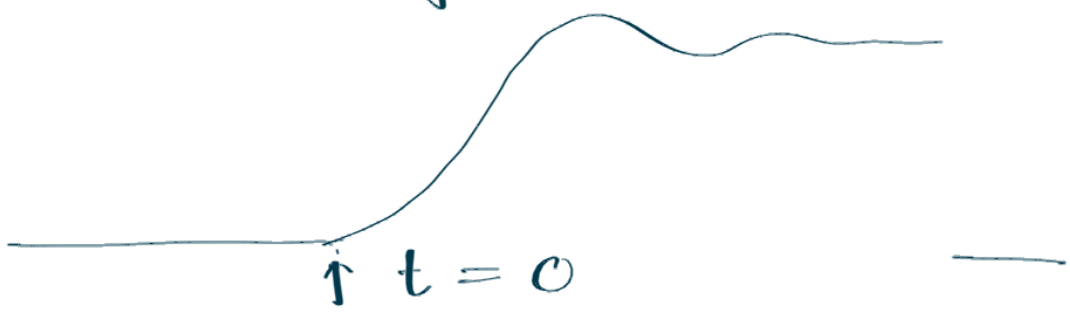
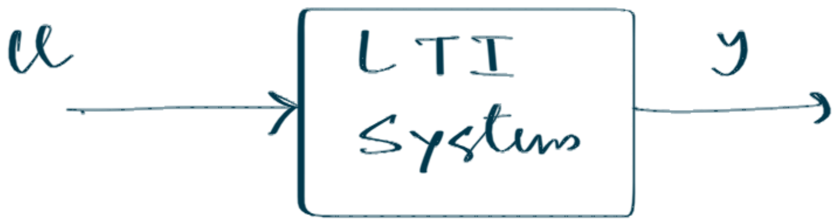
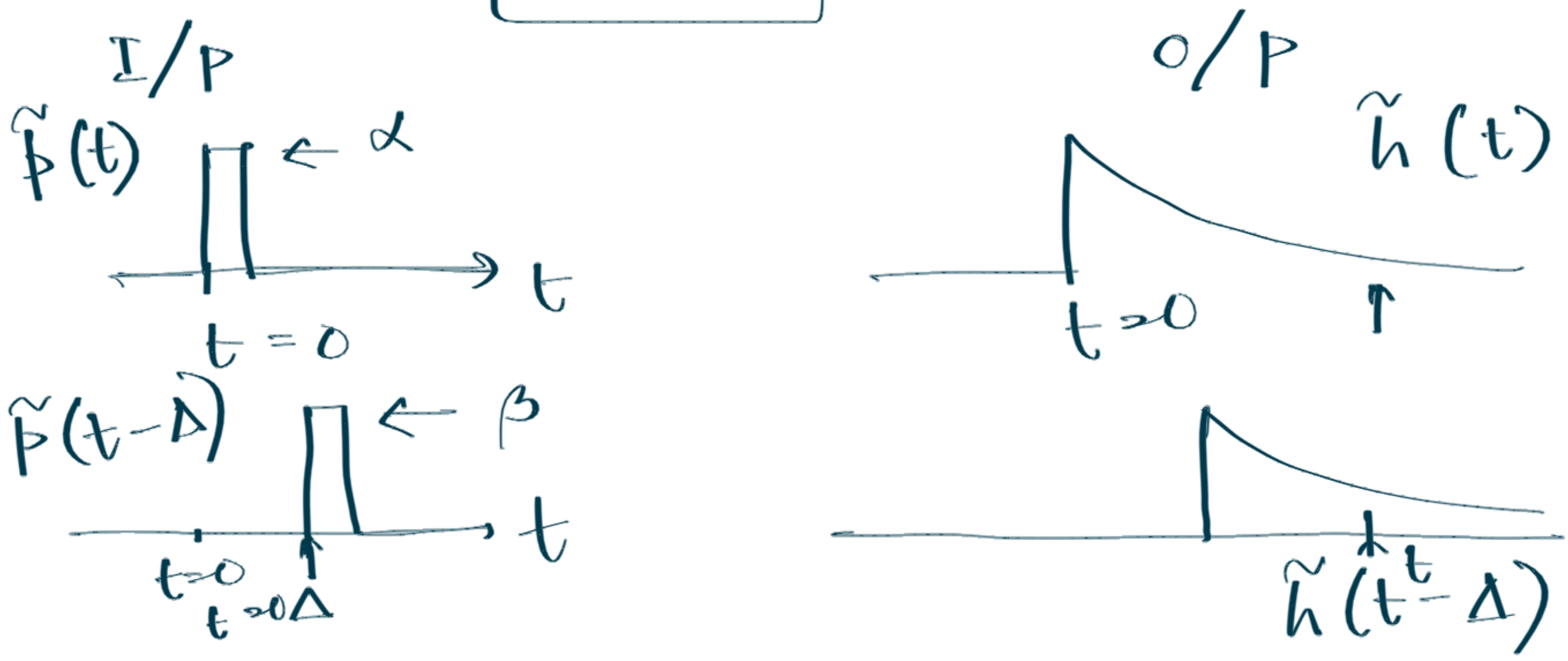
Feb 3: LTI systems.

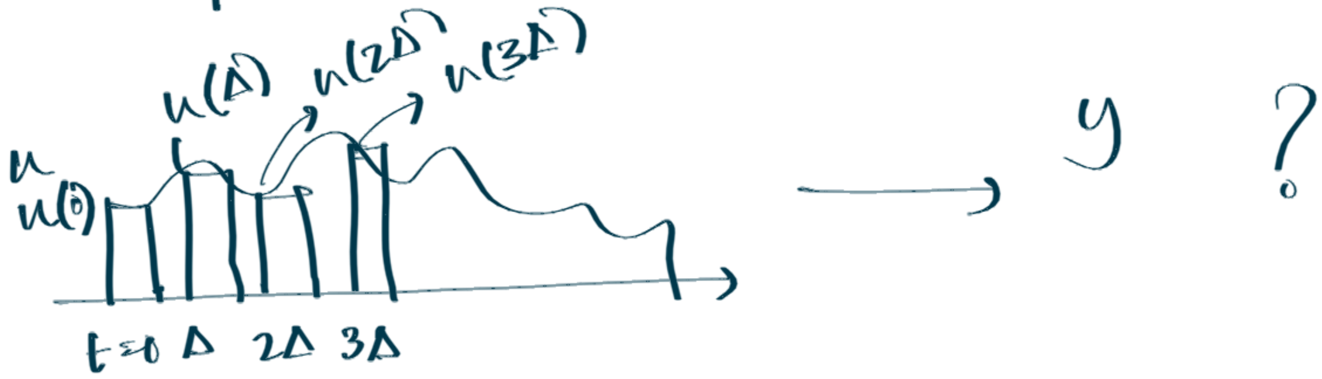






$\alpha \tilde{p}(t) + \beta \tilde{p}(t-\Delta)$ $\qquad$ $\alpha \tilde{h}(t) + \beta \tilde{h}(t-\Delta)$

$$y(t) = \alpha \tilde{h}(t) + \beta \tilde{h}(t-\Delta)$$



$$y(t) = u(0)\tilde{h}(t) + u(\Delta)\tilde{h}(t-\Delta) + u(2\Delta)\tilde{h}(t-2\Delta) + \dots + u(k\Delta)\tilde{h}(t-k\Delta) + \dots$$

In the limit

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\,\delta(t-\tau)\,d\tau$$

$$y(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

$\delta \rightarrow$ impulse

$h \rightarrow$ impulse response.

$$p\left(\frac{d}{dt}\right) y = q\left(\frac{d}{dt}\right) u$$

$p(s) \in \mathbb{R}[s]$

$q(s) \in \mathbb{R}[s]$

$$\left(a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y\right)$$

$$= \left(b_m \frac{d^m u}{dt^m} + \text{———} + b_0 u\right)$$

$$\Rightarrow \quad p(s)\, Y(s) = q(s)\, U(s)$$

$$\Rightarrow \quad Y(s) = \frac{q(s)}{p(s)}\, U(s)$$

The input $= \delta(t)$

$$\mathcal{L}(\delta(t)) = 1$$

$$\Rightarrow \quad h(t) = \mathcal{L}^{-1}\left[\frac{q(s)}{p(s)} \cdot 1\right]$$

$$= \mathcal{L}^{-1}\left[G(s)\right] \leftarrow \text{impulse response.}$$

$$\boxed{\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0)}$$

$$\int_{-\infty}^{\infty} \delta(t-\alpha) f(t)\, dt = f(\alpha)$$

$$\delta^{(k)} \xrightarrow{\mathcal{L}} s^k .$$

$$e^{\lambda t} \longrightarrow \boxed{\begin{array}{l}\text{LTI}\\ \text{system}\end{array}} \xrightarrow{\; y(t) \;}$$

$e^{\lambda t} : \mathbb{R} \to \mathbb{R}$

$t \in \mathbb{R}$

$$y(t) = \int_{-\infty}^{\infty} e^{\lambda \tau} h(t-\tau)\, d\tau$$

$t - \tau =: \xi$

$d\tau = -d\xi$

τ	$-\infty$	∞
ξ	∞	$-\infty$

$$\Rightarrow y(t) = -\int_{\infty}^{-\infty} e^{\lambda t} e^{-\lambda \xi} h(\xi)\, d\xi$$

$$= e^{\lambda t} \int_{-\infty}^{\infty} e^{-\lambda \xi} h(\xi)\, d\xi$$

$$= e^{\lambda t} G(\lambda)$$

$$G(s) = \mathcal{L}(h(t))$$

$$u \longrightarrow \boxed{\begin{array}{l}\text{LTI}\\ \text{system}\end{array}} \xrightarrow{\; y \;}$$

$$\mathcal{U} \xrightarrow{G} \mathcal{Y}$$

$$e^{\lambda t} \longmapsto \underline{\underline{G(\lambda)}} e^{\lambda t}$$

$$G(s) = \frac{1}{s+1}$$

$$G(\lambda) = \frac{1}{\lambda+1}, \quad \lambda \in \underline{\mathbb{C}}.$$

$\boxed{\text{Frequency response}}$

$$u * h := \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

$$u : [0, \infty) \to \mathbb{R}, \quad h : [0, \infty) \to \mathbb{R}$$

Causality

$$u * h := \int_{0}^{t} u(\tau)\, h(t-\tau)\, d\tau$$

$$u(t) = \frac{1}{2\pi j} \int_{\sigma_c - j\infty}^{\sigma_c + j\infty} U(s)\, \underline{\underline{e^{st}}}\, ds \quad \checkmark$$

$$A \in \mathbb{R}^{n \times n}. \quad A : \mathbb{R}^n \to \mathbb{R}^n$$
$$v \mapsto Av$$

$v_1, v_2, \ldots, v_n$ are LI eigenvectors

$$\begin{array}{cccc} v_1, & v_2, & \ldots, & v_n \\ \downarrow & \downarrow & & \downarrow \\ \lambda_1 & \lambda_2 & & \lambda_n \end{array}$$

Suppose $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

$$Av = A(\alpha_1 v_1 + \text{———} + \alpha_n v_n)$$

$$= \alpha_1 A v_1 + \text{———} + \alpha_n A v_n$$

$$= \alpha_1 \lambda_1 v_1 + \text{———} + \alpha_n \lambda_n v_n$$

$$u(t) = \frac{1}{2\pi j} \int_{\sigma_c - j\infty}^{\sigma_c + j\infty} U(s) e^{st} \, ds$$

$$y(t) = \frac{1}{2\pi j} \int_{\sigma_c - j\infty}^{\sigma_c + j\infty} G(s) U(s) e^{st} \, ds$$

$$\boxed{Y(s) = G(s) U(s)}$$
