16 April:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \checkmark$$

ch. polynomial $\chi(s) := \det(sI - A)$

$$= s^3 + 6s^2 + 11s + 6$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

$\Rightarrow \chi(s) =$

$$sI - A = \begin{bmatrix} s & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ +a_0 & +a_1 & +a_2 & \cdots & s + a_{n-1} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \rightarrow sI - A = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{bmatrix}$$

$$\det(sI - A) = s(s(s+6) + 11) + 1 \det\begin{bmatrix} 0 & -1 \\ 6 & s+6 \end{bmatrix}$$

$$= s(s^2 + 6s + 11) + 6$$

$$= s^3 + 6s^2 + 11s + 6.$$

$$\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

The general $n \times n$

– Eigenvalues/eigenvectors.

$$Av = \lambda v, \quad v \neq 0.$$

$$\Rightarrow \boxed{(\lambda I - A)v = 0} \Leftrightarrow \text{rank}(\lambda I - A) < n$$

$$\Leftrightarrow \det(\lambda I - A) = 0$$

$$\Uparrow$$

$$\Leftrightarrow \chi(s) = \det(sI - A)$$

$$= s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

$$\lambda \in \text{eig}(A) \Leftrightarrow \lambda \in \text{roots}(\chi(s))$$

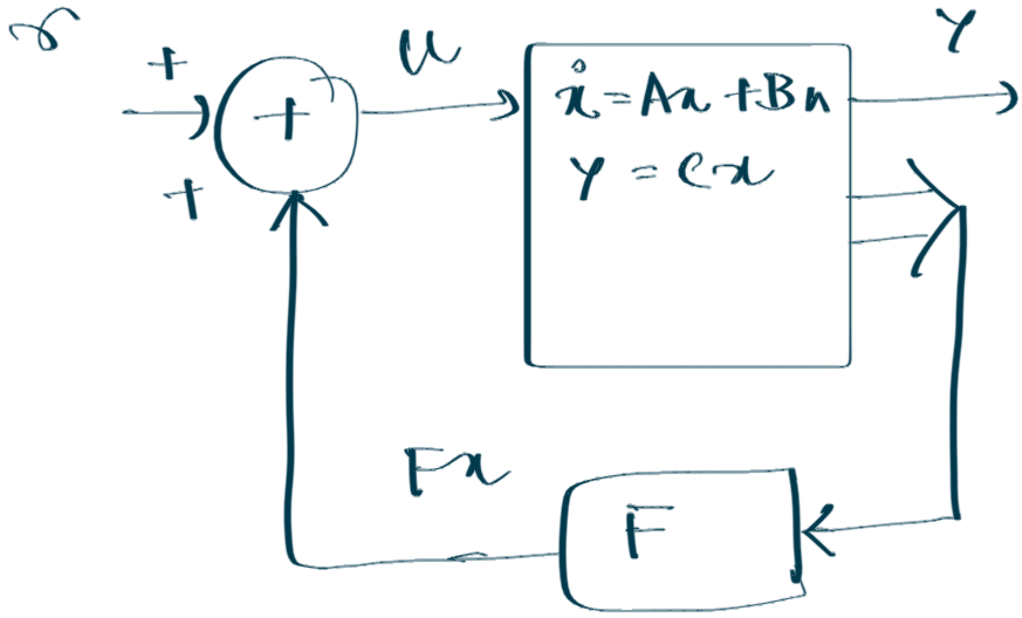
tf2ss

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1}B = \frac{b_m s^m + \cdots + b_1 s + b_0}{\chi(s)}$$

$$\text{eig}(A) = \text{roots } \chi(s) = \text{poles } G(s).$$




$x \in \mathbb{R}^n$

$F \in \mathbb{R}^{1 \times n}$

CL

$$\dot{x} = Ax + B(r + Fx), \quad y = Cx$$

$$= (A + BF)x + Br, \quad y = Cx$$

$$A+BF = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_0 & f_1 & \cdots & f_{n-1} \end{bmatrix}}_{BF}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ f_0 & f_1 & f_2 & \cdots & f_{n-1} \end{bmatrix}$$

$$A+BF = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0+f_0 & -a_1+f_1 & -a_2+f_2 & \cdots & -a_{n-1}+f_{n-1} \end{bmatrix}$$

$$\chi_{(A+BF)}(s) = s^n + (a_{n-1} - f_{n-1})s^{n-1} + \cdots + (a_1 - f_1)s + (a_0 - f_0)$$

Ex:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Desired cl poles: $-2, -4, -6$

n cl ch. poly $\chi_{(A+BF)}(s) = (s+2)(s+4)(s+6)$

$$= s^3 + 12s^2 + 44s + 48$$

$$A+BF = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -48 & -44 & -12 \end{bmatrix}$$

$$F = \begin{bmatrix} (-48+6) & (-44+11) & (-12+6) \end{bmatrix}$$

$$= \begin{bmatrix} -42 & -33 & -6 \end{bmatrix}$$

$$u = v + \begin{bmatrix} -42 & -33 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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What if the $(A, B)$ is not in CCF/PCF?

Change of variables: (linear transformation on the state space)

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

$$z := Tx \qquad T \in \mathbb{R}^{n \times n} \text{ invertible matrix.}$$

$$\Rightarrow \underline{x = T^{-1} z}.$$

$$\dot{z} = T\dot{x} = TAx + TBu$$
$$= TAT^{-1} z + TBu$$

$$y = Cx = CT^{-1} z + Du$$

$$\dot{z} = \tilde{A} z + \tilde{B} u, \quad y = \tilde{C} z + \tilde{D} u$$

$$\tilde{A} := TAT^{-1}, \quad \tilde{B} = TB, \quad \tilde{C} = CT^{-1},$$
$$\tilde{D} = D.$$

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$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathcal{C}_c := \begin{bmatrix} B_c & A_c B_c & A_c^2 B_c \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & -a_1 + a_2^2 \end{bmatrix} \Rightarrow \mathcal{C}_c^{-1} \text{ exists}$$

Suppose $(A, B) \xrightarrow{T} (A_c, B_c)$

$$\Rightarrow \quad TAT^{-1} = A_c, \quad TB = B_c$$

$$\mathcal{C} = [B \;\; AB \;\; A^2B]$$

$$\mathcal{C}_c = [B_c \;\; A_cB_c \;\; A_c^2 B_c]$$

$$= [TB \;\; TA\cancel{T^{-1}}\cancel{T}B \;\; TA^2\cancel{T^{-1}}\cancel{T}B]$$

$$= T[B \;\; AB \;\; A^2B]$$

$$= T\mathcal{C}$$

$$\mathcal{C}_c = \underline{\underline{T}}\,\mathcal{C}$$

$\mathcal{C}$ is invertible [controllability]

$$\Rightarrow \boxed{T := \mathcal{C}_c \mathcal{C}^{-1}}$$

$$\boxed{T^{-1} := \mathcal{C}\,\mathcal{C}_c^{-1}}$$

$$\boxed{u = Fz = FT^{-1}x}$$