Lecture 6 & 7: Time domain descriptions

So far we have seen some algebraic manipulations on the kernel representation matrix that helps us to reduce it to some equivalent form (maybe upper triangular or diagonal) which is easy to solve. The sole aim was to solve for the behavior trajectories explicitly. In this note we will discuss the characterization of such trajectories for scalar and vector valued behaviors.

We begin by looking at some examples.
Example 1: Consider the dynamical system given by

$$\begin{bmatrix} \frac{d}{dt}+1 & \frac{d}{dt}+2 \\ 0 & \frac{d^2}{dt^2}+3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

The kernel representation matrix is

$$R(\xi) = \begin{bmatrix} \xi+1 & \xi+2 \\ 0 & \xi^2+3 \end{bmatrix}$$

which has full row rank and full column rank. One of the methods to solve for the behavior is by evaluating the determinant of the matrix.
For this matrix the determinant is

$$\det R(\xi) = (\xi+1)(\xi^2+3)$$

The roots are $\lambda_1 = -1$ and $\lambda_{2,3} = \pm i\sqrt{3}$. The rank of the matrix evaluated at any root λ_i is less than the actual rank of the matrix. This implies that there exist vector(s) v_i in the nullspace of $R(\lambda_i)$ such that $R(\lambda_i)v = 0$.
In this example for $\lambda_1 = -1$, rank $R(-1) = 1 < 2$, thus there exists $v_1 \in \mathbb{C}^2$ such that $R(\lambda_1)v_1 = 0$. If $w(t) := v_1 e^{\lambda_1 t}$, then

$$R(\frac{d}{dt})v_1 e^{\lambda_1 t} = R(\lambda_1)v_1 e^{\lambda_1 t} = 0$$

Therefore $w(t) := v_1 e^{\lambda_1 t}$ is a trajectory in the behavior. Similarly for λ_2 and λ_3 there exists vectors $v_2, v_3 \in \mathbb{C}^2$ such that $R(\lambda_2)v_2 = 0$ and $R(\lambda_3)v_3 = 0$. Using the same approach $v_2 e^{\lambda_2 t}$ and $v_3 e^{\lambda_3 t}$ are trajectories that are in the behaviour. Since $R(\frac{d}{dt})$ is a linear operator any linear combinations of the trajectories will also be a trajectory.
But this method of calculating trajectories faces a problem when handling roots having multiplicites greater than one and then it doesnot give a complete characterization of the behaviour as is shown in the next example.

Example 2: Consider the systems given by the kernel representation matrices

$$R_1(\xi) = \begin{bmatrix} \xi+1 & 0 \\ 0 & \xi+1 \end{bmatrix} \qquad R_2(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & (\xi+1)^2 \end{bmatrix} \tag{2}$$

Here $\det R_1(\xi) = \det R_2(\xi) = (\xi+1)^2$. The roots are $\lambda_{1,2} = -1$. For $R_1(\xi)$, rank $R_1(\lambda) = 0$ which implies there are two linearly independent vectors in the nullspace of $R_1(\lambda)$. For $R_2(\xi)$, rank $R_2(\lambda) = 1$ which means only one vector ($\begin{bmatrix} 0 \\ 1 \end{bmatrix}$) can be found in the nullspace for both the roots. Therefore $\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}$ is a trajectory in the behavior. But this does not characterize the full

set of trajectories for $\begin{bmatrix} 0 \\ 1 \end{bmatrix} te^{-t}$ is also a legitimate solution which could not be calculated using this method.
To overcome this problem we need to explore some other ways of characterizing the full solution set. For this we study the scalar case first.

Characterization of trajectories for the scalar system

Lemma 1. *Let the behavior be given by the kernel representation polynomial as*

$$\begin{aligned} \mathfrak{B} &= \ker\ f(\frac{d}{dt}) \ \textit{where, } f(\xi) \in \mathbb{R}[\xi] \ \textit{and} \\ f(\xi) &= p(\xi)q(\xi) \end{aligned} \tag{3}$$

where $p(\xi)$ and $q(\xi)$ are coprime then, $\ker\ f(\frac{d}{dt}) = \ker\ p(\frac{d}{dt}) \bigoplus \ker\ q(\frac{d}{dt})$. $\bigoplus$ *represents direct sum.*

Proof: Since $p(\xi)$ and $q(\xi)$ are coprime by Aryabhatta's identity, there exists polynomials $a(\xi)$ and $b(\xi)$ such that

$$\begin{aligned} a(\xi)p(\xi) + b(\xi)q(\xi) &= 1 \\ a(\frac{d}{dt})p(\frac{d}{dt})w + b(\frac{d}{dt})q(\frac{d}{dt})w &= w \end{aligned} \tag{4}$$

where w is a function on which the polynomial acts. Let

$$v_2 := a(\frac{d}{dt})p(\frac{d}{dt})w \text{ and } v_1 := b(\frac{d}{dt})q(\frac{d}{dt})w$$

Suppose w is not just any function but is a trajectory in the behavior then, $w \in \ker\ f(\frac{d}{dt})$. If $q(\frac{d}{dt})$ acts on v_2 and $p(\frac{d}{dt})$ acts on v_1 then

$$\begin{aligned} q(\frac{d}{dt})v_2 &= q(\frac{d}{dt})(a(\frac{d}{dt})p(\frac{d}{dt})w), & p(\frac{d}{dt})v_1 &= p(\frac{d}{dt})(b(\frac{d}{dt})q(\frac{d}{dt})w) \\ q(\frac{d}{dt})v_2 &= a(\frac{d}{dt})f(\frac{d}{dt})w, & p(\frac{d}{dt})v_1 &= b(\frac{d}{dt})f(\frac{d}{dt})w \end{aligned}$$

So

$$q(\frac{d}{dt})v_2 = 0 \qquad \text{and} \qquad p(\frac{d}{dt})v_1 = 0$$

Therefore $v_2 \in \ker\ q(\frac{d}{dt})$ and $v_1 \in \ker\ p(\frac{d}{dt})$. From (4), w can be written as sum of v_1 and v_2.
For direct sum we also need to show $\ker\ p(\frac{d}{dt})$ and $\ker\ q(\frac{d}{dt})$ are disjoint i.e. both have only the zero element as the common element. This is proved using contradiction.
Suppose there exists a $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ and $w \neq 0$. Let $w \in \ker\ p(\frac{d}{dt}) \cap \ker\ q(\frac{d}{dt})$. This implies $p(\frac{d}{dt})w = 0$ and $q(\frac{d}{dt})w = 0$. Substituting in (4) we get $w = 0$ which is a contraction to the assumed fact that $w \neq 0$. Thus $\ker\ p(\frac{d}{dt})$ and $\ker\ q(\frac{d}{dt})$ have only the zero element in common and are disjoint. This completes the proves the lemma. □

Generalizing the lemma we have the following theorem.

Theorem 2. *Let the behavior be given by the kernel representation polynomial as*

$$\mathfrak{B} = \ker\ f(\frac{d}{dt}) \text{ where, } f(\xi) \in \mathbb{R}[\xi] \tag{5}$$
$$= \left\{ w(t) | f(\frac{d}{dt})w = 0 \right\}$$

where the solution set $w(t) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{C})$.
Invoking the fundamental theorem of algebra which says any polynomial over the field of complex numbers can be written as a product of linear prime factors, we write

$$f(\xi) = \prod_{i=1}^{n} (\xi - \lambda_i) \text{ where, } n = \deg\ f(\xi)$$
$$= \prod_{i=1}^{N} (\xi - \lambda_i)^{\nu_i}$$

where all λ_i*s are distinct and their respective multiplicities are denoted by* ν_i. *Let* $\mathfrak{B}_i$ *denote the behaviors of the respective factors i.e.* $\mathfrak{B}_i := \ker\ (\frac{d}{dt} - \lambda_i)^{\nu_i}$ *then*

$$\mathfrak{B} = \bigoplus_{i=1}^{N} \mathfrak{B}_i \tag{6}$$

Exercise: Modify the theorem if the solution set is defined over $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$.

Parametric representation of solutions

Theorem 3. *Let* $\mathfrak{B} = \ker\ (\frac{d}{dt} - \lambda)^\nu$. *Then the following are true*

1. $\mathfrak{B}$ *forms a finite dimensional vector space of* $\mathbb{C}$.
2. *Every trajectory in* $\mathfrak{B}$ *is a linear combination of the following functions* $w_1 = e^{\lambda t}, w_2 = te^{\lambda t}, \ldots, w_\nu = t^{\nu-1}e^{\lambda t}$.

Proof: To prove $\mathfrak{B}$ is a finite dimensional vector space we need to show

1. $w_1, w_2, \ldots, w_\nu$ are linearly independent, and
2. $\dim\ \mathfrak{B} = \nu$

1. Proof by contradiction:
Suppose $w_1, w_2, \ldots, w_\nu$ are linearly dependent. This implies there exists constants $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ not all zero such that

$$\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_\nu w_\nu = 0 \tag{7}$$
$$\alpha_1 e^{\lambda t} + \alpha_2 te^{\lambda t} + \cdots + \alpha_\nu t^{\nu-1} e^{\lambda t} = 0$$

Putting $t = 0$ we get $\alpha_1 = 0$. Differentiating (7) w.r.t. t we get

$$\alpha_2[e^{\lambda t} + \lambda te^{\lambda t}] + \cdots + \alpha_\nu[(\nu - 1)t^{\nu-2} + \lambda t^{\nu-1}e^{\lambda t}] = 0$$

Substituting $t = 0$, we get $\alpha_2 = 0$. Differentating ν times and putting $t = 0$ we get all the coefficients as zero. This is the trivial linear combination which contradicts our assumption that

some α_i should be nonzero, hence $w_1, w_2, \ldots, w_\nu$ are linearly independent.
2. We use induction to prove dim $\mathfrak{B} = \nu$.
As induction basis $(\frac{d}{dt} - \lambda)w = 0$ is true because the solution is obtained as $w = ke^{\lambda t}$ for some constant $k \in \mathbb{R}$ which is a linear combination of the first constituent.
Assuming the inductive step is true i.e. $(\frac{d}{dt} - \lambda)^n w = 0$ has every solution given by linear combinations of n constituent solutions, we need to show that every w satisfying

$$(\frac{d}{dt} - \lambda)^{(n+1)} w = 0$$

can be written as a linear combination of $n + 1$ constituent functions.

Defining

$$w' := (\frac{d}{dt} - \lambda)w \tag{8}$$

we get

$$(\frac{d}{dt} - \lambda)^n w' = 0. \tag{9}$$

If $w' = 0$ then w is given by $w = ke^{\lambda t}$. Thus, w is indeed in the linear span of $e^{\lambda t}$. On the other hand, if $w' \neq 0$ then equation (9) together with the inductive hypothesis tells us that w' will be a linear combination of n constituent functions. Solving w from (8) we get

$$w = \int_0^t e^{\lambda(t-\tau)} w'(\tau) d\tau + ke^{\lambda t}$$
$$w = \int_0^t e^{\lambda(t-\tau)} [k_1 w_1(\tau) + k_2 w_2(\tau) + \cdots + k_n w_n(\tau)] d\tau + ke^{\lambda t}$$
$$w = k_1 \int_0^t e^{\lambda(t-\tau)} w_1(\tau) d\tau + k_2 \int_0^t e^{\lambda(t-\tau)} w_2(\tau) d\tau + \cdots + k_n \int_0^t e^{\lambda(t-\tau)} w_n(\tau) d\tau + ke^{\lambda t}.$$

Therefore, every solution of $(\frac{d}{dt} - \lambda)^{(n+1)} w = 0$ is a linear combination of $n+1$ constituent functions: $\left\{ \int_0^t e^{\lambda(t-\tau)} w_1(\tau) d\tau, \int_0^t e^{\lambda(t-\tau)} w_2(\tau) d\tau, \ldots, \int_0^t e^{\lambda(t-\tau)} w_n(\tau) d\tau, e^{\lambda t} \right\}$. Since w was chosen arbitrarily, we must have dim(ker $(\frac{d}{dt} - \lambda)^{(n+1)}) \leq n + 1$.

Now, it is easy to check that each of $w_1, w_2, \ldots, w_\nu$, defined in the statement of the theorem, is a solution of $(\frac{d}{dt} - \lambda)^\nu$. Therefore, by linearity, the linear span of $\{w_1, w_2, \ldots, w_\nu\}$ is contained in $\mathfrak{B}$. Since $\{w_1, w_2, \ldots, w_\nu\}$ are linearly independent, the dimension of $\langle w_1, w_2, \ldots, w_\nu \rangle$ is ν. Therefore, $\dim(\mathfrak{B}) \geqslant \nu$. On the other hand, by the last paragraph, we know that $\dim(\mathfrak{B}) \leqslant \nu$. Therefore, $\dim(\mathfrak{B}) = \nu$, and hence

$$\langle w_1, w_2, \ldots, w_\nu \rangle = \mathfrak{B}.$$

□

Summarizing the above facts, if $\mathfrak{B} = \ker f(\frac{d}{dt})$ where $f(\frac{d}{dt}) \in \mathbb{R}[\xi]$ then the complete parametric representation of the solution is given by

$$w = p_1(t)e^{\lambda_1 t} + p_2(t)e^{\lambda_2 t} + \cdots + p_N(t)e^{\lambda_N t}$$

where $p_i(t)$ are polynomials of the form $k_{i1} + k_{i2}t + \cdots + k_{i\nu}t^{(\nu-1)}$ with ν being the multiplicity of a single factor and N the number of distinct factors.

Characterization of trajectories for vector differential equations

For the kernel representation $\mathfrak{B} = \ker R(\frac{d}{dt})$, with $R(\xi) \in \mathbb{R}^{q\times q}[\xi]$ and $\det R(\xi) \neq 0$ (full row rank) then the best way to solve for the solution is to convert $R(\xi)$ to the Smith canonical form as

$$R(\xi) = U(\xi)\begin{bmatrix} d_1(\xi) & 0 & \dots & 0 \\ 0 & d_2(\xi) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_q(\xi) \end{bmatrix} V(\xi)$$

because in that case the equations are decoupled and dependent on one variable only. With the previous knowledge of solving the scalar equations the diagonal terms can be solved for $\tilde{w}$ as

$$\begin{aligned} d_1(\frac{d}{dt})\tilde{w}_1 &= 0 \\ d_2(\frac{d}{dt})\tilde{w}_2 &= 0 \\ &\vdots \\ d_q(\frac{d}{dt})\tilde{w}_q &= 0 \end{aligned}$$

The original behavior trajectories can be obtained by premultiplying the unimodular matrix $V(\xi)$ i.e.

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_q \end{bmatrix} = V(\frac{d}{dt})\begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \vdots \\ \tilde{w}_q \end{bmatrix}$$