On characteristic cones of scalar autonomous nD systems, with general n

Mousumi Mukherjee and Debasattam Pal

***Abstract*—In this paper, we give an algebraic condition that is equivalent to a given cone being a characteristic cone for a scalar autonomous nD system, for a general n. The nD systems that we consider are described by linear partial difference equations with real constant coefficients. We obtain this result by exploring the fact that cones in the domain (the nD integer grid) have a rich algebraic structure – that of an affine semigroup. The need for a novel algebraic characterization arises because the method used for 2D systems does not extend for $n \geqslant 3$; we show this by an example. The necessary and sufficient condition that we derive can be used to check whether a given cone is a characteristic cone for a scalar autonomous nD system by standard computer algebra packages.**

***Index Terms*—Multidimensional systems, characteristic cones, affine semigroups.**

I. INTRODUCTION

Physical systems having one independent variable, namely, time, are described by ordinary differential/difference equations. However, many applications require more than one independent variables to model the dynamics of the physical system. The independent variables can be spatial coordinates, which include applications from image and video processing, or there can be some spatial variables and a time variable, which include applications from electromagnetic theory, quantum mechanics, grid sensor networks and multidimensional filtering [1]. Such systems are described by partial differential/difference equations (PDEs) in n independent variables and are known as *multidimensional* systems or, in short, nD systems.

The solution of a homogeneous k-th order ordinary difference equation is *uniquely* specified if k independent initial conditions are known. These initial conditions are composed of the values of the solution at k distinct points on the discrete time axis. This set of k points is a *characteristic set* for the system: Characteristic sets are special subsets of the domain (the set over which the trajectories evolve) with the defining property that, for every trajectory in the system, the knowledge of its values on the characteristic set uniquely identifies the trajectory over the whole domain. Characteristic sets are useful in studying system properties such as stability [2], [3], Markovian-ness [4], finite dimensionality [5], [3], etc. Unlike 1D systems, for nD systems with $n \geqslant 2$, characteristic sets need not always be a finite collection of points in the domain; rather, they come in various shapes and sizes. Indeed, for $n \geqslant 2$ characteristic sets may contain infinitely many points (for example, see [6], where it was shown that every 2D autonomous system admits a finite union of parallel lines as characteristic sets). Properties of characteristic sets, in particular, characteristic cones, and their applicability in stability analysis for 2D systems were studied by Valcher [3]. However, Valcher's method of checking whether a given set is a characteristic set for a system cannot be extended to nD systems, with $n \geqslant 3$. The principal reason behind that is as follows. Valcher's method uses a decomposition [3, Proposition 4.1] of 2D autonomous systems into a sum of two subsystems, where one is finite dimensional and the other is square. By this decomposition, the problem of checking for characteristic sets of general 2D autonomous systems reduces to doing the same for only *square* 2D autonomous systems [3, Lemma 2.6]. This decomposition, it turns out, does not extend to $n \geqslant 3$. Consequently, Valcher's methods become unusable for general nD systems with $n \geqslant 3$. We elaborate on this in Section III.

To circumvent this difficulty of decomposition, we propose an entirely new approach to determine if a given cone is a characteristic cone for an nD autonomous system with $n \geqslant 2$. Our approach explores the fact that cones in the integer grid $\mathbb{Z}^n$ have rich algebraic structures. The main result of this paper, Theorem 5, uses these algebraic structures of cones to solve the problem of determining characteristic cones for nD autonomous systems, with general $n \geqslant 2$.

The organization of the paper is as follows: Section II introduces the notation and preliminaries used in the paper. Section III explains why extension of Valcher's result for $n \geqslant 3$ is not possible. The relation between polyhedral cones, affine semigroups and the algebra generated by them is discussed in Section IV. The main result of this paper is presented in Section V. Examples to validate the main result for higher dimensions ($n \geqslant 3$) are given in Section VI.

II. NOTATION AND PRELIMINARIES

A. Notation

The ring of integers and the collection of all n-tuples of integers are denoted by $\mathbb{Z}$ and $\mathbb{Z}^n$, respectively. We use the symbols $\mathbb{R}$ and $\mathbb{C}$ to denote the fields of real numbers and complex numbers, respectively. Non-negative real numbers are represented by $\mathbb{R}_{\geqslant 0}$. The set of natural numbers, that is, $\{0, 1, 2, 3, \ldots\}$, is denoted by $\mathbb{N}$. The set of all maps from $\mathbb{Z}^n$ to $\mathbb{R}$ is denoted by $\mathcal{W}$, that is, $\mathcal{W} := \{w : \mathbb{Z}^n \longrightarrow \mathbb{R}\}$; often, we use the symbol $(\mathbb{R})^{\mathbb{Z}^n}$, too, for the set $\mathcal{W}$. The action of the i-th shift operator, denoted by σ_i, on the trajectory $w \in \mathcal{W}$ is defined by

$$\sigma_i w(\mathbf{k}) = w(k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_n),$$

The authors are with the Department of Electrical Engineering, Indian Institute of Technology Bombay, Powai, Mumbai, India. E-mail: {`mousumi, debasattam`}`@ee.iitb.ac.in`

where $\mathbf{k} = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. We use $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{-1}$ to denote the n-tuple of shift operators and that of the inverse shift operators, respectively. That is, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ and $\boldsymbol{\sigma}^{-1} = (\sigma_1^{-1}, \ldots, \sigma_n^{-1})$. The Laurent polynomial ring in n variables with real coefficients is denoted by $\mathcal{A} := \mathbb{R}[\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}] = \mathbb{R}[\boldsymbol{\sigma}^{\pm 1}]$. For $\mathbf{k} = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$, the symbol $\boldsymbol{\sigma}^{\mathbf{k}}$ denotes the monomial $\sigma_1^{k_1}\sigma_2^{k_2}\cdots\sigma_n^{k_n}$. To denote an element $f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) \in \mathcal{A}$ we often drop the argument and write just f and we hope it to be understood from the context. To denote a number, which is unspecified, the symbol $\bullet$ is used. For example, $R \in \mathcal{A}^{\bullet\times 1}$ means R is a matrix with entries from $\mathcal{A}$ that has 1 column but an unspecified number of rows. For a set $\mathcal{S}$, the symbol $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$.

B. Discrete scalar nD systems

A *scalar* nD system has only one variable of interest w, which is a real-valued function of n independent variables, $k_1, \ldots, k_n$ that take integral values. Denoting this n-tuple of integers $(k_1, \ldots, k_n)$ by $\mathbf{k}$, we note that $w(\mathbf{k}) \in \mathbb{R}$ for all $\mathbf{k} \in \mathbb{Z}^n$. We call this function, w, a *trajectory*. In this paper, we consider trajectories that are solutions of linear partial difference equations with constant real coefficients. Such difference equations are succinctly written using shift operators σ_i's for $i = 1, 2, 3, \ldots, n$. The action of a shift operator on a trajectory is defined as

$$\left(\sigma_i^j w\right)(\mathbf{k}) := w(k_1, \ldots, k_{i-1}, k_i + j, k_{i+1}, \ldots, k_n),$$

where $j \in \mathbb{Z}$. A Laurent monomial is of the form $\boldsymbol{\sigma}^{\boldsymbol{\nu}} := \sigma_1^{\nu_1}\ldots\sigma_n^{\nu_n}$, where $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$. The action of such a monomial on a trajectory is defined as

$$\left(\boldsymbol{\sigma}^{\boldsymbol{\nu}} w\right)(\mathbf{k}) = \left(\boldsymbol{\sigma}^{\boldsymbol{\nu}+\mathbf{k}} w\right)(\mathbf{0}) = w(k_1 + \nu_1, \ldots, k_n + \nu_n).$$

A Laurent polynomial is a finite linear combination of Laurent monomials, i.e.,

$$f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) \in \mathcal{A} \Rightarrow f = \sum_{\boldsymbol{\nu}\in\mathcal{S}} \alpha_{\boldsymbol{\nu}} \boldsymbol{\sigma}^{\boldsymbol{\nu}}.$$

where $\alpha_{\boldsymbol{\nu}} \in \mathbb{R}$ and $\mathcal{S} \subseteq \mathbb{Z}^n$ is finite. The action of a polynomial on a trajectory is defined as

$$f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w = \sum_{\boldsymbol{\nu}\in\mathcal{S}} \alpha_{\boldsymbol{\nu}} \boldsymbol{\sigma}^{\boldsymbol{\nu}} w. \tag{1}$$

Thus $f : \mathcal{W} \to \mathcal{W}$.

A scalar system of linear partial difference equations with constant real coefficients is written in terms of the shift operators as

$$R(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w = 0 \tag{2}$$

where $R(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) \in \mathcal{A}^{\bullet\times 1}$. The solution set of equation (2), i.e., the set of all trajectories $w \in \mathcal{W}$ that satisfy equation (2) is called the *behavior* of the system, and is denoted by $\mathfrak{B}$. In other words,

$$\mathfrak{B} = \Big\{ w \in \mathcal{W} \mid R(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w = 0 \Big\} = \ker\, R(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}). \tag{3}$$

This is called a *kernel representation* of $\mathfrak{B}$ and $R(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})$ is called a *kernel representation matrix*.

Several distinct kernel representation matrices can lead to a single behavior. To avoid this non-unique representation of a behavior, we use an alternative description. Let $\mathfrak{a} \subseteq \mathcal{A}$ be an ideal, then define

$$\mathfrak{B}(\mathfrak{a}) = \Big\{ w \in \mathcal{W} \mid r(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w = 0, \forall r \in \mathfrak{a} \Big\} \subseteq \mathcal{W}. \tag{4}$$

Note that, if $\mathfrak{B}$ is given by a kernel representation

$$\mathfrak{B} = \ker \begin{bmatrix} r_1(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) \\ \vdots \\ r_m(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) \end{bmatrix}, \tag{5}$$

then $\mathfrak{B} = \mathfrak{B}(\mathfrak{a})$, where

$$\mathfrak{a} := \Big\{ \sum_{i=1}^{m} f_i(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) r_i(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) \mid f_i(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) \in \mathcal{A} \Big\}.$$

In this case, the ideal $\mathfrak{a}$ is called the *equation ideal* of $\mathfrak{B}$, and is also denoted as $\mathfrak{a} = \langle r_1, \ldots, r_m \rangle$. Conversely, given an ideal $\mathfrak{a} \subseteq \mathcal{A}$, a kernel representation of $\mathfrak{B}(\mathfrak{a})$ is given by $\mathfrak{B} = \ker\, R(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})$, with $R(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) = \begin{bmatrix} r_1(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) & \cdots & r_m(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) \end{bmatrix}^T$, where $\mathfrak{a} = \langle r_1, \ldots, r_m \rangle$.

A behavior $\mathfrak{B}$ defined by a kernel representation, or, equivalently, by an equation ideal, has the structure of a vector space over $\mathbb{R}$. Indeed, $\mathfrak{B}$ is clearly closed under addition, and is also closed under multiplication by scalars in $\mathbb{R}$. Further, it will be important in the sequel to note that $\mathfrak{B}$ also has the structure of a module over $\mathcal{A}$, where scalar multiplications by an $f \in \mathcal{A}$ to a $w \in \mathfrak{B}$ is defined as the action $f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w$. It is not difficult to check that $\mathfrak{B}$ is closed under this scalar multiplication: for $w \in \mathfrak{B}$, we have $f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w \in \mathfrak{B}$ for all $f \in \mathcal{A}$.

C. The quotient ring

The algebraic notion of a *quotient ring* associated with a behavior $\mathfrak{B}$ (or, equivalently, an equation ideal $\mathfrak{a}$) will be of crucial importance in the sequel. Given an ideal $\mathfrak{a} \subseteq \mathcal{A}$, the quotient ring $\mathcal{A}/\mathfrak{a}$ is the set of all equivalence classes originating from the equivalence relation defined as follows: two elements $f_1, f_2 \in \mathcal{A}$ are related if $f_1 - f_2 \in \mathfrak{a}$. For an $f \in \mathcal{A}$, its equivalence class is denoted by $\overline{f}$. It is straightforward to show that $\mathcal{A}/\mathfrak{a}$ has the structure of a commutative ring ($\mathbb{R}$-algebra) where the addition and multiplication are inherited from those of the parent ring $\mathcal{A}$. In this paper, we use the symbol $\mathcal{M}$ to denote the quotient ring $\mathcal{A}/\mathfrak{a}$. Note that $\mathcal{M}$ also has the structures of an $\mathcal{A}$-module and an $\mathbb{R}$-vector space.

The action of an element $m \in \mathcal{M}$ on a trajectory $w \in \mathfrak{B}$ will be of crucial importance in this paper. It is defined in the following way: Suppose $m \in \mathcal{M}$ and $w \in \mathfrak{B}$, then

$$m(w) := (\widehat{m}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w) \tag{6}$$

where $\widehat{m} \in \mathcal{A}$ is a lift of m. Note that this action is well defined. Indeed, if $\widehat{m}_1$ and $\widehat{m}_2$ are two distinct lifts of m, then $\widehat{m}_1 - \widehat{m}_2 \in \mathfrak{a}$. Therefore, the action on $w \in \mathfrak{B}$ is

$$\widehat{m}_1(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w - \widehat{m}_2(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w = \\ \left(\widehat{m}_1(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) - \widehat{m}_2(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})\right) w = 0,$$

because $\widehat{m}_1 - \widehat{m}_2 \in \mathfrak{a}$. Hence, the actions of both the lifts are the same on w, that is, $\widehat{m}_1(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w = \widehat{m}_2(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})w$. Thus, the action of m defined using a lift is well-defined.

We often need the *canonical surjection* $\mathcal{A} \twoheadrightarrow \mathcal{M}$ that sends every element in $\mathcal{A}$ to its equivalence class in $\mathcal{M}$, that is, $\mathcal{A} \ni f \mapsto \overline{f} \in \mathcal{M}$ under the canonical surjection.

D. Autonomous systems

Autonomous nD systems are characterized in various ways. In [7], it was shown that 2D autonomous systems have a full column rank kernel representation matrix. This is equivalent to the condition that 2D autonomous systems have proper cones of $\mathbb{R}^2$ intersected with $\mathbb{Z}^2$ as characteristic sets (see [3]). Further equivalent algebraic conditions for characterizing autonomy of (continuous) nD systems were given in [2].

From the equivalent conditions, any scalar nD behavior with non-zero kernel representation matrix is autonomous. Therefore, a scalar nD behavior $\mathfrak{B}$ is always autonomous if the equation ideal $\mathfrak{a}$ is nonzero. Furthermore, an autonomous behavior is said to be *strongly autonomous* if and only if the quotient ring $\mathcal{M}$ is a finite dimensional vector space over $\mathbb{R}$. Another way of characterizing strongly autonomous behaviors is by *characteristic variety*. Characteristic variety of a behavior $\mathfrak{B}$, with equation ideal $\mathfrak{a}$, is defined as the set

$$\mathbb{V}(\mathfrak{B}) = \left\{ \boldsymbol{\xi} \in \mathbb{C}^n \mid r(\boldsymbol{\xi}) = 0 \ \forall \ r \in \mathfrak{a} \right\}.$$

Given an equation ideal $\mathfrak{a}$, the corresponding behavior $\mathfrak{B}$ is strongly autonomous if and only if its characteristic variety $\mathbb{V}(\mathfrak{B})$ is a finite set [2].

E. Characteristic sets

For an autonomous discrete nD system with behavior $\mathfrak{B}$, characteristic sets are special subsets of the domain (here $\mathbb{Z}^n$) such that every trajectory $w \in \mathfrak{B}$ can be uniquely extended with the knowledge of w restricted to this set. The notion of restriction of trajectories to subsets of the domain would be helpful in order to define characteristic sets formally. Given a trajectory $w : \mathbb{Z}^n \to \mathbb{R}$ and a subset $\mathcal{C} \subseteq \mathbb{Z}^n$, the restriction of w to $\mathcal{C}$, denoted by $w|_{\mathcal{C}}$, is defined as

$$\begin{aligned} w|_{\mathcal{C}} \quad &: \quad \mathcal{C} \to \mathbb{R} \\ (w|_{\mathcal{C}})(\mathbf{k}) \quad &= \quad w(\mathbf{k}) \text{ for all } \mathbf{k} \in \mathcal{C}. \end{aligned} \tag{7}$$

Now we formally define a characteristic set [3].

Definition 1. *Given a behavior $\mathfrak{B}$, a subset $\mathcal{C}$ of $\mathbb{Z}^n$ is said to be a* characteristic set *for $\mathfrak{B}$ if for every trajectory w in $\mathfrak{B}$, the restriction of w to the set $\mathcal{C}$, allows to uniquely determine the remaining portion of w, that is, $w|_{\mathbb{Z}^n \setminus \mathcal{C}}$ can be uniquely determined if $w|_{\mathcal{C}}$ is known.*

Example 2. Consider the scalar 2D behavior having kernel representation as $\mathfrak{B} = \ker R$ with

$$R = \begin{bmatrix} \sigma_2 - 5 \\ \sigma_1 - 3 \end{bmatrix} \in \mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}]^{2 \times 1}.$$

Explicitly writing the equations we get,

$$\begin{bmatrix} \sigma_2 - 5 \\ \sigma_1 - 3 \end{bmatrix} w(k_1, k_2) = 0.$$

$$\begin{aligned} \sigma_2 w(k_1,k_2) &= 5w(k_1,k_2), & \sigma_1 w(k_1,k_2) &= 3w(k_1,k_2) \\ w(k_1,k_2+1) &= 5w(k_1,k_2), & w(k_1+1,k_2) &= 3w(k_1,k_2) \end{aligned}$$

If $w(0,0)$ is known, it is possible to generate the values of the trajectories at every point in the discrete grid using the relation $w(k_1,k_2) = 3^{k_1}5^{k_2}w(0,0)$. Here the value at one point (i.e., at $(0,0)$) is sufficient for knowing the full trajectory. Thus a characteristic set for this behavior is a single point, namely, the origin $\{(0,0)\} \subseteq \mathbb{Z}^2$.

III. Why Valcher's results do not extend to $n \geqslant 3$?

In [3], Valcher proposes a method of determining whether a given cone is a characteristic cone for a 2D autonomous behavior $\mathfrak{B} = \ker R$ where, $R \in \mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}]^{g \times q}$. The method crucially depends on a decomposition of autonomous 2D behaviors as a sum of two special type of autonomous behaviors. These two special subclasses of autonomous behaviors are as follows: *finite dimensional behaviors* and *square behaviors*. A finite dimensional behavior is nothing but a strongly autonomous behavior. On the other hand, square autonomous behaviors are defined as kernels of nonsingular square Laurent polynomial matrices. It was shown in [3, Proposition 4.1] that every discrete 2D autonomous behavior $\mathfrak{B}$ can be decomposed as $\mathfrak{B} = \mathfrak{B}_{\text{fd}} + \mathfrak{B}_{\text{sq}}$, where $\mathfrak{B}_{\text{fd}}$ is a finite dimensional behavior and $\mathfrak{B}_{\text{sq}}$ is a square behavior. This decomposition is done in the following manner. Let $R \in \mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}]^{g \times q}$ be a kernel representation matrix for $\mathfrak{B}$. Then R can always be factorized as $R = \widetilde{R}\Delta$ where, $\widetilde{R} \in \mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}]^{g \times q}$ is *right-factor-prime* (see [8] for a definition of right-factor-prime) and $\Delta \in \mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}]^{q \times q}$ is square and non-singular. It then follows that the said decomposition $\mathfrak{B} = \mathfrak{B}_{\text{fd}} + \mathfrak{B}_{\text{sq}}$ is obtained by defining $\mathfrak{B}_{\text{fd}} := \ker \widetilde{R}$ and $\mathfrak{B}_{\text{sq}} := \ker \Delta$. While it is clear why $\ker \Delta$ is square, the fact that $\ker \widetilde{R}$ is finite dimensional (strongly autonomous) follows from $\widetilde{R}$ being right-factor-prime [9].

Using this decomposition, it was shown in [3, Proposition 2.6] that a *proper* cone (a closed, pointed, solid convex cone is called a proper cone) is a characteristic cone for $\mathfrak{B}$ if and only if it is a characteristic cone for $\mathfrak{B}_{\text{sq}}$. It was further shown that a proper cone is characteristic for the square behavior $\mathfrak{B}_{\text{sq}}$ if and only if it is a characteristic for the scalar behavior $\mathfrak{B}_\delta$, where $\mathfrak{B}_\delta := \ker(\det \Delta)$. Thus the problem of determining if a given proper cone is a characteristic cone for a 2D behavior reduces to checking if the cone is a characteristic cone for such a scalar behavior, which is the kernel of a single polynomial. Checking whether a proper cone is a characteristic cone for a scalar behavior is then done by a neat graphical method [3, Proposition 2.8].

Obviously, this analysis holds if the above-mentioned decomposition exists. Thus, in order to extend Valcher's graphical method of checking for characteristic cones to nD systems, with $n \geqslant 3$, an extension of the decomposition result becomes mandatory. Unfortunately, the decomposition does not extend for $n \geqslant 3$ as we show in Example 3 below.

Example 3. Consider the 3D discrete autonomous system $\mathfrak{B} = \ker R$, where $R = \begin{bmatrix} 1+\sigma_1 \\ 1+\sigma_2 \end{bmatrix} \in \mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \sigma_3^{\pm 1}]$. Note that R is already right-factor-prime. So, as per the above-mentioned decomposition of 2D behaviors, the square part of $\mathfrak{B}$ here is just $\{0\}$. However, note that $\mathfrak{B}$ is not finite dimensional either, although R is right-factor-prime. (Indeed, $\mathfrak{B} = \ker R$ cannot be finite dimensional because the characteristic variety is not a finite collection of points.)

IV. Cones, affine semigroups and semigroup-algebras

One of the prime reasons for studying convex cones as characteristic sets is because of their applicability to stability analysis [3]. However, as pointed out in the last section, the existing methods of handling characteristic cones becomes inapplicable to the case of $n \geqslant 3$. Interestingly, proper cones in $\mathbb{Z}^n$ have rich algebraic structures. For example, a rational proper cone in $\mathbb{Z}^n$ has the structure of a semigroup. The main contribution of this paper is in showing how these algebraic structures can be exploited to resolve the issue of characteristic cones for scalar discrete nD autonomous systems for general n. In this section, we delineate these algebraic structures associated with polyhedral cones in $\mathbb{Z}^n$: namely, their structure as affine semigroups and the algebra generated by them.

A. Cones and affine semigroups

A set $\mathcal{C} \subseteq \mathbb{R}^n$ is called a *cone* if $\lambda\mathcal{C} \subseteq \mathcal{C}$ for all $\lambda \in \mathbb{R}_{\geqslant 0}$. If a cone admits the following representation

$$\mathcal{C} = \left\{\lambda_1 \mathbf{c}_1 + \cdots + \lambda_d \mathbf{c}_d | \lambda_1, \ldots, \lambda_d \in \mathbb{R}_{\geqslant 0}\right\}$$

then it is said to be finitely generated by $\mathbf{c}_1, \ldots, \mathbf{c}_d \in \mathbb{R}^n$ and is known as a *polyhedral cone*. Further, $\mathcal{C}$ is called *rational* if $\mathbf{c}_1, \ldots, \mathbf{c}_d$ can be chosen to be vectors of rational numbers.

A cone is said to be *convex* if the line segment joining any two points in the cone is also contained the cone. A convex cone is *solid* if it contains an open ball of $\mathbb{R}^n$ and it is *pointed* if $\mathcal{C} \cap -\mathcal{C} = \{0\}$. A closed, pointed, solid, convex cone is called a *proper* cone.

A subset of a group, which is closed under the group operation and follows associativity, is called a *semigroup*. A semigroup is an *affine semigroup* if it is isomorphic to a sub-semigroup of $\mathbb{Z}^d$ for some d. According to Gordan's Lemma ([10, Theorem 7.16]), for every rational cone $\mathcal{C} \subseteq \mathbb{R}^n$, the intersection $\mathcal{C} \cap \mathbb{Z}^n$ is an affine subsemigroup of the Abelian group $\mathbb{Z}^n$ (under addition as the group operation). It further follows from [10, Proposition 7.15, Theorem 7.16] that such a cone $\mathcal{C} \cap \mathbb{Z}^n$ admits a representation

$$\mathcal{C} \cap \mathbb{Z}^n = \left\{\lambda_1 \mathbf{c}_1 + \cdots + \lambda_r \mathbf{c}_r | \lambda_1, \ldots, \lambda_r \in \mathbb{N}\right\},$$

where $\mathbf{c}_1, \ldots, \mathbf{c}_r \in \mathbb{Z}^n$. In this paper, by a cone in $\mathbb{Z}^n$ we mean the intersection of a rational proper cone $\mathcal{C} \subseteq \mathbb{R}^n$ with $\mathbb{Z}^n$. From now on we slightly abuse the notation $\mathcal{C}$ to mean $\mathcal{C} \cap \mathbb{Z}^n$, where $\mathcal{C}$ is a rational proper cone in $\mathbb{R}^n$.

B. Semigroup algebras

Let $\mathcal{C}$ be a cone (or, equivalently, an affine semigroup) in $\mathbb{Z}^n$. The semigroup algebra, denoted by $\mathbb{R}[\mathcal{C}]$, plays a crucial role in this paper. The algebra $\mathbb{R}[\mathcal{C}]$ is defined in the following manner

$$\mathbb{R}[\mathcal{C}] := \left\{ \sum_{\boldsymbol{\nu} \in \mathcal{S}} \alpha_{\boldsymbol{\nu}} \boldsymbol{\sigma}^{\boldsymbol{\nu}} \;\middle|\; \mathcal{S} \subseteq \mathcal{C},\ |\mathcal{S}| < \infty,\ \alpha_{\boldsymbol{\nu}} \in \mathbb{R} \right\}. \tag{8}$$

In other words, $\mathbb{R}[\mathcal{C}]$ is the $\mathbb{R}$-vector space of finite linear combinations of monomials having their indices in $\mathcal{C}$. Note that $\mathcal{C}$ being closed under addition (because of its semigroup structure) implies that $\mathbb{R}[\mathcal{C}]$ is closed under multiplication. Moreover, $\mathbb{R}[\mathcal{C}]$ is clearly closed under addition. Thus, $\mathbb{R}[\mathcal{C}]$ is a *subring* (or, equivalently, a *subalgebra* over $\mathbb{R}$)[1] of $\mathcal{A}$.

Cones in 2D are generated by two linearly independent vectors. It is interesting to note that for $n \geqslant 3$, a cone can have a generating set whose cardinality is more than n. For example, in $\mathbb{R}^3$ a cone can be given by the intersection of four half-spaces thus forming a cone with a quadrilateral base, that is, 4 generating vectors.

V. Algebraic characterization of characteristic cones

This section presents the main result of the paper. Given a cone $\mathcal{C}$ in $\mathbb{Z}^n$, let $\mathbb{R}[\mathcal{C}]$ be the algebra defined by $\mathcal{C}$ (see equation (8) above). Recall that $\mathbb{R}[\mathcal{C}]$ is a subalgebra of $\mathcal{A}$; we define the natural inclusion map, $\widetilde{\Psi}$, as

$$\widetilde{\Psi} : \mathbb{R}[\mathcal{C}] \hookrightarrow \mathcal{A}. \tag{9}$$

Let $\mathfrak{a} \subseteq \mathcal{A}$ be the equation ideal of a scalar autonomous behavior $\mathfrak{B}$. Note that, $\widetilde{\Psi}^{-1}(\mathfrak{a}) = \mathfrak{a} \cap \mathbb{R}[\mathcal{C}] \subseteq \mathbb{R}[\mathcal{C}]$ is an ideal of $\mathbb{R}[\mathcal{C}]$. We denote by $\mathcal{Q}$ the quotient ring $\frac{\mathbb{R}[\mathcal{C}]}{\mathfrak{a} \cap \mathbb{R}[\mathcal{C}]}$. Clearly, $\mathcal{Q}$ has the structure of an $\mathbb{R}$-algebra; in particular, $\mathcal{Q}$ is a vector space over $\mathbb{R}$.

We define the $\mathbb{R}$-linear map

$$\Psi : \mathcal{Q} \to \mathcal{M} \tag{10}$$

in the following way: for $m \in \mathcal{Q}$, let $\widehat{m}$ be a lift of m in $\mathbb{R}[\mathcal{C}]$. By the natural inclusion map $\widetilde{\Psi}$, $\widehat{m} \in \mathcal{A}$. Let $\overline{\widehat{m}} \in \mathcal{M}$ be the image of $\widehat{m}$ under the canonical surjection $\mathcal{A} \twoheadrightarrow \mathcal{M}$. Then Ψ is defined as

$$\Psi : m \mapsto \overline{\widehat{m}}. \tag{11}$$

To show Ψ is well defined, suppose m has two distinct lifts $\widehat{m}_1$ and $\widehat{m}_2$ in $\mathbb{R}[\mathcal{C}]$ satisfying $\widehat{m}_1 - \widehat{m}_2 \in \mathfrak{a} \cap \mathbb{R}[\mathcal{C}]$. By the natural inclusion $\widetilde{\Psi}$, $\widehat{m}_1 \neq \widehat{m}_2$ in $\mathcal{A}$. However, under the surjection $\mathcal{A} \twoheadrightarrow \mathcal{M}$, $\overline{\widehat{m}_1} = \overline{\widehat{m}_2}$ because $\widehat{m}_1$ and $\widehat{m}_2$ are equivalent modulo $\mathfrak{a}$. Thus Ψ is well defined. The definition of Ψ is illustrated by the commutative diagram (Figure 1) below.

Lemma 4. *The $\mathbb{R}$-linear map $\Psi : \mathcal{Q} \to \mathcal{M}$ is injective.*

Proof: Let $m_1, m_2 \in \mathcal{Q}$ be such that $\Psi(m_1) = \Psi(m_2)$. It follows from the definition of Ψ that $\overline{\widehat{m}}_1 = \overline{\widehat{m}}_2$, where

[1] In fact, in the notation of the semigroup algebra, $\mathcal{A} = \mathbb{R}[\mathbb{Z}^n]$.

$$
\begin{tikzcd}
\mathbb{R}[\mathcal{C}] \arrow[r, "\widetilde{\Psi}"] \arrow[d] & \mathcal{A} \arrow[d] \\
\mathcal{Q} \arrow[r, "\Psi"] & \mathcal{M}
\end{tikzcd}
$$

Fig. 1. Commutative diagram showing Ψ.

$\widehat{m}_1, \widehat{m}_2 \in \mathbb{R}[\mathcal{C}]$ are lifts of m_1, m_2, respectively. However, $\overline{\widehat{m}_1} = \overline{\widehat{m}_2}$ implies that $\widehat{m}_1 - \widehat{m}_2 \in \mathfrak{a}$. Since $\mathbb{R}[\mathcal{C}]$ is a ring $\widehat{m}_1 - \widehat{m}_2 \in \mathbb{R}[\mathcal{C}]$. It then follows that $\widehat{m}_1 - \widehat{m}_2 \in \mathfrak{a} \cap \mathbb{R}[\mathcal{C}]$. Hence $m_1 - m_2 = 0 \in \mathcal{Q}$. $\square$

We now state the main result of this paper, Theorem 5. While the map Ψ is always injective – as shown in Lemma 4 above – Theorem 5 shows that in order for a cone $\mathcal{C}$ to be a characteristic cone, Ψ must be surjective as well.

Theorem 5. *Let $\mathfrak{B}$ be a scalar nD autonomous behavior with equation ideal $\mathfrak{a} \subseteq \mathcal{A}$. Then a cone (or, equivalently, an affine semigroup) $\mathcal{C} \subseteq \mathbb{Z}^n$ is a characteristic cone for the behavior $\mathfrak{B}$ if and only if the $\mathbb{R}$-linear map $\Psi : \mathcal{Q} \to \mathcal{M}$ as defined in equation* (11)*, is surjective.*

We postpone the proof of Theorem 5 now, for we need the following background development for the proof.

A. Duality between behaviors and $\mathbb{R}$-algebras

Suppose $\mathfrak{B}$ is a behavior with equation ideal $\mathfrak{a}$. Recall that the quotient ring $\mathcal{M}$ has the structure of an $\mathbb{R}$-vector space and an $\mathcal{A}$-module. We define, $\mathcal{M}^* := \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}, \mathbb{R})$, the *algebraic dual* of $\mathcal{M}$ as a vector space over $\mathbb{R}$. In other words, $\mathcal{M}^*$ is the set of all $\mathbb{R}$-linear functionals on $\mathcal{M}$. The proof of the following result, Proposition 6, is straightforward.

Proposition 6. *$\mathcal{M}^*$ has the structure of an $\mathcal{A}$-module, where multiplication by scalars from $\mathcal{A}$ is defined as follows: for $\varphi \in \mathcal{M}^*$,*

$$(f\varphi)(m) := \varphi(fm) \text{ for all } f \in \mathcal{A}.$$

The set of $\mathcal{A}$-module morphisms from $\mathcal{M}$ to $\mathcal{W}$ and the behavior $\mathfrak{B}$ are isomorphic as $\mathcal{A}$-modules, that is, $\mathfrak{B} \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{W})$; this is the well-known Malgrange's Theorem [11]. Here we prove a variant of Malgrange's Theorem, that the behavior $\mathfrak{B}$ and the algebraic dual $\mathcal{M}^*$ of $\mathcal{M}$ are also isomorphic as $\mathcal{A}$-modules. This result is not new; it can be found in various earlier works, see for example [11]. However, we give a proof of this result for the sake of completeness and easy referencing in the sequel.

Proposition 7. *Let $\mathfrak{B}$ be a discrete autonomous nD behavior with equation ideal $\mathfrak{a} \subseteq \mathcal{A}$. Let $\mathcal{M}$ be the quotient ring $\mathcal{A}/\mathfrak{a}$ and $\mathcal{M}^*$ its algebraic dual. Recall the definition of action of $\mathcal{M}$ on $\mathfrak{B}$ as defined in equation* (6)*. Define the $\mathcal{A}$-module morphism $\Gamma : \mathfrak{B} \to \mathcal{M}^*$ in the following manner: for $w \in \mathfrak{B}$ and $m \in \mathcal{M}$,*

$$(\Gamma(w))(m) := (m(w))(\mathbf{0}).$$

Then Γ is an isomorphism of $\mathcal{A}$-modules.

Proof: It is enough to show that Γ is injective and surjective. **(Injectivity)** Suppose, for a $w \in \mathfrak{B}$ we have $\Gamma(w) = 0 \in \mathcal{M}^*$, that is $(\Gamma(w))(m) = 0$ for all $m \in \mathcal{M}$. We want to show that this means $w \equiv 0$, that is, $w(\mathbf{k}) = 0$ for all $\mathbf{k} \in \mathbb{Z}^n$. In order for that, let $\mathbf{k} \in \mathbb{Z}^n$ be arbitrary. Then $w(\mathbf{k}) = \left(\boldsymbol{\sigma}^{\mathbf{k}} w\right)(\mathbf{0})$. It then follows from the definition of Γ and the definition of action of $\mathcal{M}$ on $\mathfrak{B}$ (equation (6)) that $\left(\boldsymbol{\sigma}^{\mathbf{k}} w\right)(\mathbf{0}) = \left(\overline{\boldsymbol{\sigma}^{\mathbf{k}}} w\right)(\mathbf{0}) = (\Gamma(w))\left(\overline{\boldsymbol{\sigma}^{\mathbf{k}}}\right) = 0$ because $\Gamma(w)$ has been assumed to be the zero map on $\mathcal{M}$. This proves that Γ is injective.

(Surjectivity) Suppose $\varphi \in \mathcal{M}^*$, we want to show that there exists $w \in \mathfrak{B}$ such that $\Gamma(w) = \varphi$ on $\mathcal{M}$. We show this by constructing such a w. Define, for $\mathbf{k} \in \mathbb{Z}^n$

$$w(\mathbf{k}) := \varphi\left(\overline{\boldsymbol{\sigma}^{\mathbf{k}}}\right).$$

We first claim that $w \in \mathfrak{B}$. Note that $w \in \mathcal{W}$ and hence action of the shift operator $\boldsymbol{\sigma}^{\mathbf{k}}$ on w is given by $\left(\boldsymbol{\sigma}^{\mathbf{k}} w\right)(\mathbf{0}) = w(\mathbf{k}) = \varphi\left(\overline{\boldsymbol{\sigma}^{\mathbf{k}}}\right)$. Let $f \in \mathcal{A}$ be an arbitrary Laurent polynomial given by

$$f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) = \sum_{\boldsymbol{\nu} \in \mathcal{S}} \alpha_{\boldsymbol{\nu}} \boldsymbol{\sigma}^{\boldsymbol{\nu}},$$

where $\mathcal{S} \subseteq \mathbb{Z}^n$ is finite and $\alpha_{\boldsymbol{\nu}} \in \mathbb{R}$. Using $\mathbb{R}$-linearity of φ, it then follows that, for the $f \in \mathcal{A}$ defined above, we must have

$$
\begin{aligned}
\left(f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\mathbf{0}) &= \sum_{\boldsymbol{\nu} \in \mathcal{S}} \alpha_{\boldsymbol{\nu}} \left(\left(\boldsymbol{\sigma}^{\boldsymbol{\nu}} w\right)(\mathbf{0})\right) \\
&= \sum_{\boldsymbol{\nu} \in \mathcal{S}} \alpha_{\boldsymbol{\nu}} \left(\varphi\left(\overline{\boldsymbol{\sigma}^{\boldsymbol{\nu}}}\right)\right) \\
&= \varphi\left(\sum_{\boldsymbol{\nu} \in \mathcal{S}} \alpha_{\boldsymbol{\nu}} \overline{\boldsymbol{\sigma}^{\boldsymbol{\nu}}}\right) \\
&= \varphi\left(\overline{f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})}\right) \qquad (12)
\end{aligned}
$$

Now, suppose $f \in \mathfrak{a}$, then from equation (12) we get that

$$\left(f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\mathbf{0}) = \varphi\left(\overline{f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})}\right) = \varphi(0) = 0,$$

because $f \in \mathfrak{a}$ implies that $\overline{f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})} = 0$. Thus, for all $f \in \mathfrak{a}$ we have $\left(f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\mathbf{0}) = 0$. Now, given $f \in \mathfrak{a}$, observe that $\boldsymbol{\sigma}^{\mathbf{k}} f \in \mathfrak{a}$ for all $\mathbf{k} \in \mathbb{Z}^n$. Therefore, it follows that

$$\left(f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\mathbf{k}) = \left(\boldsymbol{\sigma}^{\mathbf{k}} f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\mathbf{0}) = 0.$$

Thus, for all $f \in \mathfrak{a}$, we have $f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w \equiv 0$, which means $w \in \mathfrak{B}$.

Next, we claim that for this w we must have $\Gamma(w) = \varphi$ on $\mathcal{M}$. Let $m \in \mathcal{M}$ be arbitrary. Suppose $\widehat{m} \in \mathcal{A}$ is a lift of m. Then from the definition of Γ we have

$$(\Gamma(w))(m) = (m(w))(\mathbf{0}).$$

However, by equation (6), we have $(m(w))(\mathbf{0}) = \left(\widehat{m}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\mathbf{0})$. It follows from equation (12) that $\left(\widehat{m}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\mathbf{0}) = \varphi\left(\overline{\widehat{m}(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})}\right) = \varphi(m)$. Since

$m \in \mathcal{M}$ was chosen arbitrarily, we have $\Gamma(w) = \varphi$ on $\mathcal{M}$. This proves that Γ is surjective. $\square$

Proposition 7 enables us to devise an algorithm for obtaining trajectories in a behavior given an equation ideal. We elaborate on this algorithm in Lemma 8 below. Similar methods have been presented in various earlier works; see for example, [11].

Lemma 8. *Let $\mathfrak{a} \subseteq \mathcal{A}$ be an equation ideal with behavior $\mathfrak{B}$. Further, let $\mathcal{E} = \{m_1, m_2, m_3 \ldots\} \subseteq \mathcal{M}$ be a (Hamel) basis*[2] *of $\mathcal{M}$ as a vector space over $\mathbb{R}$. Suppose $\varphi \in \mathcal{M}^*$. Define $w : \mathbb{Z}^n \to \mathbb{R}$ in the following manner: for $\mathbf{k} \in \mathbb{Z}^n$*

$$w(\mathbf{k}) := \sum_i \alpha_i \varphi(m_i), \tag{13}$$

where $\overline{\boldsymbol{\sigma}^{\mathbf{k}}} = \sum_i \alpha_i m_i$. Since $\mathcal{E}$ is a basis of $\mathcal{M}$, the above-mentioned sums are finite. Then, $w \in \mathcal{W}$ thus defined is a trajectory in $\mathfrak{B}$.

Proof: The proof immediately follows from Proposition 7 by noting that w, as defined in equation (13) above, is nothing but $\Gamma^{-1}(\varphi)$. $\square$

B. Proof of the main result

With Proposition 7 and Lemma 8 in place, we are now in a position to prove our main result, Theorem 5.

Proof of Theorem 5: **(If)** Suppose Ψ is surjective, we have to show that $\mathcal{C}$ is a characteristic cone for $\mathfrak{B}$. It is enough to show that for all $w \in \mathfrak{B}$ we have, $w|_{\mathcal{C}} = 0$ implies $w \equiv 0$. That is, for $w \in \mathfrak{B}$,

$$w(\mathbf{k}) = 0 \text{ for all } \mathbf{k} \in \mathcal{C} \;\Rightarrow\; w(\mathbf{k}) = 0 \text{ for all } \mathbf{k} \in \mathbb{Z}^n.$$

In order to show this let $w \in \mathfrak{B}$ be such that $w|_{\mathcal{C}} = 0$, and let $\mathbf{k} \in \mathbb{Z}^n$ be arbitrary. Now, since Ψ is surjective, it follows from the definition of Ψ that there exists $f \in \mathbb{R}[\mathcal{C}]$ such that

$$\boldsymbol{\sigma}^{\mathbf{k}} - f \in \mathfrak{a}.$$

It then follows that $\left(\boldsymbol{\sigma}^{\mathbf{k}} - f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1})\right)(w) \equiv 0$. Therefore,

$$w(\mathbf{k}) = \left(\boldsymbol{\sigma}^{\mathbf{k}} w\right)(\mathbf{0}) = \left(f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\mathbf{0}). \tag{14}$$

However, note that $w|_{\mathcal{C}} = 0$ implies that for all $f \in \mathbb{R}[\mathcal{C}]$ we must have $\left(f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w\right)(\mathbf{0}) = 0$. Hence we get from equation (14)

$$w(\mathbf{k}) = \left(\boldsymbol{\sigma}^{\mathbf{k}} w\right)(\mathbf{0}) = (f(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}) w)(\mathbf{0}) = 0.$$

Since $\mathbf{k} \in \mathbb{Z}^n$ was arbitrary, it follows that $w \equiv 0$.

(Only if) We assume that Ψ is *not* surjective, we want show that $\mathcal{C}$ then cannot be a characteristic set for $\mathfrak{B}$. Recall that $\mathcal{Q}$ denotes the $\mathbb{R}$-algebra $\frac{\mathbb{R}[\mathcal{C}]}{\mathfrak{a} \cap \mathbb{R}[\mathcal{C}]}$. We first note that there exists $\mathcal{S} \subseteq \mathcal{C}$ such that $\mathcal{Q}$, as a vector space over $\mathbb{R}$, admits a (Hamel) basis of the following type:

$$\widetilde{\mathcal{E}} := \{m \in \mathcal{Q} \mid m \text{ has a lift } \boldsymbol{\sigma}^{\boldsymbol{\nu}}, \boldsymbol{\nu} \in \mathcal{S}\}$$

(see [13, Proposition 1.1]). Define

$$\mathcal{E} := \{\overline{\boldsymbol{\sigma}^{\boldsymbol{\nu}}} \in \mathcal{M} \mid \boldsymbol{\nu} \in \mathcal{S}\}.$$

Clearly, $\mathcal{E} = \Psi\left(\widetilde{\mathcal{E}}\right)$. Since Ψ is injective (see Lemma 4), $\mathcal{E}$ is a linearly independent set in $\mathcal{M}$. It then follows that $\mathcal{M}$ admits a (Hamel) basis $\mathcal{E}'$ such that $\mathcal{E} \subseteq \mathcal{E}'$ (see [12, Corollary 2.2]). Note that we must have $\mathcal{E} \subsetneq \mathcal{E}'$ because we have assumed that Ψ is not surjective. Furthermore, Ψ being not surjective also implies that there exists $\mathbf{k}^* \in \mathbb{Z}^n \setminus \mathcal{C}$ such that

$$\overline{\boldsymbol{\sigma}^{\mathbf{k}^*}} \notin \operatorname{span} \mathcal{E}.$$

In other words, there exists $m \in \mathcal{E}' \setminus \mathcal{E}$ such that

$$\overline{\boldsymbol{\sigma}^{\mathbf{k}^*}} = \alpha m + \sum_{m_i \in \mathcal{E}'} \alpha_i m_i, \tag{15}$$

where the sum is finite and $\alpha \neq 0$.

Now, we shall define a $\varphi \in \mathcal{M}^*$ in the following manner. Since φ is $\mathbb{R}$-linear and $\mathcal{E}'$ is a basis of $\mathcal{M}$ as a vector space over $\mathbb{R}$, in order to define φ, it is enough to define its action on the elements of $\mathcal{E}'$. Furthermore, this action of φ on the elements of $\mathcal{E}'$ can be defined independently because elements in $\mathcal{E}'$ are linearly independent. Therefore, we can define $\varphi \in \mathcal{M}^*$ to be such that

$$\begin{aligned} \varphi(m) &= 1 \\ \varphi|_{\mathcal{E}' \setminus \{m\}} &= 0. \end{aligned}$$

Then we construct a trajectory $w : \mathbb{Z}^n \to \mathbb{R}$ from this φ following equation (13) in Lemma 8. By Lemma 8, this $w \in \mathfrak{B}$. Now note that, for all $\mathbf{k} \in \mathcal{C}$, $\overline{\boldsymbol{\sigma}^{\mathbf{k}}} \in \operatorname{span} \mathcal{E}$. Therefore, from the construction of φ it follows that

$$w(\mathbf{k}) = 0 \text{ for all } \mathbf{k} \in \mathcal{C}.$$

In other words, $w|_{\mathcal{C}} = 0$. However, $w \not\equiv 0$, because $w(\mathbf{k}^*) = \alpha \neq 0$. This shows that $\mathcal{C}$ cannot be a characteristic set for $\mathfrak{B}$. $\square$

VI. Examples

This section provides some examples to validate Theorem 5.

Example 9. Consider the 3D behavior with kernel representation

$$\mathfrak{B} = \ker \begin{bmatrix} \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 1 \\ \sigma_2 \sigma_3 - 1 \end{bmatrix}.$$

The equation ideal is $\mathfrak{a} = \langle \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 1, \sigma_2\sigma_3 - 1 \rangle \subseteq \mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \sigma_3^{\pm 1}]$. Let $\mathcal{C} \subseteq \mathbb{Z}^3$ be the cone generated by $\mathbf{c}_1 = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^T$, $\mathbf{c}_2 = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T$ and $\mathbf{c}_3 = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^T$, that is, $\mathcal{C}$ is the cone generated by non-negative integral combination of the columns of the following matrix.

$$M_{\mathcal{C}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

[2] A Hamel basis of a possibly infinite dimensional vector space $\mathcal{V}$ over a field $\mathbb{K}$ is a subset $\mathcal{E}$ of $\mathcal{V}$ that satisfies:

1) elements in $\mathcal{E}$ are *linearly independent* over $\mathbb{K}$, that is, no finite non-zero linear combination of elements in $\mathcal{E}$ equals zero, and
2) every element of $\mathcal{V}$ can be written as a finite linear combination of elements from $\mathcal{E}$.

See [12, Section 2].

We claim that $\mathcal{C}$ is a characteristic cone for the behavior $\mathfrak{B}$.

This is because the monomials $\sigma_1^{-1}, \sigma_2^{-1}$ and σ_3^{-1} already belong to $\mathbb{R}[\mathcal{C}]$ and the monomials $\sigma_i^{\nu}, \nu \in \mathbb{R}_{>0}$, can be written as $\sigma_i^{\nu} \equiv q_i \mod \mathfrak{a}$ where $q_i \in \mathbb{R}[\mathcal{C}]$. Indeed,

$$\begin{aligned}
\sigma_1^{\nu} &\equiv (-\sigma_1^{-1}\sigma_2^{-2} - \sigma_1^{-1}\sigma_3^{-2} + \sigma_1^{-1})^{\nu} \mod \mathfrak{a} \\
\sigma_2^{\nu} &\equiv (\sigma_3^{-1})^{\nu} \mod \mathfrak{a} \\
\sigma_3^{\nu} &\equiv (\sigma_2^{-1})^{\nu} \mod \mathfrak{a}.
\end{aligned}$$

Therefore for every $\overline{m} \in \mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \sigma_3^{\pm 1}]/\mathfrak{a}$ there exists a $q \in \mathbb{R}[\mathcal{C}]$ such that $\Psi(\overline{q}) = \overline{m}$.

Example 10. This example is to check if the positive orthant is a characteristic cone for the following 4D behavior given by the kernel representation

$$\mathfrak{B} = \ker \begin{bmatrix} \sigma_4^2 - 1 \\ \sigma_1\sigma_2\sigma_3 - 1 \end{bmatrix}.$$

The equation ideal is $\mathfrak{a} = \langle \sigma_4^2 - 1, \sigma_1\sigma_2\sigma_3 - 1 \rangle \subseteq \mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \sigma_3^{\pm 1}, \sigma_4^{\pm 1}]$. The matrix representation of the cone is

$$M_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since monomials σ_1, σ_2, σ_3 and σ_4 belong to the cone it is important to check if $\sigma_1^{-1}, \sigma_2^{-1}$, σ_3^{-1} and σ_4^{-1} can be written as linear combination of monomials from the cone satisfying the system equations. Simple calculations show that

$$\begin{aligned}
(\sigma_1^{-1})^{\nu} &\equiv (\sigma_2\sigma_3)^{\nu} \mod \mathfrak{a} \\
(\sigma_2^{-1})^{\nu} &\equiv (\sigma_1\sigma_3)^{\nu} \mod \mathfrak{a} \\
(\sigma_3^{-1})^{\nu} &\equiv (\sigma_1\sigma_2)^{\nu} \mod \mathfrak{a} \\
(\sigma_4^{-1})^{\nu} &\equiv \sigma_4^{\nu} \mod \mathfrak{a}
\end{aligned}$$

where, $\sigma_2\sigma_3$, $\sigma_1\sigma_3$, $\sigma_1\sigma_2$ and σ_4 belong to the cone. Therefore the positive orthant is a characteristic cone for the system.

VII. Discussions and Conclusions

This paper gives a necessary and sufficient algebraic condition to check if a given cone is a characteristic cone for a scalar autonomous multidimensional behavior. This approach is general, unlike previous results of [3], as it is applicable for nD systems with $n \geqslant 2$. The analysis uses the fact that cones have the structure of an affine semigroup.

The added advantage of using algebraic methods is that computational aspects are structured and algorithmic and results from computational commutative algebra can be used. This is a possible direction of future work. We also wish to explore characteristic cones for vector valued autonomous behaviors.

Acknowledgement

The authors wish to thank Mr. Chirayu D. Athalye for his many technical inputs.

References

[1] E. Rogers, K. Galkowski, W. Paszke, K. L. Moore, P. H. Bauer, L. Hladowski, and P. Dabkowski, "Multidimensional control systems: case studies in design and evaluation," *Multidimensional Systems and Signal Processing*, vol. 26, pp. 895 – 939, July, 2015.

[2] H. K. Pillai and S. Shankar, "A behavioral approach to control of distributed systems," *SIAM Journal on Control and Optimization*, vol. 37, no. 2, pp. 388–408, November, 1998.

[3] M. E. Valcher, "Characteristic cones and stability properties of two-dimensional autonomous behaviors," *IEEE Transactions On Circuits and Systems - Part I: Fundamental Theory and Applications*, vol. 47, no. 3, pp. 290–302, March, 2000.

[4] P. Rocha and J. C. Willems, "Markov properties for systems described by PDEs and first-order representations," *Systems and Control Letters*, vol. 55, pp. 538–542, 2006.

[5] P. Rocha and J. C. Willems, "State for 2-D systems," *Linear Algebra and Its Applications*, vol. 122/123/124, pp. 1003–1038, 1989.

[6] D. Pal, "Every discrete 2D autonomous system admits a finite union of parallel lines as a characteristic set," *Multidimensional Systems and Signal Processing*, vol. DOI:10.1007/s11045-015-0330-y, pp. 1–25, May, 2015.

[7] E. Fornasini, P. Rocha, and S. Zampieri, "State space realizations of 2-D finite-dimensional behaviours," *SIAM Journal on Control and Optimization*, vol. 31, no. 6, pp. 1502–1517, November, 1993.

[8] D. C. Youla and G. Gnavi, "Notes on n-dimensional system theory," *IEEE Transactions on Circuits and Systems*, vol. Cas-26, no. 2, pp. 105–111, February, 1979.

[9] E. Fornasini and M. E. Valcher, "nD polynomial matrices with applications to multidimensional signal analysis," *Multidimensional Systems and Signal Processing*, vol. 8, pp. 387 – 407, 1997.

[10] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*. USA: Springer, 2004.

[11] U. Oberst, "Multidimensional constant linear systems," *Acta Applicandae Mathematicae*, vol. 20, pp. 1–175, 1990.

[12] B. V. Limaye, *Functional Analysis*. New Delhi: New Age International (P) Ltd., Publishers, 1996.

[13] B. Sturmfels, *Gröbner Bases and Convex Polytopes*. American Mathematical Society: University Lecture Series, 1996.