (28)*.*

Proof: To show $\mathrm{im}\ \Phi \subseteq \mathbb{R}[\mathcal{C}]$, let $f \in \mathrm{im}\ \Phi$. Then there exists $\mathbb{R}[\boldsymbol{\delta}] \ni t = \sum_{i=1}^{p} \alpha_{\boldsymbol{\nu}_i} \boldsymbol{\delta}^{\boldsymbol{\nu}_i}$, where, $\boldsymbol{\nu}_i \in \mathbb{Z}^r_{\geqslant 0}$ and $\alpha_{\boldsymbol{\nu}_i} \in \mathbb{R}$, such that

$$\begin{aligned} f = \Phi(t) &= \Phi\left(\sum_{i=1}^{p} \alpha_{\boldsymbol{\nu}_i} \boldsymbol{\delta}^{\boldsymbol{\nu}_i}\right) \\ &= \sum_{i=1}^{p} \alpha_{\boldsymbol{\nu}_i} \Phi\left(\boldsymbol{\delta}^{\boldsymbol{\nu}_i}\right) \\ &= \sum_{i=1}^{p} \alpha_{\boldsymbol{\nu}_i} \boldsymbol{\xi}^{\sum_{j=1}^{r} \nu_{ij} \boldsymbol{c}_j}. \end{aligned}$$

Since $\boldsymbol{\xi}^{\boldsymbol{c}_j}$ are monomials in $\mathcal{C}$ and $\nu_{ij} \in \mathbb{Z}_{\geqslant 0}$, $\sum_{j=1}^{r} \nu_{ij} \boldsymbol{c}_j \in \mathcal{C}$. Thus $f \in \mathbb{R}[\mathcal{C}]$.

Conversely, for an element $s \in \mathbb{R}[\mathcal{C}]$, $s = \sum_{j=1}^{q} \alpha_{\boldsymbol{\nu}_j} \boldsymbol{\xi}^{\boldsymbol{\nu}_j}$ where $\boldsymbol{\nu}_j$'s are indices in $\mathcal{C}$ and $\alpha_{\boldsymbol{\nu}_j} \in \mathbb{R}$. Therefore, each $\boldsymbol{\nu}_j$ can be expressed as a linear combination of $\boldsymbol{c}_i$'s with coefficients from $\mathbb{Z}_{\geqslant 0}$. That is, $\boldsymbol{\nu}_j = \sum_{i=1}^{r} \gamma_{ij} \boldsymbol{c}_i$, with γ_{ij}'s $\in \mathbb{Z}_{\geqslant 0}$. Thus

$$s = \sum_{j=1}^{q} \alpha_{\boldsymbol{\nu}_j} \boldsymbol{\xi}^{\sum_{i=1}^{r} \gamma_{ij} \boldsymbol{c}_i} = \sum_{j=1}^{q} \alpha_{\boldsymbol{\nu}_j} \left(\prod_{i=1}^{r} \boldsymbol{\xi}^{\boldsymbol{c}_i \gamma_{ij}}\right).$$

Hence $s = \Phi(f)$ where,

$$f(\boldsymbol{\delta}) = \sum_{j=1}^{q} \alpha_{\boldsymbol{\nu}_j} \left(\prod_{i=1}^{r} \delta_i^{\gamma_{ij}}\right) \in \mathbb{R}[\boldsymbol{\delta}].$$

Therefore, $\mathbb{R}[\mathcal{C}] = \text{im } \Phi$. $\square$

From Lemma 11 and standard results from commutative algebra (See [25]) it follows,

$$\mathbb{R}[\mathcal{C}] = \text{im } \Phi \cong \frac{\mathbb{R}[\boldsymbol{\delta}]}{\ker \Phi}.$$

For the vector case, consider the free module $\mathbb{R}[\boldsymbol{\delta}]^{1\times q}$ over $\mathbb{R}[\boldsymbol{\delta}]$. We define the $\mathbb{R}$-linear map $\Phi^\star$ which is a homomorphism of $\mathbb{R}$-algebra modules (via the $\mathbb{R}$-algebra homomorphism $\Phi : \mathbb{R}[\boldsymbol{\delta}] \to \mathcal{A}$). Thus, $\Phi^\star$ is defined for all $j = 1, 2, \ldots, n$, as

$$\begin{aligned} \Phi^\star : \mathbb{R}[\boldsymbol{\delta}]^{1\times q} &\longrightarrow \mathcal{A}^{1\times q} \\ \delta_1 \boldsymbol{e}_{\boldsymbol{j}}^T &\mapsto \boldsymbol{\xi}^{\boldsymbol{c}_1} \boldsymbol{e}_{\boldsymbol{j}}^T, \\ \delta_2 \boldsymbol{e}_{\boldsymbol{j}}^T &\mapsto \boldsymbol{\xi}^{\boldsymbol{c}_2} \boldsymbol{e}_{\boldsymbol{j}}^T, \\ &\vdots \\ \delta_r \boldsymbol{e}_{\boldsymbol{j}}^T &\mapsto \boldsymbol{\xi}^{\boldsymbol{c}_r} \boldsymbol{e}_{\boldsymbol{j}}^T. \end{aligned} \tag{29}$$

The following Lemma is a vector version of Lemma 11 and the proof follows using similar arguments as in the scalar case.

Lemma 12. *Consider the free module $\mathbb{R}[\mathcal{C}]^{1\times q}$ of rank q over $\mathbb{R}[\mathcal{C}]$. Let $\Phi^\star$ be the homomorphism of $\mathbb{R}$-algebra modules via the $\mathbb{R}$-algebra homomorphism Φ as defined in equation* (29)*. Then,*

$$\mathbb{R}[\mathcal{C}]^{1\times q} = \text{im } \Phi^\star \cong \frac{\mathbb{R}[\boldsymbol{\delta}]^{1\times q}}{\ker \Phi^\star},$$

where, the isomorphism is between modules over $\mathbb{R}$-algebras via the $\mathbb{R}$-algebra map $\Phi : \mathbb{R}[\boldsymbol{\delta}] \to \mathcal{A}$.

Therefore, for an element $f(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathbb{R}[\mathcal{C}]^{1\times q}$ there exists r-variable polynomials z_i's in $\mathbb{R}[\boldsymbol{\xi}^{\boldsymbol{c}_1}, \ldots, \boldsymbol{\xi}^{\boldsymbol{c}_r}]$ such that, for $i \in \{1, 2, \ldots, q\}$,

$$f(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) = \begin{bmatrix} z_1(\boldsymbol{\xi}^{\boldsymbol{c}}) & z_2(\boldsymbol{\xi}^{\boldsymbol{c}}) & \ldots & z_q(\boldsymbol{\xi}^{\boldsymbol{c}}) \end{bmatrix}. \tag{30}$$

Recall that the generators of the cone $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_r$ were in $\mathbb{Z}^n$. Let $\boldsymbol{c}_{i+} \in \mathbb{Z}^n_{\geqslant 0}$ denote the n-tuple of non-negative integers that contains the non-negative components of $\boldsymbol{c}_i$ with the negative components replaced by zero. Similarly, $\boldsymbol{c}_{i-} \in \mathbb{Z}^n_{\geqslant 0}$ represents the n-tuple of non-negative integers that contains the negative of the negative components of $\boldsymbol{c}_i$ with the positive components replaced by zero. That is, every $\boldsymbol{c}_i \in \mathbb{Z}^n$ can be written as

$$\boldsymbol{c}_i = \boldsymbol{c}_{i+} - \boldsymbol{c}_{i-},$$

where $\boldsymbol{c}_{i+}, \boldsymbol{c}_{i-} \in \mathbb{Z}^n_{\geqslant 0}$. Now, under the map π as defined in equation (25) every monomial $\boldsymbol{\xi}^{\boldsymbol{c}_i} \in \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}]$ has a preimage in $\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}]$. Define $m_i(\boldsymbol{\xi}, \boldsymbol{\eta}) := \pi^{-1}(\boldsymbol{\xi}^{\boldsymbol{c}_i})$. Then the monomial $m_i(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}]$ can be written as

$$\pi^{-1}(\boldsymbol{\xi}^{\boldsymbol{c}_i}) = m_i(\boldsymbol{\xi}, \boldsymbol{\eta}) := \boldsymbol{\xi}^{\boldsymbol{c}_{i+}} \boldsymbol{\eta}^{\boldsymbol{c}_{i-}}.$$

In other words, since the generators of the cone $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r$ are in $\mathbb{Z}^n$, the following $\mathbb{R}$-algebra map can be defined

$$\begin{aligned}
\widehat{\Phi} : \mathbb{R}[\boldsymbol{\delta}] \quad &\longrightarrow \quad \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}] \\
\delta_1 \quad &\mapsto \quad m_1(\boldsymbol{\xi}, \boldsymbol{\eta}), \\
\delta_2 \quad &\mapsto \quad m_2(\boldsymbol{\xi}, \boldsymbol{\eta}), \\
&\vdots \\
\delta_r \quad &\mapsto \quad m_r(\boldsymbol{\xi}, \boldsymbol{\eta}).
\end{aligned} \tag{31}$$

The $\mathbb{R}$-algebra map $\widehat{\Phi}$ induces a homomorphism of $\mathbb{R}$-algebra modules $\widehat{\Phi}^\star$. We define $\widehat{\Phi}^\star$ as, for all $j = 1, 2, \ldots, n$,

$$\begin{aligned}
\widehat{\Phi}^\star : \mathbb{R}[\boldsymbol{\delta}]^{1\times q} \quad &\longrightarrow \quad \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}]^{1\times q} \\
\delta_1 \boldsymbol{e}_{\boldsymbol{j}}^T \quad &\mapsto \quad m_1(\boldsymbol{\xi}, \boldsymbol{\eta}) \boldsymbol{e}_{\boldsymbol{j}}^T, \\
\delta_2 \boldsymbol{e}_{\boldsymbol{j}}^T \quad &\mapsto \quad m_2(\boldsymbol{\xi}, \boldsymbol{\eta}) \boldsymbol{e}_{\boldsymbol{j}}^T, \\
&\vdots \\
\delta_r \boldsymbol{e}_{\boldsymbol{j}}^T \quad &\mapsto \quad m_r(\boldsymbol{\xi}, \boldsymbol{\eta}) \boldsymbol{e}_{\boldsymbol{j}}^T.
\end{aligned} \tag{32}$$

The complete commutative diagram is shown in Fig. 2 where, $\widetilde{\Phi}^\star$ is, by

$$\begin{tikzcd}
 & \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}]^{1\times q} \arrow[d, "\Pi"] \\
\mathbb{R}[\boldsymbol{\delta}]^{1\times q} \arrow[ur, "\widehat{\Phi}^\star"] \arrow[r, "\Phi^\star"] \arrow[dr, "\widetilde{\Phi}^\star"'] & \mathcal{A}^{1\times q} \arrow[d] \\
 & \mathcal{M}
\end{tikzcd}$$

Figure 2: Complete commutative diagram.

construction, the composition of $\Phi^\star$ with the canonical surjection $\mathcal{A}^{1\times q} \twoheadrightarrow \mathcal{M}$.

Proposition 13. *Consider the homomorphism of* $\mathbb{R}$*-algebra modules* $\Psi : \mathcal{Q} \to \mathcal{M}$ *as defined in equation* (14) *and* $\widetilde{\Phi}^\star$ *as shown in the commutative diagram of Fig. 2. Then,* Ψ *is surjective if and only if* $\widetilde{\Phi}^\star$ *is surjective.*

Proof: Recall that an element $\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathbb{R}[\mathcal{C}]^{1\times q}$ can be written as

$$\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) = \begin{bmatrix} z_1(\boldsymbol{\xi^c}) & z_1(\boldsymbol{\xi^c}) & \ldots & z_1(\boldsymbol{\xi^c}) \end{bmatrix}.$$

Define $\boldsymbol{z}(\boldsymbol{\xi^c}) := \begin{bmatrix} z_1(\boldsymbol{\xi^c}) & z_1(\boldsymbol{\xi^c}) & \ldots & z_1(\boldsymbol{\xi^c}) \end{bmatrix}$. Also by Lemma 12, $\mathbb{R}[\mathcal{C}]^{1\times q} = \text{im } \Phi^\star$, therefore, there exists $\boldsymbol{g}(\boldsymbol{\delta}) \in \mathbb{R}[\boldsymbol{\delta}]^{1\times q}$ such that $\boldsymbol{z}(\boldsymbol{\xi^c}) = \Phi^\star(\boldsymbol{g}(\boldsymbol{\delta}))$.
(Only if) Ψ being surjective implies for all $\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A}^{1\times q}$, there exists $\boldsymbol{z}(\boldsymbol{\xi^c}) \in \mathbb{R}[\mathcal{C}]^{1\times q}$ such that

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) &\equiv \boldsymbol{z}(\boldsymbol{\xi^c}) \mod \mathcal{R} \\ &\equiv \Phi^\star(\boldsymbol{g}(\boldsymbol{\delta})) \mod \mathcal{R}. \end{aligned}$$

Under the canonical surjection,

$$\begin{aligned} \overline{\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})} &= \overline{\Phi^\star(\boldsymbol{g}(\boldsymbol{\delta}))} \\ &= \widetilde{\Phi}^\star(\boldsymbol{g}(\boldsymbol{\delta})). \end{aligned}$$

Since $\boldsymbol{f} \in \mathcal{A}^{1\times q}$ was chosen arbitrarily, $\widetilde{\Phi}^\star$ is surjective.
(If) The map $\widetilde{\Phi}^\star$ being surjective implies for every $m \in \mathcal{M}$, there exists $\boldsymbol{g}(\boldsymbol{\delta}) \in \mathbb{R}[\boldsymbol{\delta}]^{1\times q}$ such that

$$\widetilde{\Phi}^\star(\boldsymbol{g}(\boldsymbol{\delta})) = m$$

This implies that for all $\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A}^{1\times q}$, there exists $\boldsymbol{g}(\boldsymbol{\delta}) \in \mathbb{R}[\boldsymbol{\delta}]^{1\times q}$ such that

$$\overline{\Phi^\star(\boldsymbol{g}(\boldsymbol{\delta}))} = \overline{\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})}.$$

Now, from Lemma 12 we have $\mathbb{R}[\mathcal{C}]^{1\times q} = \text{im } \Phi^\star$. Also any element in $\mathbb{R}[\mathcal{C}]^{1\times q}$ can be written as $\boldsymbol{z}(\boldsymbol{\xi^c})$ using equation (30). Therefore, $\Phi^\star(\boldsymbol{g}(\boldsymbol{\delta})) = \boldsymbol{z}(\boldsymbol{\xi^c})$, thus, $\overline{\boldsymbol{z}(\boldsymbol{\xi^c})} = \overline{\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})}$. Hence, Ψ is surjective. $\square$

According to Theorem 4, for a cone $\mathcal{C} \subseteq \mathbb{Z}^n$ to be a characteristic cone for a behavior $\mathfrak{B}$ defined by the equation module $\mathcal{R}$, the map $\Psi : \mathcal{Q} \to \mathcal{M}$ must be surjective. We have shown in Proposition 13 that this is equivalent to $\widetilde{\Phi}^\star$ being surjective. In other words, the homomorphism of $\mathbb{R}$-algebra modules

$$\widetilde{\Phi}^\star : \mathbb{R}[\boldsymbol{\delta}]^{1\times q} \longrightarrow \mathcal{M} \tag{33}$$

must be surjective. Recall that $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ and $\Pi : \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}]^{1\times q} \to \mathcal{A}^{1\times q}$. Define $\widetilde{\mathcal{R}} := \Pi^{-1}(\mathcal{R})$. Therefore, $\widetilde{\mathcal{R}} \subseteq \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}]^{1\times q}$. From the commutative diagram, we have

$$\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}]^{1\times q} \xrightarrow{\Pi} \mathcal{A}^{1\times q} \longrightarrow \mathcal{M}.$$

Using the First Isomorphism theorem, $\mathcal{A}^{1\times q} \cong \frac{\mathbb{R}[\boldsymbol{\xi},\boldsymbol{\eta}]^{1\times q}}{\ker \Pi}$ as Π is surjective. Combining it with the canonical surjection we have

$$\mathcal{M} = \frac{\mathcal{A}^{1\times q}}{\mathcal{R}} \cong \frac{\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}]^{1\times q}}{\widetilde{\mathcal{R}} + \ker \Pi}.$$

To check surjectivity of $\widetilde{\Phi}^{\star}$, we require the following construction. This facilitates the use of Gröbner bases techniques. Recall the homomorphism $\widehat{\Phi}^{\star}$ of $\mathbb{R}$-algebra modules induced by the $\mathbb{R}$-algebra map $\widehat{\Phi}$. Consider the free module $\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]^{1\times q}$. Define a submodule $\mathcal{T}$ of $\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]^{1\times q}$ as

$$\mathcal{T} = \mathrm{rowspan}_{\mathbb{R}[\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\delta}]} T$$

where the matrix

$$T = \ \mathrm{diag}\left(\underbrace{\begin{bmatrix}\delta_1 - m_1\\ \vdots \\ \delta_r - m_r\end{bmatrix}, \begin{bmatrix}\delta_1 - m_1\\ \vdots \\ \delta_r - m_r\end{bmatrix}, \ldots, \begin{bmatrix}\delta_1 - m_1\\ \vdots \\ \delta_r - m_r\end{bmatrix}}_{q \text{ entries}}\right).$$

Define the submodule

$$\mathcal{K} = \widetilde{\mathcal{R}}\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}] + (\ker \Pi)\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}] + \mathcal{T} \subseteq \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]^{1\times q}. \tag{34}$$

Let $\succ$ be an elimination term ordering on $\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]$ such that $\boldsymbol{\xi} \succ \boldsymbol{\eta} \succ \boldsymbol{\delta}$. Define the corresponding "term over position" module ordering $\succ_{\mathrm{TOP}}$. Let $\mathcal{G} = \{\boldsymbol{g_1}, \ldots, \boldsymbol{g_s}\}$ be a Gröbner basis of $\mathcal{K}$ with respect to the term ordering $\succ_{\mathrm{TOP}}$.

Lemma 14. *Let $\widetilde{\Phi}^{\star}$ be the homomorphism of $\mathbb{R}$-algebra modules defined in equation* (33). *Consider the free module $\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]^{1\times q}$ and the submodule $\mathcal{K} \subseteq \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]^{1\times q}$ as defined in equation* (34). *Let $\mathcal{G}$ be a Gröbner basis of $\mathcal{K}$ with respect to the elimination module ordering $\succ_{\mathrm{TOP}}$. For an element $\boldsymbol{f} \in \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}]^{1\times q}$, $\overline{\boldsymbol{f}} \in \mathrm{im}\ \widetilde{\Phi}^{\star}$ if and only if $\overline{\boldsymbol{f}}^{\mathcal{G}} \in \mathbb{R}[\boldsymbol{\delta}]^{1\times q}$, where, $\overline{\boldsymbol{f}}^{\mathcal{G}}$ denotes the remainder of $\boldsymbol{f}$ obtained after dividing it by elements of $\mathcal{G}$.*

Proof: The element $\boldsymbol{f} \in \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]^{1\times q}$ can be written as

$$\boldsymbol{f} = q_1 \boldsymbol{g_1} + \cdots + q_s \boldsymbol{g_s} + \boldsymbol{r}$$

where $q_i \in \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]$ and $\boldsymbol{r} = \overline{\boldsymbol{f}}^{\mathcal{G}} \in \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]^{1\times q}$.
(If) Suppose $\boldsymbol{r} \in \mathbb{R}[\boldsymbol{\delta}]^{1\times q}$. This implies $\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) - \boldsymbol{r}(\boldsymbol{\delta}) \in \mathcal{K}$ (see Proposition 9). Under $\Phi^\star$ we have

$$\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) - \boldsymbol{r}(m_1, \ldots, m_r) \in \widetilde{\mathcal{R}} + \ker \Pi.$$

Applying the equivalence relation on $\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}]$, that is, Π followed by the canonical surjection, we get

$$\begin{aligned} 0 &= \overline{\boldsymbol{f}} - \overline{\boldsymbol{r}(m_1, \ldots, m_r)} \\ \Rightarrow \overline{\boldsymbol{f}} &= \overline{\boldsymbol{r}(m_1, \ldots, m_r)} \\ \Rightarrow \overline{\boldsymbol{f}} &= \widetilde{\Phi}^\star \left(\boldsymbol{r}(\delta_1, \ldots, \delta_r)\right) \end{aligned}$$

which shows $\overline{\boldsymbol{f}} \in \operatorname{im} \widetilde{\Phi}^\star$.
(Only if) Conversely, suppose $\overline{\boldsymbol{f}} \in \operatorname{im} \widetilde{\Phi}^\star$. Then, there exists $\boldsymbol{h} \in \mathbb{R}[\boldsymbol{\delta}]^{1\times q}$ such that $\overline{\boldsymbol{f}} = \widetilde{\Phi}^\star(\boldsymbol{h})$. This implies there exists $\boldsymbol{p} \in \widetilde{\mathcal{R}} + \ker \Pi$ such that,

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) + \boldsymbol{p}(\boldsymbol{\xi}, \boldsymbol{\eta}) &= \widehat{\Phi}^\star(\boldsymbol{h}(\boldsymbol{\delta})) \\ \boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) + \boldsymbol{p}(\boldsymbol{\xi}, \boldsymbol{\eta}) &= \boldsymbol{h}(m_1, \ldots, m_r) \end{aligned}$$

Now,

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) &\equiv \boldsymbol{h}(m_1, \ldots, m_r) \mod \mathcal{K} \\ \boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) &\equiv \boldsymbol{h}(\boldsymbol{\delta}) \mod \mathcal{K}. \end{aligned}$$

Thus

$$\begin{aligned} \overline{\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\eta})}^{\mathcal{G}} &= \overline{\boldsymbol{h}(m_1, \ldots, m_r)}^{\mathcal{G}} \\ &= \overline{\boldsymbol{h}(\delta_1, \ldots, \delta_r)}^{\mathcal{G}}. \end{aligned}$$

Since $\boldsymbol{\xi} \succ \boldsymbol{\eta} \succ \boldsymbol{\delta}$, from elimination theorem (Proposition 10) it follows that a Gröbner basis for $\mathcal{K} \cap \mathbb{R}[\boldsymbol{\delta}]^{1\times q}$ is $\mathcal{G} \cap \mathbb{R}[\boldsymbol{\delta}]^{1\times q}$. Therefore,

$$\overline{\boldsymbol{h}(\delta_1, \ldots, \delta_r)}^{\mathcal{G}} = \overline{\boldsymbol{h}(\delta_1, \ldots, \delta_r)}^{\mathcal{G} \cap \mathbb{R}[\boldsymbol{\delta}]^{1\times q}}$$

because $\boldsymbol{h}(\delta_1, \ldots, \delta_r) \in \mathbb{R}[\boldsymbol{\delta}]^{1\times q}$.
But, $\overline{\boldsymbol{h}(\delta_1, \ldots, \delta_r)}^{\mathcal{G} \cap \mathbb{R}[\boldsymbol{\delta}]^{1\times q}} \in \mathbb{R}[\boldsymbol{\delta}]^{1\times q}$. Therefore $\overline{f(\boldsymbol{\xi}, \boldsymbol{\eta})}^{\mathcal{G}} \in \mathbb{R}[\boldsymbol{\delta}]^{1\times q}$. $\square$

In order to check if a given cone is a characteristic cone for a behavior $\mathfrak{B}$ the first algebraic characterization states that the homomorphism of $\mathbb{R}$-algebra modules $\Psi : \mathcal{Q} \to \mathcal{M}$ needs to be surjective (Theorem 4). This condition was transformed into another equivalent algebraic condition in Proposition 13 which states that Ψ being surjective is equivalent to $\widetilde{\Phi}^\star$ being surjective. To check surjectivity of $\widetilde{\Phi}^\star$ we defined the submodule $\mathcal{K} \subseteq \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]^{1\times q}$ as in equation (34). Using the previous lemma (Lemma 14) we now state the second main result which gives an equivalent condition for a cone to be a characteristic cone using a Gröbner basis for the submodule $\mathcal{K}$. This in turn provides a directly implementable algorithm for testing.

Theorem 15. *Let $\mathfrak{B}$ be an nD discrete autonomous behavior with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$. Consider the free module $\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]^{1\times q}$ and the submodule $\mathcal{K} = \widetilde{\mathcal{R}}\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}] + (\ker \Pi)\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}] + \mathcal{T} \subseteq \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]^{1\times q}$. Then a cone (or, equivalently, an affine semigroup) $\mathcal{C} \subseteq \mathbb{Z}^n$ is a characteristic cone for the behavior $\mathfrak{B}$ if and only if the elements*

$$\xi_1 \boldsymbol{e}_{\boldsymbol{j}}^T, \dots, \xi_n \boldsymbol{e}_{\boldsymbol{j}}^T, \eta_1 \boldsymbol{e}_{\boldsymbol{j}}^T, \dots, \eta_n \boldsymbol{e}_{\boldsymbol{j}}^T$$

on division by the Gröbner basis of $\mathcal{K}$ under the elimination module term ordering $\succ_{\mathrm{TOP}}$ contains elements only in $\mathbb{R}[\boldsymbol{\delta}]^{1\times q}$ for all $j \in \{1, 2, \dots, q\}$.

Proof: (Only if) By Theorem 4, $\mathcal{C}$ being a characteristic cone implies Ψ is surjective. This in turn implies $\widetilde{\Phi}^\star$ is surjective by Proposition 13. Now, $\widetilde{\Phi}^\star$ being surjective implies for every $\xi_1 \boldsymbol{e}_{\boldsymbol{j}}^T, \dots, \xi_n \boldsymbol{e}_{\boldsymbol{j}}^T, \eta_1 \boldsymbol{e}_{\boldsymbol{j}}^T, \dots, \eta_n \boldsymbol{e}_{\boldsymbol{j}}^T \in \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}]^{1\times q}$, there exists a preimage in $\mathbb{R}[\boldsymbol{\delta}]^{1\times q}$. By Lemma 14, $\overline{\boldsymbol{f}} \in \operatorname{im} \widetilde{\Phi}^\star$ implies $\overline{\boldsymbol{f}}^{\mathcal{G}} \in \mathbb{R}[\boldsymbol{\delta}]^{1\times q}$. Since $\widetilde{\Phi}^\star$ is surjective this holds for all $\xi_1 \boldsymbol{e}_{\boldsymbol{j}}^T, \dots, \xi_n \boldsymbol{e}_{\boldsymbol{j}}^T, \eta_1 \boldsymbol{e}_{\boldsymbol{j}}^T$, $\dots, \eta_n \boldsymbol{e}_{\boldsymbol{j}}^T$ for all $j \in \{1, 2, \dots, q\}$.

(If) If $\xi_1 \boldsymbol{e}_{\boldsymbol{j}}^T, \dots, \xi_n \boldsymbol{e}_{\boldsymbol{j}}^T, \eta_1 \boldsymbol{e}_{\boldsymbol{j}}^T, \dots, \eta_n \boldsymbol{e}_{\boldsymbol{j}}^T$ on division by the Gröbner basis of $\mathcal{K}$ contains elements in $\mathbb{R}[\boldsymbol{\delta}]$ then by Lemma 14, $\overline{\xi_1 \boldsymbol{e}_{\boldsymbol{j}}^T}, \dots, \overline{\xi_n \boldsymbol{e}_{\boldsymbol{j}}^T}, \overline{\eta_1 \boldsymbol{e}_{\boldsymbol{j}}^T}, \dots, \overline{\eta_n \boldsymbol{e}_{\boldsymbol{j}}^T}$ $\in \operatorname{im} \widetilde{\Phi}^\star$ for all $j \in \{1, 2, \dots, q\}$ which implies $\widetilde{\Phi}^\star$ is surjective. By Proposition 13 this implies Ψ is surjective which in turn implies, by Theorem 4, that $\mathcal{C}$ is a characteristic cone. $\square$

This algorithm checks if a given cone $\mathcal{C} \subseteq \mathbb{Z}^n$ is a characteristic cone for a behavior $\mathfrak{B}$ given by the equation module $\mathcal{R}$.

Algorithm 16. *Input:*

1. *The system equations given in kernel representation as* $\mathfrak{B} = \ker R$ *where,* $R(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A}^{r\times q}$.

2. *The cone* $\mathcal{C}$ *generated by* $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r \in \mathbb{Z}^n$.

Output:

Yes, if $\mathcal{C}$ *is a characteristic cone for* $\mathfrak{B}$.

No, if $\mathcal{C}$ *is not a characteristic cone.*

Algorithm

1. *Define the free module* $\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]^{1\times q}$ *and the ideal* $\mathcal{K} \subseteq \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]^{1\times q}$ *as*

$$\mathcal{K} = \widetilde{\mathcal{R}}\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}] + (\ker \Pi)\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}] + \mathcal{T}.$$

Note that if $\boldsymbol{r_1}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}), \boldsymbol{r_2}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}), \ldots, \boldsymbol{r_m}(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1}) \in \mathcal{A}^{1\times q}$ *are rows of* $R(\boldsymbol{\xi}, \boldsymbol{\xi}^{-1})$ *then*

$$\mathcal{R} = \{a_1\boldsymbol{r_1} + \cdots + a_m\boldsymbol{r_m} \mid a_i \in \mathcal{A}, i = 1, \ldots, m\} \subseteq \mathcal{A}^{1\times q}.$$

2. *Calculate the Gröbner basis*[5] $\mathcal{G} = \{\boldsymbol{g_1}, \ldots, \boldsymbol{g_s}\}$ *of* $\mathcal{K}$ *with elimination term ordering* $\boldsymbol{\xi} \succ \boldsymbol{\eta} \succ \boldsymbol{\delta}$ *on* $\mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\delta}]$ *and the corresponding elimination module term ordering* $\succ_{\mathrm{TOP}}$.

3. *Calculate the remainders of* $\xi_1\boldsymbol{e_j^T}, \ldots, \xi_n\boldsymbol{e_j^T}, \eta_1\boldsymbol{e_j^T}, \ldots, \eta_n\boldsymbol{e_j^T}$ *for all* $j \in \{1, 2, \ldots q\}$ *by division with* $\mathcal{G}$.

4. *If* $\overline{\xi_1\boldsymbol{e_j^T}}^{\mathcal{G}}, \ldots, \overline{\xi_n\boldsymbol{e_j^T}}^{\mathcal{G}}, \overline{\eta_1\boldsymbol{e_j^T}}^{\mathcal{G}}, \ldots, \overline{\eta_n\boldsymbol{e_j^T}}^{\mathcal{G}} \in \mathbb{R}[\boldsymbol{\delta}]^{1\times q}$ *for all* $j \in \{1, 2, \ldots q\}$, *then* $\mathcal{C}$ *is a characteristic cone for* $\mathfrak{B}$.

5. *If not, then* $\mathcal{C}$ *is not a characteristic cone.*

7 Examples

In this section we illustrate the above ideas with some examples.

[5] An algorithm for calculating the Gröbner basis of a module can be found in [22, Algorithm 3.5.2].

Example 17. Here we verify Example 4 of [3] to show that the proposed method is indeed a generalization of the method proposed in [3]. Consider the 2D behavior with kernel representation

$$\mathfrak{B} = \ker \begin{bmatrix} 1+\sigma_1^{-1} & -\sigma_1^{-1}\sigma_2^{-1} \\ -2-\sigma_2^{-1} & \sigma_2^{-1} \end{bmatrix}. \tag{35}$$

The equation module $\mathcal{R} = \text{rowspan}_{\mathcal{A}} \begin{bmatrix} 1+\xi_1^{-1} & -\xi_1^{-1}\xi_2^{-1} \\ -2-\xi_2^{-1} & \xi_2^{-1} \end{bmatrix} \subseteq \mathbb{R}[\xi_1^{\pm 1}, \xi_2^{\pm 1}]^{1\times 2}$. According to [3] we need to check if the cones $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are characteristic cones for $\mathfrak{B}$.

$$\mathcal{C}_1 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \mathcal{C}_2 = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}, \quad \mathcal{C}_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

The claims are validated using the Algorithm 16. To check if $\mathcal{C}_1$ is characteristic cone we obtain

$$\begin{array}{lll lll}
\overline{\xi_1 \boldsymbol{e_1}^T}^{\mathcal{G}} &=& \delta_2 \boldsymbol{e_2}^T - \boldsymbol{e_1}^T, & \overline{\xi_2 \boldsymbol{e_1}^T}^{\mathcal{G}} &=& -\frac{1}{2}\boldsymbol{e_1}^T + \frac{1}{2}\boldsymbol{e_2}^T, \\
\overline{\eta_1 \boldsymbol{e_1}^T}^{\mathcal{G}} &=& (-\delta_1 - \frac{1}{2})\boldsymbol{e_1}^T + \frac{1}{2}\boldsymbol{e_2}^T, & \overline{\eta_2 \boldsymbol{e_1}^T}^{\mathcal{G}} &=& \delta_2 \boldsymbol{e_2}^T - 2\boldsymbol{e_1}^T, \\
\overline{\xi_1 \boldsymbol{e_2}^T}^{\mathcal{G}} &=& (\delta_2 + 1)\boldsymbol{e_2}^T, & \overline{\xi_2 \boldsymbol{e_2}^T}^{\mathcal{G}} &=& (\delta_1 - \frac{1}{2})\boldsymbol{e_1}^T + (\delta_1 + \frac{1}{2})\boldsymbol{e_2}^T, \\
\overline{\eta_1 \boldsymbol{e_2}^T}^{\mathcal{G}} &=& (\delta_1 - \frac{1}{2})\boldsymbol{e_1}^T + \frac{1}{2}\boldsymbol{e_2}^T, & \overline{\eta_2 \boldsymbol{e_2}^T}^{\mathcal{G}} &=& \delta_2 \boldsymbol{e_2}^T,
\end{array}$$

which ensures $\mathcal{C}_1$ is a characteristic cone by Theorem 15. For verifying if $\mathcal{C}_2$ is a characteristic cone, we obtain

$$\begin{array}{lll lll}
\overline{\xi_1 \boldsymbol{e_1}^T}^{\mathcal{G}} &=& (\frac{1}{2}\delta_1 + 1)\boldsymbol{e_1}^T + \frac{1}{2}\delta_1 \boldsymbol{e_2}^T, & \overline{\xi_2 \boldsymbol{e_1}^T}^{\mathcal{G}} &=& -\frac{1}{2}\boldsymbol{e_1}^T + \frac{1}{2}\boldsymbol{e_2}^T, \\
\overline{\eta_1 \boldsymbol{e_1}^T}^{\mathcal{G}} &=& (\frac{1}{2}\delta_1 + 1)\boldsymbol{e_1}^T - \frac{1}{2}\delta_1 \boldsymbol{e_2}^T, & \overline{\eta_2 \boldsymbol{e_1}^T}^{\mathcal{G}} &=& \frac{1}{2}\delta_1 \boldsymbol{e_1}^T + \frac{1}{2}\delta_2 \boldsymbol{e_2}^T, \\
\overline{\xi_1 \boldsymbol{e_2}^T}^{\mathcal{G}} &=& (\frac{1}{2}\delta_1 + 2)\boldsymbol{e_1}^T + (\frac{1}{2}\delta_1 + 1)\boldsymbol{e_2}^T, & \overline{\xi_2 \boldsymbol{e_2}^T}^{\mathcal{G}} &=& (\delta_2 - 1)\boldsymbol{e_2}^T, \\
\overline{\eta_1 \boldsymbol{e_2}^T}^{\mathcal{G}} &=& (-\frac{1}{2}\delta_1 - 2)\boldsymbol{e_1}^T + (\frac{1}{2}\delta_1 + 1)\boldsymbol{e_2}^T, & \overline{\eta_2 \boldsymbol{e_2}^T}^{\mathcal{G}} &=& (\frac{1}{2}\delta_1 + 2)\boldsymbol{e_1}^T + \frac{1}{2}\delta_1 \boldsymbol{e_2}^T,
\end{array}$$

which verifies $\mathcal{C}_2$ to be a characteristic cone. Finally, for the cone $\mathcal{C}_3$ we follow the same exercise of computing the remainders $\overline{\xi_1 \boldsymbol{e_j}^T}^{\mathcal{G}}, \overline{\xi_2 \boldsymbol{e_j}^T}^{\mathcal{G}}, \overline{\eta_1 \boldsymbol{e_j}^T}^{\mathcal{G}}, \overline{\eta_2 \boldsymbol{e_j}^T}^{\mathcal{G}}$ for $j = 1, 2$. We obtain

$$\begin{array}{lll lll}
\overline{\xi_1 \boldsymbol{e_1}^T}^{\mathcal{G}} &=& (\delta_1 - 1)\boldsymbol{e_2}^T - \boldsymbol{e_1}^T, & \overline{\xi_2 \boldsymbol{e_1}^T}^{\mathcal{G}} &=& -\frac{1}{2}\boldsymbol{e_1}^T + \frac{1}{2}\boldsymbol{e_2}^T, \\
\overline{\eta_1 \boldsymbol{e_1}^T}^{\mathcal{G}} &=& (-\delta_2 - \frac{1}{2})\boldsymbol{e_1}^T + \frac{1}{2}\boldsymbol{e_2}^T, & \overline{\eta_2 \boldsymbol{e_1}^T}^{\mathcal{G}} &=& (\delta_1 - 1)\boldsymbol{e_2}^T - 2\boldsymbol{e_1}^T, \\
\overline{\xi_1 \boldsymbol{e_2}^T}^{\mathcal{G}} &=& \delta_1 \boldsymbol{e_2}^T, & \overline{\xi_2 \boldsymbol{e_2}^T}^{\mathcal{G}} &=& (\delta_2 - \frac{1}{2})\boldsymbol{e_1}^T + (\delta_2 + \frac{1}{2})\boldsymbol{e_2}^T, \\
\overline{\eta_1 \boldsymbol{e_2}^T}^{\mathcal{G}} &=& (\delta_2 - \frac{1}{2})\boldsymbol{e_1}^T + \frac{1}{2}\boldsymbol{e_2}^T, & \overline{\eta_2 \boldsymbol{e_2}^T}^{\mathcal{G}} &=& (\delta_1 - 1)\boldsymbol{e_2}^T,
\end{array}$$

which validates the claim of $\mathcal{C}_3$ being a characteristic cone.

Example 18. Consider the 3D behavior with kernel representation

$$\mathfrak{B} = \ker \begin{bmatrix} \sigma_2 & \sigma_2^{-1}\sigma_3^{-1} - 1 \\ \sigma_1^{-1}\sigma_3^{-1} & \sigma_1 + \sigma_2 \\ \sigma_3 & \sigma_2^3 \end{bmatrix} \tag{36}$$

The equation module

$$\mathcal{R} = \mathrm{rowspan}_{\mathcal{A}} \begin{bmatrix} \xi_2 & \xi_2^{-1}\xi_3^{-1} - 1 \\ \xi_1^{-1}\xi_3^{-1} & \xi_1 + \xi_2 \\ \xi_3 & \xi_2^3 \end{bmatrix} \subseteq \mathbb{R}[\xi_1^{\pm 1}, \xi_2^{\pm 1}, \xi_3^{\pm 1}]^{1\times 2}.$$

The claim is that the all positive orthant, that is, the cone generated by $\boldsymbol{c}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $\boldsymbol{c}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ and $\boldsymbol{c}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ is a characteristic cone for the behavior $\mathfrak{B}$. To verify the claim we compute the remainders of $\xi_1\boldsymbol{e_j}^T, \xi_2\boldsymbol{e_j}^T, \xi_3\boldsymbol{e_j}^T, \eta_1\boldsymbol{e_j}^T, \eta_2\boldsymbol{e_j}^T, \eta_3\boldsymbol{e_j}^T$ for $j = 1, 2$ by division with a Gröbner basis $\mathcal{G}$ of $\mathcal{K}$ constructed using equation (34). We find that

$$\begin{array}{lcllcl}
\overline{\xi_1\boldsymbol{e_1}^T}^{\mathcal{G}} &=& \delta_1\boldsymbol{e_1}^T, & \overline{\xi_2\boldsymbol{e_1}^T}^{\mathcal{G}} &=& \delta_2\boldsymbol{e_1}^T, \\
\overline{\xi_3\boldsymbol{e_1}^T}^{\mathcal{G}} &=& \delta_3\boldsymbol{e_1}^T, & \overline{\eta_1\boldsymbol{e_1}^T}^{\mathcal{G}} &=& (-\delta_1\delta_3 - \delta_2\delta_3)\boldsymbol{e_2}^T, \\
\overline{\eta_2\boldsymbol{e_1}^T}^{\mathcal{G}} &=& (\delta_2^4 + \delta_3)\boldsymbol{e_1}^T, & \overline{\eta_3\boldsymbol{e_1}^T}^{\mathcal{G}} &=& (-\delta_1^2 - \delta_1\delta_2)\boldsymbol{e_2}^T, \\
\overline{\xi_1\boldsymbol{e_2}^T}^{\mathcal{G}} &=& \delta_1\boldsymbol{e_2}^T, & \overline{\xi_2\boldsymbol{e_2}^T}^{\mathcal{G}} &=& \delta_2\boldsymbol{e_2}^T, \\
\overline{\xi_3\boldsymbol{e_2}^T}^{\mathcal{G}} &=& \delta_3\boldsymbol{e_2}^T, & \overline{\eta_1\boldsymbol{e_2}^T}^{\mathcal{G}} &=& (-\delta_1\delta_2\delta_3^5 - \delta_1\delta_3^4 - \delta_2\delta_3^4 - \\
&&&&& \delta_3^3)\boldsymbol{e_1}^T + (\delta_1\delta_2^2\delta_3^2 + \delta_1\delta_3^5 + \delta_3^4)\boldsymbol{e_2}^T, \\
\overline{\eta_2\boldsymbol{e_2}^T}^{\mathcal{G}} &=& -\delta_2\delta_3\boldsymbol{e_1}^T + \delta_3\boldsymbol{e_2}^T, & \overline{\eta_3\boldsymbol{e_2}^T}^{\mathcal{G}} &=& -\delta_2^2\boldsymbol{e_1}^T + \delta_2\boldsymbol{e_2}^T.
\end{array}$$

Thus, by Algorithm 16, $\mathcal{C}$ is a characteristic cone for the behavior $\mathfrak{B}$.

8 Concluding Remarks

This paper gives an algebraic characterization of characteristic cones for discrete autonomous nD systems. The crucial observation of the fact that cones in $\mathbb{Z}^n$ have the structure of an affine semigroup have been explored. First a necessary and sufficient condition for checking if a given cone is a characteristic cone for an autonomous nD behavior is given. This condition is converted to another equivalent algebraic condition which enables us to provide an algorithm for doing this check using standard computer algebra packages. Lastly we provide an algorithm which can be used to do this test. Thus a complete solution to the problem of determining if a given cone is a characteristic cone for a discrete autonomous nD system is given here.

Investigating the computational aspects associated with Gröbner basis calculation for modules, such as, improvement of speed and computational efficiency is a future goal. Other possible future directions of research include carrying forward this analysis of characterizing characteristic sets to various subsets other than cones. We would like to extend the results on stability analysis of autonomous nD systems with respect to characteristic cones. Design of stabilizing controllers in the presence of inputs is yet another direction of future research.

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