Matrix Computations
EE636
Assignment 2

Q 1 Given a positive definite matrix $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, define

$$\|x\|_A := (x^T A x)^{1/2}.$$

(a) Let R be the Cholesky factor of A, so that $A = R^T R$. Verify that for all $x \in \mathbb{R}^{\mathtt{n}}$, $\|x\|_A = \|Rx\|_2$.

(b) Using the fact that the 2-norm is indeed a norm on $\mathbb{R}^{\mathtt{n}}$, prove that $\|x\|_A$ is a norm on $\mathbb{R}^{\mathtt{n}}$.

Q 2 The *Frobenius norm* is defined by:

$$\|A\|_F = \left(\sum_{i=1}^{\mathtt{n}} \sum_{j=1}^{\mathtt{n}} |a_{ij}|^2 \right)^{1/2}.$$

Show that the Frobenius norm is a matrix norm (i.e., it is a norm on the vector-space of $\mathtt{n} \times \mathtt{n}$ matrices, which also satisfies the sub-multiplicative property).
Give an example of a norm on the space of real $\mathtt{n} \times \mathtt{n}$ matrices that is not a matrix norm.

Q 3 (a) Show that for all $x \in \mathbb{R}^{\mathtt{n}}$

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{\mathtt{n}}\|x\|_2 \leq \mathtt{n}\|x\|_\infty.$$

(b) Make systematic use of the inequalities from (a) to prove that for all $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$:

$$\|A\|_1 \leq \sqrt{\mathtt{n}}\|A\|_2 \leq \mathtt{n}\|A\|_1$$

and

$$\|A\|_\infty \leq \sqrt{\mathtt{n}}\|A\|_2 \leq \mathtt{n}\|A\|_\infty.$$

Q 4 Let A be nonsingular, and let x and $\hat{x} = x + \delta x$ be the solutions of $Ax = b$ and $A\hat{x} = b + \delta b$, respectively. Then show that:

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|},$$

where $\kappa(A)$ denotes the condition number of A. Show that this inequality is tight.

Q 5 Let $A = \begin{bmatrix} 375 & 374 \\ 752 & 750 \end{bmatrix}$.

(a) Calculate A^{-1} and $\kappa_\infty(A)$.

(b) Find b, δb, x and δx such that $Ax = b$, $A(x + \delta x) = b + \delta b$, $\|\delta b\|_\infty / \|b\|_\infty$ is small, and $\|\delta x\|_\infty / \|x\|_\infty$ is large.

(c) Find b, δb, x and δx such that $Ax = b$, $A(x + \delta x) = b + \delta b$, $\|\delta x\|_\infty / \|x\|_\infty$ is small, and $\|\delta b\|_\infty / \|b\|_\infty$ is large.

Q 6 In MATLAB we can type `A = hilb(3)` to get the 3×3 Hilbert matrix H_3, for example. Use MATLAB's condition number estimator `condest` to estimate $\kappa_1(H_{\mathtt{n}})$ for $\mathtt{n} = 3, 6, 9,$ and 12. Compare it with the true condition number, as computed by `cond(A,1)`.

Q 7 Let A be nonsingular, and suppose x and $\hat{x}$ satisfy $Ax = b$ and $\widehat{A}\hat{x} = \hat{b}$, respectively, where $\widehat{A} = A + \delta A$, $\hat{x} = x + \delta x \neq 0$, and $\hat{b} = b + \delta b \neq 0$. Then prove that

$$\frac{\|\delta x\|}{\|\hat{x}\|} \leq \kappa(A)\left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|\hat{b}\|} + \frac{\|\delta A\|}{\|A\|}\frac{\|\delta b\|}{\|\hat{b}\|}\right). \tag{1}$$

Q 8 If A is nonsingular, $\|\delta A\| \,/\, \|A\| < l/\kappa(A)$, $b \neq 0$, $Ax = b$, and $(A+\delta A)(x+\delta x) = b+\delta b$, using inequality (1) prove that

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\kappa(A)\left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right)}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}}.$$

Q 9 A metric space is a set X together with a function $d : X \times X \to \mathbb{R}$ (called a *metric* or *distance function*) satisfying the axioms:

i. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,

ii. $d(x, y) = d(y, x)$,

iii. $d(x, y) + d(y, z) \geq d(x, z)$.

Show that a vector space with a well defined norm is a metric space.

Q 10 Define $\|A\|_{\mathtt{p},\mathtt{q}} := \max\limits_{x\neq 0} \frac{\|Ax\|_{\mathtt{q}}}{\|x\|_{\mathtt{p}}}$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Show that $\|A\|_{\mathtt{p},\mathtt{q}}$ satisfies the axioms of norm. Derive formulae for $\|A\|_{1,2}$, $\|A\|_{2,1}$, and $\|A\|_{1,\infty}$.

Q 11 A function $f : \mathbb{R}^{\mathtt{n}} \to \mathbb{R}$ is said to be continuous at $x_0 \in \mathbb{R}^{\mathtt{n}}$ if for all $\epsilon > 0, \epsilon \in \mathbb{R}$ there exists $\delta(\epsilon, x_0)$ such that

$$\begin{aligned} &\|x - x_0\| < \delta(\epsilon, x_0) \\ \text{implies}\quad &|f(x) - f(x_0)| < \epsilon. \end{aligned}$$

f is said to be continuous if the above statement holds for all $x_0 \in \mathbb{R}^{\mathtt{n}}$.
Show that the vector norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ are continuous functions.

Q 12 Show that the condition number of an orthogonal matrix is 1.

Q 13 Consider the circuit in Figure 1.

(a) Write down a linear system $Ax = b$ with seven equations for the seven unknown nodal voltages and solve the system for x using the MATLAB command `linsolve`. Also compute the 1-norm of the residual and condition number $\kappa_1(A)$.

(b) The following MATLAB code solves a system $Ax = b$, where A is a large discrete Laplacian operator:

```
m=50;
A=delsq(numgrid('N',m));
n=size(A,1);
b=ones(n,1);
xhat=A\b;
```

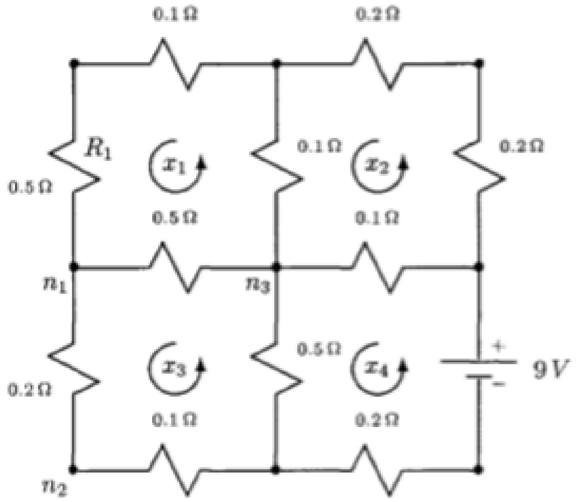

Figure 1: Electrical circuit with loop currents

Compute the norm of the residual $\hat{r}$. Use the MATLAB command `condest` and the inequality

$$\frac{\|x-\hat{x}\|_1}{\|x\|_1} \leqslant \kappa_1(A)\frac{\|\hat{r}\|_1}{\|b\|_1}$$

to estimate the error.

(c) What do you conclude by comparing the results found in part (a) and part (b)? Give your answer in one or two sentences.