Matrix Computations
EE636
Assignment 1

Q 1 Given an $\mathtt{n}\times\mathtt{n}$ symmetric matrix A, calculate the number of flops required for finding:

(a) LU factorization of A,

(b) Cholesky decomposition.

Q 2 Calculate the flop count for forward substitution and backward substitution.

Q 3 Let $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ be a (nonsingular) matrix whose leading principal submatrices are all nonsingular. Partition A as:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \text{where} \quad A_{11} \in \mathbb{R}^{\mathtt{k}\times\mathtt{k}}, A_{12} \in \mathbb{R}^{\mathtt{k}\times(\mathtt{n}-\mathtt{k})}, A_{21} \in \mathbb{R}^{(\mathtt{n}-\mathtt{k})\times\mathtt{k}}, A_{22} \in \mathbb{R}^{(\mathtt{n}-\mathtt{k})\times(\mathtt{n}-\mathtt{k})}.$$

Show that there is exactly one matrix $M \in \mathbb{R}^{(\mathtt{n}-\mathtt{k})\times\mathtt{k}}$ such that:

$$\begin{bmatrix} I_{\mathtt{k}} & 0 \\ -M & I_{\mathtt{n}-\mathtt{k}} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \tag{1}$$

Also calculate $\tilde{A}_{22}$.

Q 4 Let

$$A = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 1 & 5 \\ 1 & 3 & 6 \end{bmatrix} \text{ and } b = \begin{bmatrix} 10 \\ -2 \\ -5 \end{bmatrix} \tag{2}$$

Use Gaussian elimination with partial pivoting to find matrices L and U such that U is upper triangular, L is unit lower triangular with $|l_{ij}| \leq 1$ for all $i > j$, and $LU = \widehat{A}$, where $\widehat{A}$ can be obtained from A by making row interchanges. Use LU decomposition to solve the system $Ax = b$.

Q 5 Let A be positive definite. Then prove that A can be expressed in exactly one way as a product $A = LDL^T$, such that L is unit lower triangular, and D is a diagonal matrix whose main-diagonal entries are positive.

Q 6 For a symmetric matrix A, show that the following conditions are equivalent:

(a) $A \succeq 0$.

(b) $A = U^T U$ for some matrix U.

(c) All principal minors of A are nonnegative.

Note: A minor of of order $\mathtt{k}$ is principal if it is obtained by deleting $(\mathtt{n}-\mathtt{k})$ rows and the $(\mathtt{n}-\mathtt{k})$ columns with the same index numbers. For example: if one deletes the 3rd row and 4th row of a matrix $A \in \mathbb{C}^{6\times 6}$, then he/she should delete the 3rd and 4th column to get a principal minor of order 4.

Q 7 Consider the circuit in Figure 1.

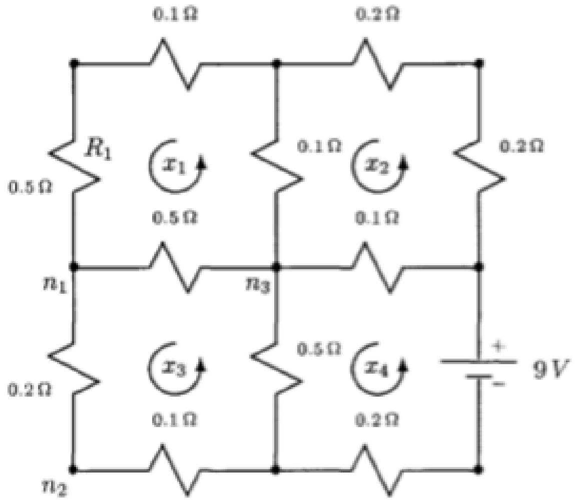

Figure 1: Electrical circuit with loop currents

(a) Write down a linear system $Ax = b$ of four equations for the four unknown loop currents.

(b) Solve the system for x using the MATLAB command *linsolve*. Calculate the residual $r = b - A\widehat{x}$, where $\widehat{x}$ denotes your computed solution.

(c) Using Ohm's law and the loop currents calculated in (b), find the voltage drops from the node labeled n_1 to the nodes labeled n_2 and n_3.

Q 8 Suppose $x(0 : \mathtt{n}) \in \mathbb{R}^{\mathtt{n}+1}$. A matrix $V \in \mathbb{R}^{(\mathtt{n}+1)\times(\mathtt{n}+1)}$ of the form:

$$V = V(x_0, \dots, x_{\mathtt{n}}) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_1 & \dots & x_{\mathtt{n}} \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{\mathtt{n}} & x_1^{\mathtt{n}} & \dots & x_{\mathtt{n}}^{\mathtt{n}} \end{bmatrix} \tag{3}$$

is said to be a Vandermonde matrix. Suppose $x_i \neq x_j$ for $i \neq j$. This question is an illustration of how systems of the form $V^T a = f = f(0 : \mathtt{n})$ can be solved using $O(\mathtt{n}^2)$ flops.

(a) Solving $V^T a = f$ is equivalent to polynomial interpolation:

$$p(x) = \sum_{j=0}^{\mathtt{n}} a_j x^j \tag{4}$$

where $p(x_i) = f_i$ for $i = 0 : \mathtt{n}$. The first step in computing a_j of (5) is to calculate the Newton representation of the interpolating polynomial p:

$$p(x) = \sum_{k=0}^{\mathtt{n}} c_k \Big(\prod_{i=0}^{k-1} (x - x_i)\Big) \tag{5}$$

Devise an algorithm to determine the constants c_k.

(b) The next task is to generate the coefficients $a_0, \dots, a_{\mathtt{n}}$ in (6) from the Newton representation coefficients $c_0, \dots, c_{\mathtt{n}}$. Define an iteration for the polynomials $p_{\mathtt{n}}(x), \dots, p_0(x)$ which in turn provides a way to obtain coefficients a_i recursively.

(c) Determine the flop count for the algorithm you devised combining (a) and (b).

Q 9 Consider the following system of equations:

$$\begin{bmatrix} 4 & 8 & -4 & -4 \\ 2 & 4 & 8 & -6 \\ -3 & -3 & 8 & -2 \\ -1 & 1 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -8 \\ 10 \\ 7 \\ 7 \end{bmatrix}.$$

Use partial pivoting technique to solve for x.

Q 10 Use complete pivoting technique to solve the following system of equations:

$$\begin{bmatrix} 3 & 4 & 3 \\ 1 & 5 & -1 \\ 6 & 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 15 \end{bmatrix}.$$

Q 11 Consider the matrix $F_{\mathtt{n}} \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ with the entries $f_{\mathtt{jk}} = \omega_{\mathtt{n}}^{\mathtt{jk}}$, where $\mathtt{j},\mathtt{k} \in \{\mathtt{0},\mathtt{1},\ldots,\mathtt{n}-\mathtt{1}\}$ and $\omega_{\mathtt{n}} = e^{-\frac{2\pi i}{\mathtt{n}}}$.

(a) Suppose $\mathtt{n} = \mathtt{2m}$, where $\mathtt{m}$ is a nutural number. Show that there exists a matrix P such that

$$F_{\mathtt{n}}P = \begin{bmatrix} F_{\mathtt{m}} & \Omega_{\mathtt{m}}F_{\mathtt{m}} \\ F_{\mathtt{m}} & -\Omega_{\mathtt{m}}F_{\mathtt{m}} \end{bmatrix},$$

$$\text{where} \quad \Omega_{\mathtt{m}} = \mathtt{diag}(1, \omega_{\mathtt{n}}, \omega_{\mathtt{n}}^2, \ldots, \omega_{\mathtt{n}}^{\mathtt{m}-1}).$$

(b) Use the result found in part (a) to construct an algorithm to solve the system of equations

$$\bar{F}_{\mathtt{n}}x = b$$

where $\mathtt{n} = 2^{\mathtt{q}}$ for some $\mathtt{q} \in \{0, 1, 2, \ldots\}$. $\bar{F}_{\mathtt{n}}$ is found by replacing each entry of $F_{\mathtt{n}}$ by its complex conjugate.

(c) What will be the flop count for such an algorithm? Justify your answer.

Q 12 For any $v \in \mathbb{R}^{\mathtt{n}}$ define the vectors $v_+ = (v + E_{\mathtt{n}}v)/2$ and $v_- = (v - E_{\mathtt{n}}v)/2$. Suppose $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is symmetric and persymmetric. Show that if $Ax = b$ then $Ax_+ = b_+$ and $Ax_- = b_-$.

Note: $E_{\mathtt{n}} := \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}.$