Behavioral Theory of Systems (EE 714)

Problem Set 1

1. Let the polynomial matrix $U(\xi)$ be given by

$$\begin{bmatrix} -1+\xi^2+\xi^3 & \xi \\ \xi+\xi^2 & 1 \end{bmatrix}$$

Is $U(\xi)$ unimodular ? Write it as a product of elementary matrices.

2. Consider the electrical circuit shown in below figure. The volage and current at the port are manifest variables. Take current through inductors and voltages across capacitors as latent variables. Then, find behavioral equations defining the relation between manifest variables.

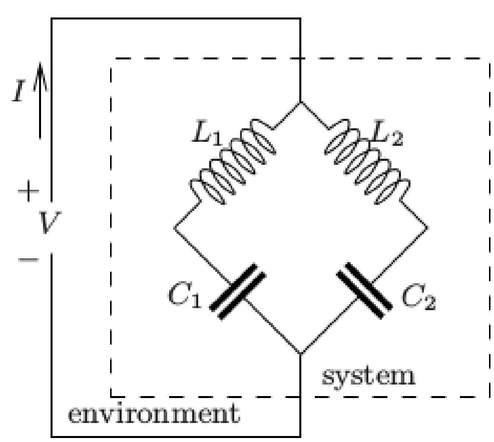

Figure 1: Electrical circuit

3. Let $r_1(\xi) = 2 - 3\xi + \xi^2, r_2(\xi) = 6 - 5\xi + \xi^2$ and $r_3(\xi) = 12 - 7\xi + \xi^2$.

   (a) Find a unimodular matrix $U(\xi) \in \mathbb{R}^{3\times 3}[\xi]$ such that the last row of $U(\xi)$ equals $[r_1(\xi), r_2(\xi), r_3(\xi)]$.

   (b) Find polynomials $a_1(\xi), a_2(\xi)$ and $a_3(\xi)$ such that

$$r_1(\xi)a_1(\xi) + r_2(\xi)a_2(\xi) + r_3(\xi)a_3(\xi) = 1$$

4. Prove that -
   A polynomial matrix $U(\xi) \in \mathbb{R}^{g\times g}[\xi]$ has a polynomial inverse $V(\xi) \in \mathbb{R}^{g\times g}[\xi]$ if and only if $\det U(\xi)$ is equal to a nonzero constant.

5. Consider the time-invariant dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}, \mathfrak{B})$. Define it to be *time-reversible* if $(w \in \mathfrak{B}) \Leftrightarrow (rev(w) \in \mathfrak{B})$ with the map rev defined by

$$(rev(w))(t) := w(-t)$$

   Which of the following scalar differential equations define a time-reversible system ?

   (a) $\frac{d^2}{dt^2}w + w = 0$

   (b) $\frac{d}{dt}w + \alpha w = 0$ (the answer may depend on the parameter α)

   (c) $\frac{d^2}{dt^2}w - w = 0$

   (d) $\frac{d^n}{dt^n}w = 0$ (the answer may depend on the parameter n)

6. Consider the differential system

$$-w_1 + \frac{d^2}{dt^2}w_1 + w_2 + \frac{d}{dt}w_2 = 0$$

$$-\frac{d}{dt}w_1 + \frac{d^2}{dt^2}w_1 + \frac{d}{dt}w_2 = 0$$

   Is it a full row rank representation ? If not, construct an equivalent full row rank representation.

7. Let $a(\xi), b(\xi) \in \mathbb{R}[\xi]$ be polynomials. Prove that $a(\xi)$ and $b(\xi)$ are coprime if and only if there exist polynomials $p(\xi)$ and $q(\xi)$ such that

$$a(\xi)p(\xi) + b(\xi)q(\xi) = 1$$

8. Consider the set $\mathbb{R}^{g \times g}[\xi]$ with 2 binary operations, addition $+$ and multiplication $\bullet$. Prove that $(\mathbb{R}[\xi], +, \bullet)$ defines a ring. Let

$$G := \{U(\xi) \in \mathbb{R}^{g \times g}[\xi] | U(\xi) \text{ is unimodular}\}$$

Prove that $(G, \bullet)$ forms a group.

9. If the real vectors $v_1, ..., v_k \in \mathbb{R}^n$ are linearly dependent over $\mathbb{R}$, then at least one of these vectors can be written as a linear combination (over $\mathbb{R}$) of the others. Give an example to show that this is not true for polynomial vectors in $\mathbb{R}^{1 \times n}[\xi]$. (Recall that (in)dependence of polynomial vectors in $\mathbb{R}^{1 \times n}[\xi]$ assumes that coefficients are coming from the ring $\mathbb{R}[\xi]$.

10. Let $P_i(\xi) \in \mathbb{R}[\xi], (i = 1, 2)$. Denote the corresponding behaviors by $\mathfrak{B}_i$. Assume that $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$. Prove that the polynomial $P_1(\xi)$ divides $P_2(\xi)$.

11. Determine the behavior $\mathfrak{B}$ associated with the differential equation

$$-32w + 22\frac{d^2}{dt^2}w + 9\frac{d^3}{dt^3}w + \frac{d^4}{dt^4}w = 0$$

12. Consider the set of differential equations

$$w_1 + \frac{d^2}{dt^2}w_1 - 3w_2 - \frac{d}{dt}w_2 + \frac{d^2}{dt^2}w_2 + \frac{d^3}{dt^3}w_2 = 0$$

$$w_1 - \frac{d}{dt}w_1 - w_2 + \frac{d}{dt}w_2 = 0$$

   (a) Determine the matrix $P(\xi) \in \mathbb{R}^{2 \times 2}[\xi]$ such that the set of above equations is equivalent to $P(\frac{d}{dt})w = 0$.

   (b) Determine the roots of $\det P(\xi)$.

   (c) Prove that every solution of above set of equations can be written as

$$w(t) = \begin{bmatrix} \alpha_1 - 3\alpha_2 \\ \alpha_1 \end{bmatrix} e^t + \begin{bmatrix} \alpha_2 \\ \alpha_2 \end{bmatrix} te^t + \begin{bmatrix} \beta \\ \beta \end{bmatrix} e^{-2t} + \begin{bmatrix} \gamma \\ \gamma \end{bmatrix} e^{-t}$$

13. (a) Show that the polynomial matrix $U(\xi) \in \mathbb{R}^{2\times 2}[\xi]$ given by

$$U(\xi) := \begin{bmatrix} 1+3\xi+\xi^2 & -2\xi-\xi^2 \\ -2-\xi & 1+\xi \end{bmatrix}$$

is unimodular and determine $(U(\xi))^{-1}$.

(b) Write $U(\xi)$ as a product of elementary unimodular matrices.

(c) Determine the behavior of $U(\frac{d}{dt})w = 0$. What general principle lies behind your answer ?

14. Determine the behavior $\mathfrak{B}$ associated with $P(\frac{d}{dt})w = 0$, where

$$P(\xi) = \begin{bmatrix} 2+\xi^2 & 1 \\ 2-2\xi-4\xi^2 & 1+\xi \end{bmatrix}$$

15. Different polynomial matrices may have the same determinant. Let $P(\xi) \in \mathbb{R}^{2\times 2}[\xi]$ be a diagonal matrix. Given $\det P(\xi) = -2-\xi+2\xi^2+\xi^3$, how many different behaviors correspond to this determinant ?

16. Many differential equations occuring in physical applications, e.g.- in mechanics, contain *even* derivatives only. Consider the behavioral equation

$$P(\frac{d^2}{dt^2})w = 0$$

with $P(\xi) \in \mathbb{R}^{q\times q}$, $\det P(\xi) \neq 0$. Assume that the roots of $\det P(\xi)$ are real and simple (multiplicity one). Describe the *real* behavior of this system in terms of the roots λ_k of $\det P(\xi)$ and the kernel of $P(\lambda_k)$.