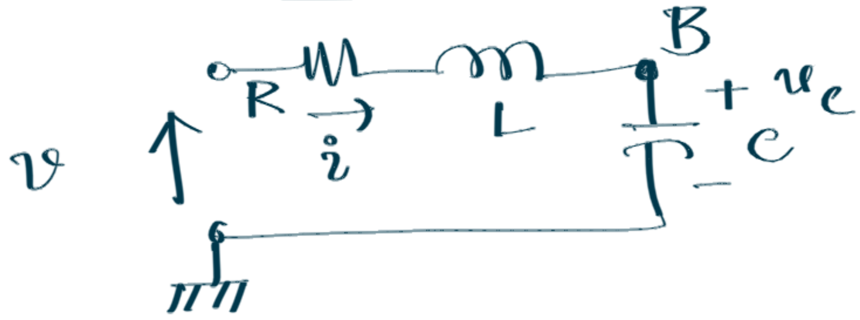
13 April Extra:



$x_1 := i$

$x_2 := v_c$

$y = v_c = x_2$

$\dot{x}_2 = \dot{v}_C = \frac{1}{C} i = \frac{1}{C} x_1$ (KCL @ node B)

$\dot{x}_1 = \frac{di}{dt} = \frac{1}{L}\left[-v_c - iR\right] + \frac{v}{L}$

$= -\frac{1}{L} x_2 - \frac{R}{L} x_1 + \frac{v}{L}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

SS model for the Speaker system.

- Solutions: (ZIS)

$\dot{x} = Ax$; $\underbrace{x(0) = x_0}_{\text{Initial cond}^n} \in \mathbb{R}^n$

$$\boxed{x(t) = e^{At} x_0}$$

$$e^{At} := I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots$$

$$e^{At}x_0 = x_0 + t\,\underline{Ax_0} + \frac{t^2}{2!}A^2x_0 + \frac{t^3}{3!}A^3x_0 + \cdots$$

$$\frac{d}{dt}\left(\underline{e^{At}x_0}\right) = 0 + Ax_0 + tA^2x_0 + \frac{t^2}{2!}A^3x_0 + \cdots$$

$$= A\left[x_0 + tAx_0 + \frac{t^2}{2!}A^2 x_0 + \cdots\right]$$

$$= A(\underline{e^{At} x_0})$$

$\underline{e^{At}}$.

A diagonalizable $\Leftrightarrow$ $\exists$ n LI eigenvectors of A in $\mathbb{R}^n$.
(over $\mathbb{R}$)

"eigenvector" $\lambda \in \mathbb{R}$, $v \in \mathbb{R}^n$, $v \neq 0$.

$$\boxed{Av = \lambda v}$$

$\{v_1, —, v_n\}$ LI set of eigenvectors

$T := [v_1 \ v_2 \ \cdots \ v_n] \in \mathbb{R}^{n \times n}$

T invertible.

$$AT = A[v_1 \ v_2 \ \cdots \ v_n] = [Av_1 \ \cdots \ Av_n]$$

$$= [\lambda_1 v_1 \ \cdots \ \lambda_n v_n]$$

$$= [v_1 \ \cdots \ v_n] \underbrace{\begin{bmatrix} \lambda_1 & 0 & & 0 \\ & \lambda_2 & \ddots & \\ 0 & 0 & & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$\Rightarrow AT = T\Lambda$$

$$\Rightarrow T^{-1}AT = \Lambda$$

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots$$

$$\Rightarrow T^{-1}e^{At}T = \underbrace{T^{-1}IT} + T^{-1}AtT + T^{-1}A^2\frac{t^2}{2!}T + T^{-1}A^3\frac{t^3}{3!}T + \cdots$$

$$= I + \Lambda t + \Lambda^2\frac{t^2}{2!} + \Lambda^3\frac{t^3}{3!} + \cdots$$

$$\boxed{T^{-1}A^2T = T^{-1}ATT^{-1}AT = \Lambda^2}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} = e^{\Lambda t}$$

$e^{\Lambda t}$ exists [scalar exponentials exist]

$$\Rightarrow T^{-1}e^{At}T = e^{\Lambda t}$$

$$\Rightarrow \boxed{e^{At} = Te^{\Lambda t}T^{-1}}$$

Warning: Not every matrix is diagonalizable.

$$\boxed{x(t) = e^{At}x_0}$$

$$x(t+T) = e^{A(t+T)}x_0$$

$$= e^{At}(e^{AT}x_0)$$

$$= e^{At}\cdot x(T)$$

$$x(t) = e^{At} x(0)$$

$$\Rightarrow \quad x(0) = e^{-At} x(t)$$

<u>Input</u>

$$\dot{x} = Ax + Bu$$

$$x(t) = \underline{e^{At} x(0)} + \int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau$$

$$x(T+t) = e^{A(T+t)} x(0) + \int_0^{T+t} e^{A(T+t-\tau)} Bu(\tau)\,d\tau$$

$$= e^{At} e^{AT} x(0) + \int_0^{T} e^{A(T+t-\tau)} Bu(\tau)\,d\tau + \int_T^{T+t} e^{A(T+t-\tau)} Bu(\tau)\,d\tau$$

$$= e^{At} e^{AT} x(0) + e^{At} \int_0^{T} e^{A(T-\tau)} Bu(\tau)\,d\tau + \int_T^{T+t} e^{A(T+t-\tau)} Bu(\tau)\,d\tau$$

$$= e^{At} \left[e^{AT} x(0) + \int_0^{T} e^{A(T-\tau)} Bu(\tau)\,d\tau \right] + \int_T^{T+t} e^{A(T+t-\tau)} Bu(\tau)\,d\tau$$

$$= e^{At} x(T) + \int_{T}^{T+t} e^{A(T+t-\tau)} B u(\tau) d\tau$$