13 April:

$$\underline{\dot{x} = Ax + Bu}, \quad y = Cx + Du$$

SISO:

"tf2ss"

$$G(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = u \quad (*)$$

Define: $x_1 := y,\ x_2 := \dot{x}_1 = \dot{y},\ x_3 := \dot{x}_2 = \ddot{y},$

$\ldots,\ x_n := \dot{x}_{n-1} = y^{(n-1)} = \frac{d^{n-1}y}{dt^{n-1}}$

$$\underline{x} := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\underline{\dot{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ -a_{n-1}x_n - a_{n-2}x_{n-1} - \cdots - a_1x_2 - a_0x_1 \\ +u \end{bmatrix}$$

$$\dot{x}_n = \frac{d^n y}{dt^n} \quad (**)$$

(*) Rewritten in terms of x and u is

$$\dot{x}_n + a_{n-1}x_n + a_{n-2}x_{n-1} + \cdots + a_1x_2 + a_0x_1 = u$$

(**) Rewritten in matrix-vector form as

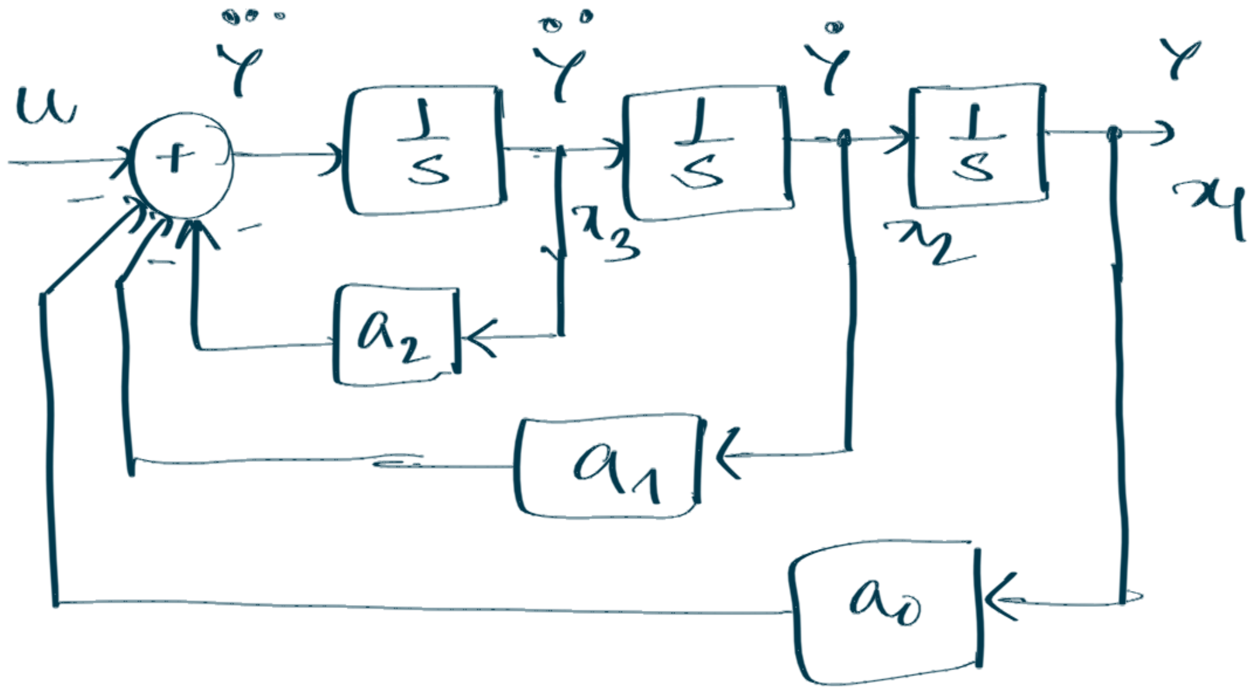
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{B} u$$

$$\dot{\underline{x}} = A\underline{x} + Bu.$$

$$Y = x_1 = \underbrace{[1\ 0\ 0 \cdots 0]}_{C} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{0}_{D}\cdot u$$

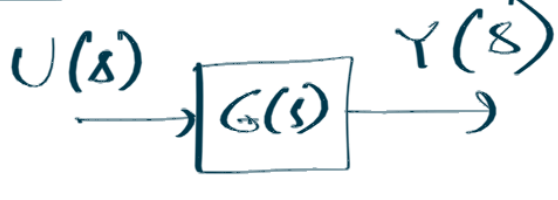
Block diagram:

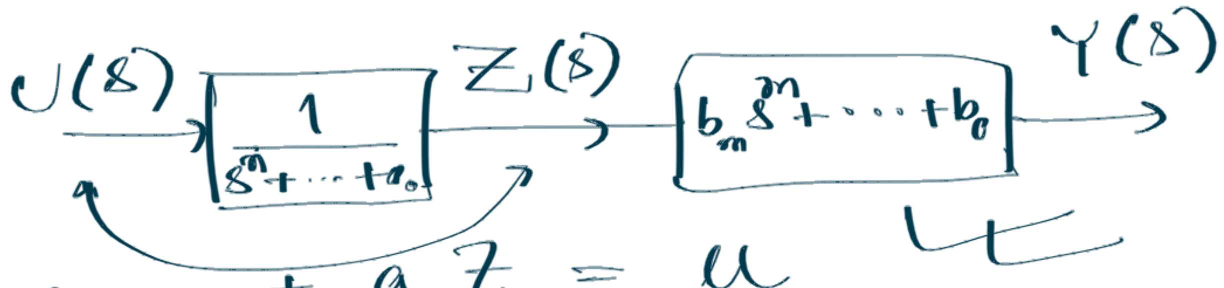
$$\frac{d^3Y}{dt^3} + a_2\frac{d^2Y}{dt^2} + a_1\frac{dY}{dt} + a_0 Y = u$$



$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$\boxed{m < n}$$





$$\frac{d^n z}{dt^n} + \cdots + a_0 z = u$$

$$x_1 := z,\ x_2 := \dot{z},\ \ldots,\ x_n := z^{(n-1)}$$

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$z = [1 \quad 0 \quad \cdots \quad 0]\, \underline{x} \qquad \times \text{ Not needed}$$

From the block diagram we get

$$Y(s) = (b_m s^m + \cdots + b_1 s + b_0)\, Z(s)$$

$$\Rightarrow y = b_m \frac{d^m z}{dt^m} + \cdots + b_1 \frac{dz}{dt} + b_0 z$$

$$= b_m x_{m+1} + \cdots + b_1 x_2 + b_0 x_1$$

$$y = \underbrace{[b_0 \quad b_1 \quad \cdots \quad b_m \quad 0 \cdots 0]}_{C}\, \underline{x}$$

$\boxed{m = n}$ Homework.

$$G(s) = K + \tilde{G}(s)$$

$\tilde{G}(s)$ is strictly proper

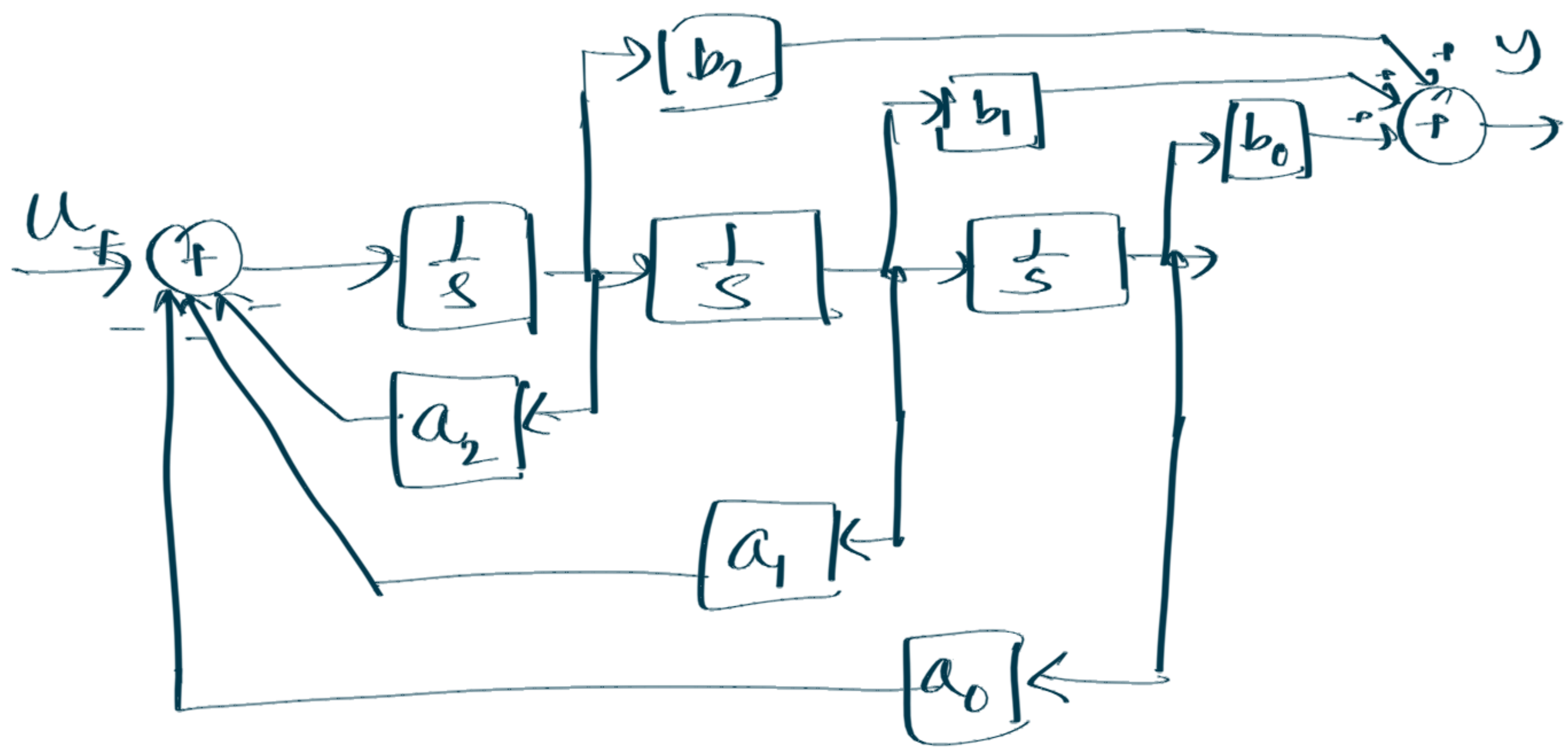
$$Y(s) = \underline{\underline{K U(s)}} + \tilde{G}(s) U(s)$$

PCF $\leftrightarrow$ Phase variable Canonical Form

Controller Canonical Form

Block diagram:

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$



$$G(s) = \frac{A_1}{s+\lambda_1} + \frac{A_2}{s+\lambda_2} + \cdots + \frac{A_n}{s+\lambda_n}$$

$$Y(s) = G(s)\,U(s)$$

$$= \frac{A_1}{s+\lambda_1}U(s) + \frac{A_2}{s+\lambda_2}U(s) + \cdots + \frac{A_n}{s+\lambda_n}U(s)$$

$$\left.\begin{aligned} &\dot{x}_1 = -\lambda_1 x_1 + u,\quad \dot{x}_2 = -\lambda_2 x_2 + u, \\ &\quad \ldots,\quad \dot{x}_n = -\lambda_n x_n + u \end{aligned}\right\}$$

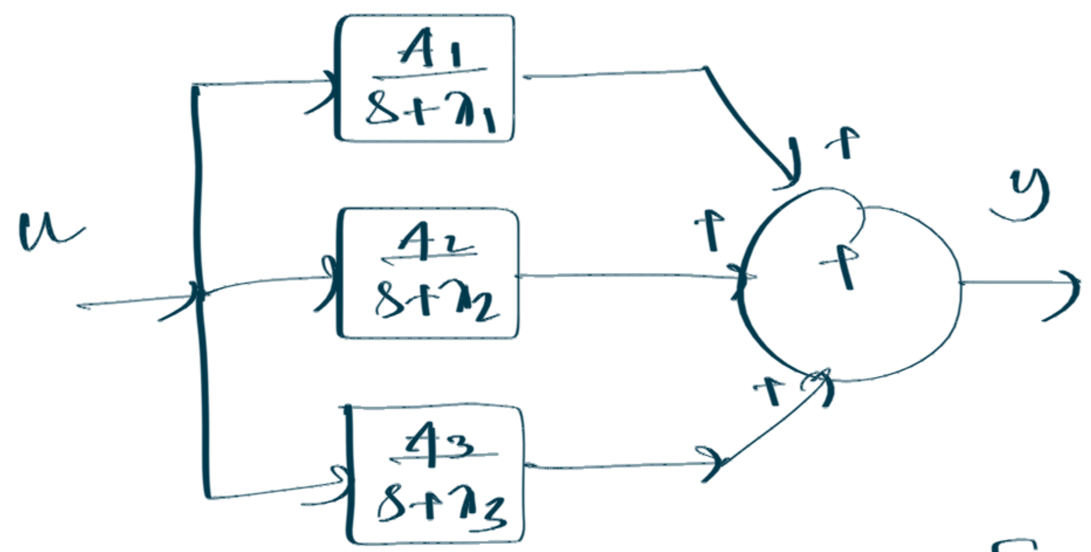
$$y = A_1 x_1 + A_2 x_2 + \cdots + A_n x_n$$

$$sX_1(s) = -\lambda_1 X_1(s) + U(s)$$

$$\Rightarrow (s+\lambda_1)X_1(s) = U(s)$$

$$\Rightarrow X_1(s) = \frac{1}{s+\lambda_1}U(s)$$

$$G(s) = \frac{A_1}{s+\lambda_1} + \frac{A_2}{s+\lambda_2} + \frac{A_3}{s+\lambda_3}$$



$$\underline{\dot{x}} = \begin{bmatrix} -\lambda_1 & & & 0 \\ & -\lambda_2 & & \\ & & \ddots & \\ 0 & & & -\lambda_n \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = [A_0 \; A_1 \; \cdots \; A_n] \; \underline{x}$$

Modal Canonical form

MCF

SS2tf:

$$\underline{\dot{x}} = A\underline{x} + Bu$$

$$y = Cx + Du$$

$$sX(s) = AX(s) + BU(s)$$

$$\Rightarrow [sI - A]X(s) = BU(s)$$

$$\Rightarrow X(s) = [sI - A]^{-1} BU(s)$$

$$\Rightarrow Y(s) = CX(s) + DU(s)$$

$$= \left(C[sI - A]^{-1}B + D\right)U(s)$$

$$\Rightarrow \boxed{G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D}$$