Lecture Notes of EE 714

Lecture 1

Motivation

Systems theory that we have studied so far deals with the notion of specified input and output spaces. But there are systems which do not have a clear demarcation between the input and output (e.g. Economists believe that there exists a relation between the production P of a particular economic resource, the capital K invested and the labor L expended towards its production. It becomes difficult to differentiate between the input and output for such models.) or do not have an input or output at all (e.g. solar system, system of linear equations). To model and analyse such systems existing classical methods in frequency domain or the state space methods in time domain are not sufficient.

Further, the transfer function, which is a map from input space to output space works well for a system having only one independent variable. For two variables the transfer function becomes a ratio of polynomials n and d in two indeterminates s_1 and s_2 as $\frac{n(s_1,s_2)}{d(s_1,s_2)}$. If

$$\frac{n(s_1,s_2)}{d(s_1,s_2)} = \frac{s_1}{s_2}$$

then it can take a 0/0 form which cannot exist and is difficult to handle. These were some of the causes that lead to the study of systems from another viewpoint - according to their behaviors.

Mathematical Models

A mathematical model is a collection of exclusion laws. Mathematical modeling is usually done for physical systems which are governed by some physical laws. These laws define which conditions can occur and which are impossible to occur. They are the exclusion laws. We start by looking at simple examples.

Example 1: Consider the electrical resistor. Before Ohm' law came into existence it was believed that any $(v, i) \in \mathbb{R}^2$ defines the system but Ohm's law defined a proportional relationship between v and i. Such physical laws that governs the system are known as **exclusion laws**. All values $(v, i) \in \mathbb{R}^2$ defines the **universum** $\mathbb{U}$. The property that uniquely defines the system is the **behavior** given by

$$\mathbb{U} \supseteq \mathfrak{B} = \Big\{(v, i) \in \mathbb{R}^2 | v = iR\Big\}$$

Example 2: Mathematical model of the states of water. If the state is represented by s it belongs to { ice, water, steam }, the other variable required is temperature t in degree Celsius which belongs to [-273, ∞). All outcomes are $\mathbb{U} = (s, t) =$ { ice, water, steam } $\times$ [-273, ∞). Out of these the possible outcomes that form the behavior is given by

$$\mathfrak{B} = \Big\{(\{ice\} \times [-273, 0]) \cup (\{water\} \times [0, 100]) \cup (\{steam\} \times [100, \infty))\Big\}$$

Example 3: For a system of linear equations in $\mathbb{R}^n$, the behavior is given by

$$\mathfrak{B} = \Big\{x \in \mathbb{R}^n | Ax = b\Big\}$$

This forms an affine set in $\mathbb{R}^n$.
Now we formally define what a mathematical model is.

Definition 1. *A* mathematical model *is a pair* $(\mathbb{U}, \mathfrak{B})$ *with* $\mathbb{U}$ *as the* universum*, which is the set of all* outcomes *of a given phenomenon (which we want to model) and* $\mathfrak{B}$ *a subset of* $\mathbb{U}$ *called the* behavior*.*

Dynamical Systems

In dynamical systems the system variables evolve with time.
Example: In the most elementary case of a solar system with Sun at the centre and the only planet as Earth whose distance in three coordinate system is given by the triple $x(t) \in \mathbb{R}^3$ the universum and the behavior is defined by the following relations

$$\mathbb{U} = \{x : \mathbb{R} \longrightarrow \mathbb{R}^3\}$$

$$\mathfrak{B} = \{x \in \mathbb{U} | \text{ Kepler's laws hold}\}$$

and this defines the dynamical system.

Thus a dynamical system Σ is defined by the triple

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

where $\mathbb{T}$ is the indexing set, $\mathbb{W}$ is the signal space and $\mathfrak{B}$ is the behavior.
Depending on the system description, whether continous time or discrete time, the indexing set can be $\mathbb{R}$ or $\mathbb{R}_+$ or $\mathbb{Z}$ or $\mathbb{Z}_+$ or more generally some interval in $\mathbb{R}$ or $\mathbb{Z}$. Here there is only one independent variable, time. For two independent variables the indexing set becomes $\mathbb{T}^2$.
The signal space depends on the number of manifest variables q which evolve with time. Manifest variables are those whose behavior is described by the model. A single input, single output system has two variables (input and output) taking real values therefore $\mathbb{W} = \mathbb{R}^2$. $\mathbb{W} = \mathbb{R}^3$ for the solar system. Generally $\mathbb{W} = \mathbb{R}^q$. $\mathbb{W}$ is a finite dimensional vector space for lumped system and is infinite dimensional for distributed system. $\mathbb{W}$ can take values from finite field also. Such systems are known as discrete event systems. The collection of maps from the indexing set to the signal space is represented as $\mathbb{W}^{\mathbb{T}} = \{\mathbb{T} \longrightarrow \mathbb{W}\}$ and the behavior $\mathfrak{B}$ is a subset of $\mathbb{W}^{\mathbb{T}}$. Behaviors are also known as trajectories of the system.

Linearity

A dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ for $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$ is linear if the behavior $\mathfrak{B}$ is a $\mathbb{R}$-vector space (linear vector space over the field of real numbers).
The superposition principle is followed, i.e. if $w_1, w_2 \in \mathfrak{B}$ and $\alpha, \beta \in \mathbb{F} \Rightarrow \alpha w_1 + \beta w_2 \in \mathfrak{B}$.
Example 1:

$$\mathfrak{B} = \left\{x \in \mathbb{R}^{\mathbb{R}} | \dot{x} = ax\right\} \subseteq \mathbb{R}^{\mathbb{R}}$$

It is a linear system because the solution set $\{x(t) = e^{at}k, k \in \mathbb{R}\}$ forms a linear vector space.

Example 2:

$$\mathfrak{B} = \left\{x \in \mathbb{R}^{\mathbb{R}} | \dot{x} = ax^2\right\}$$

This is not a linear system because if $x_1(t) \in \mathfrak{B}$, $x_2(t) \in \mathfrak{B}$ then $\dot{x_1} = ax_1^2$ and $\dot{x_2} = ax_2^2$. But $\dot{x_1} + \dot{x_2} = a(x_1^2 + x_2^2) \neq a(x_1 + x_2)^2$. Therefore $(x_1 + x_2) \notin \mathfrak{B}$.

Time (In)Variance

A dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$ is said to be time invariant if $\sigma_T \mathfrak{B} = \mathfrak{B}$ for all $T \in \mathbb{T}$. The shift operator σ_T is defined as $\sigma_T(w(t)) = w(t - T)$. Physically it means, if $w(t)$ is a behavior any shifted version of it is also a behavior. If $\mathbb{T} = \mathbb{Z}_+$ or $\mathbb{R}_+$, then for time invariance $\sigma_T \mathfrak{B} \subseteq \mathfrak{B}$.

Example 1:

$$\mathfrak{B} = \left\{ x \in \mathbb{R}^{\mathbb{R}^n} | \dot{x} = Ax \right\}$$

The solution set is $\{x(t) = e^{At}x(0), x(0) \in \mathbb{R}^n\}$ which is linear as already seen. To check if it is time invariant take a time shift $T \in \mathbb{R}$ then,

$$\sigma_T(e^{At}x(0)) = e^{A(t-T)}x(0) = e^{At}e^{-AT}x(0)$$

e^{-AT} is a nonsingular, invertible map and $e^{-AT}x(0)$ can be viewed as the initial condition at time $t = -T$. Thus this shifted trajectory is also a behavior and since $x(0)$ is arbitrary this applies for all trajectories. Hence the system is time invariant.
By specifying any other initial condition, not necessarily at $t = 0$, it is also possible to specify the full trajectory.

Example 2: Check if the system described by the following behavior is linear and time invariant.

$$\mathfrak{B} = \left\{ x(t) | \dot{x} = tx \right\}$$

If $x_1(t) \in \mathfrak{B}$, $x_2(t) \in \mathfrak{B}$ then $\dot{x_1} = tx_1$ and $\dot{x_2} = tx_2$. If $x_3 = (x_1 + x_2)$ then $\dot{x_3} = tx_3 = t(x_1 + x_2) = \dot{x_1} + \dot{x_2}$. Therefore $(x_1 + x_2) \in \mathfrak{B}$ and the system is linear.
To check time invariance defining the shifted trajectory as $\tilde{x}(t) := \sigma_T(x(t)) = x(t - T)$. If it is a behavior then

$$\frac{d\tilde{x}(t)}{dt} = \dot{x}(t - T)$$
$$\frac{d\tilde{x}(t)}{dt} = (t - T)x(t - T) = (t - T)\tilde{x}(t)$$

Since $(t - T)\tilde{x}(t) \neq t\tilde{x}(t)$, it is not a behavior and the system is time varying.

Lecture 2

Linear Time Invariant Systems

Linear time invariant systems which are described by linear constant coefficient differential equations are studied because many nonlinear systems can be approximated by linear systems and linear systems are found in many applications like

1. Newton's law of motion, $m\ddot{x} = F$, where, m is the mass, $\ddot{x}$ is the acceleration and F is the applied force.

2. A spring mass damper system described by $m\ddot{x} + D\dot{x} + Kx = F$ where, D is the coefficient of dynamic friction and K is the spring constant. This can be extended to the case where multiple blocks are connected with each other.

3. Hamiltonian systems defined by
$$\begin{bmatrix}\dot{p}\\ \dot{q}\end{bmatrix} = \begin{bmatrix}\frac{\partial H}{\partial q}\\ -\frac{\partial H}{\partial p}\end{bmatrix}$$
where p is the momentum, q is the position and H is a scalar function known as the Hamiltonian of the system.

The behaviour $\mathfrak{B} \supseteq \mathbb{U} = \mathbb{W}^{\mathbb{T}}$. When the signal space is one dimensional, let $w(t) \in \mathbb{R}$ be a trajectory. Then $(p(\frac{d}{dt}))w = 0$ defines a behavior, where p is a polynomial in $\frac{d}{dt}$. For a second order system

$$\frac{d^2w}{dt^2} + 5\frac{dw}{dt} + 6w = 0$$
$$(\frac{d^2}{dt^2} + 5\frac{d}{dt} + 6)w = 0$$

Substituting $\frac{d}{dt} = \xi$ gives $p(\xi) = \xi^2 + 5\xi + 6 \in \mathbb{R}[\xi]$ where $\mathbb{R}[\xi]$ represents the ring of polynomials with real coefficients.

For $w(t) \in \mathbb{R}^n$, the corresponding ordinary differential equation (ODE) is

$$r_1(\frac{d}{dt})w_1 + r_2(\frac{d}{dt})w_2 + \cdots + r_q(\frac{d}{dt})w_q = 0$$
$$\begin{bmatrix} r_1(\frac{d}{dt}) & r_2(\frac{d}{dt}) & \ldots & r_q(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} w_1\\ w_2\\ \ldots\\ w_q \end{bmatrix} = 0$$

Thus a row vector of differential operator acts on the trajectory vector. There can be many such differential equations which relate the variables w_i. Stacking these equations row-wise we get a polynomial matrix R with q columns and as many rows as the number of equations (say g). Each term of this matrix is a polynomial in $\frac{d}{dt}$. Therefore we have

$$R(\frac{d}{dt})w = 0$$

By substituting $\frac{d}{dt} = \xi, R(\xi) \in \mathbb{R}[\xi]^{g\times q}$.

The behavior is the solution set of $R(\frac{d}{dt})w = 0$. Therefore

$$\mathfrak{B} = \left\{ w \in \mathbb{W}^{\mathbb{U}} | R(\xi)w = 0 \right\}$$
$$= \ker R(\frac{d}{dt})$$

This is known as the **Kernel Representation** and $R(\frac{d}{dt})$ is the kernel representation matrix.
Challenging Problem: Find examples for linear time invariant (LTI) systems in continuous time that are not described by ordinary differential equation (ODE). What are the specifications (apart from (LTI)) required for the system to be described by an ODE?

Strong and Weak Solutions of a Differential Equation

Consider the differential equation of a R-C circuit

$$\dot{y} + y - \dot{u} = 0$$

where y is the output and u is the input. The transfer function of this system is given by

$$\frac{Y(s)}{U(s)} = \frac{s}{(s+1)}$$

If input voltage is a step function then the output current is an exponentially decaying signal as shown in (1).

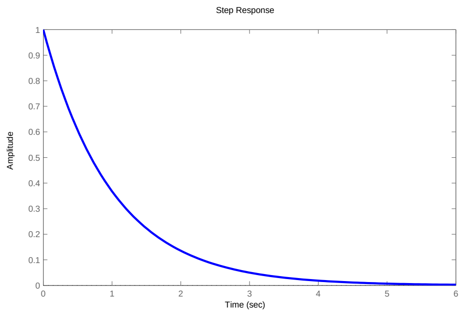

Figure 1: Output signal of the differential equation

Rewriting the differential equations in the behavioral context where the voltage and current are manifest variables w_2 and w_1 respectively we have

$$\frac{dw_1}{dt} + w_1 - \frac{dw_2}{d_t} = 0 \tag{1}$$

Let the solution of this differential equation be the 2-tuple $\begin{bmatrix} w_1(t) & w_2(t) \end{bmatrix}^T$. So it must satisfy the differential equation. But for the differential equation to be satisfied both w_1 and w_2 must be atleast once differentiable with respect to time. However neither the voltage nor the current as seen previously are smooth (differentiable), but they are indeed the solution to this differential equation. How do we justify this? For this the notion of weak and strong solution of a differential equation is used. Before defining these terms we define the following term.

Definition 2. ***Locally Integrable Function:*** *A function $w : \mathbb{R} \to \mathbb{R}^q$ is called* locally integrable *if for all $a, b \in \mathbb{R}$,*

$$\int_a^b \|w(t)\|_2 dt < \infty$$

where $\|.\|_2$ denotes the Euclidean norm of the vector. The space of locally integrable functions $w : \mathbb{R} \to \mathbb{R}^q$ is denoted by $\mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q)$. Functions with finite jumps are locally integrable like step and sawtooth.

Since both the step function and the function of figure (1) are locally integrable functions we integrate the differential equation (1) for some time interval t,

$$\int_0^t \frac{dw_1(\tau)}{d\tau}dt + \int_0^t w_1(\tau)dt - \int_0^t \frac{dw_2(\tau)}{d\tau}dt = C_1 \tag{2}$$
$$w_1(t) - w_1(0) + \int_0^t w_1(\tau)dt - w_2(t) - w_2(0) = C_1$$
$$w_1(t) + \int_0^t w_1(\tau)dt - w_2(t) = C$$

If $(w_1,\, w_2)$ is a solution for the differential equation (1) then it should also be a solution for (2). For the integral equation the restriction of differentiability does not apply. Therefore if w is sufficiently smooth and satisfies $R(\frac{d}{dt})w = 0$, $w(t)$ are the **strong** solutions of the differential equation. If w is not smooth but satisfies the integral equation (2) then $w(t)$ is a **weak** solutions of the differential equation.

Remark 3. When looking for weak solutions of the integral equation (that is, (2)) one must not look for satisfation of the equation *for all* time $t \in \mathbb{R}$. This is in contrast to the situation of strong solutions, where the differential equation must be satisfied for all $t \in \mathbb{R}$. The reason for this seeming double standard in dealing with weak and strong solutions lies in the description of the respective solution spaces. While $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ functions are specified point-wise, it is not done so for $\mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R})$ functions. More precisely, two $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ functions, say w_1, w_2, are identified to be the same if $w_1(t) = w_2(t)$ for all $t \in \mathbb{R}$. On the other hand, $w_1, w_2 \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R})$ are identified to be the same if they satisfy

$$\int_a^b \|w_1(t) - w_2(t)\|_2 dt = 0$$

for all $a, b \in \mathbb{R}$. This implies that $w_1 - w_2$ is allowed to be non-zero over a **set of measure zero**. Therefore, $w_1 = w_2$ as $\mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R})$ functions if $w_1(t) = w_2(t)$ for all $t \in \mathbb{R}$ but over a set of measure zero, in other words, $w_1(t) = w_2(t)$ for almost all $t \in \mathbb{R}$. In the same way, when $w \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ is tested for whether it is a weak solution or not, the LHS of equation (2) needs be equal to the RHS only for all $t \in \mathbb{R}$ minus a set of measure zero. That is, a weak solution satisfies equation (2) for almost all $t \in \mathbb{R}$.

Obtaining integral equations from the differential representation

Let $R(\xi) \in \mathbb{R}[(\xi)]^{g \times q}$, then R can be represented as

$$R(\xi) = \begin{bmatrix} p_{11}(\xi) & p_{12}(\xi) & \dots & p_{1q}(\xi) \\ p_{21}(\xi) & p_{22}(\xi) & \dots & p_{2q}(\xi) \\ \vdots & & \ddots & \\ p_{g1}(\xi) & p_{g2}(\xi) & \dots & p_{gq}(\xi) \end{bmatrix}$$

where each $p_{ij} = a_0 + a_1\xi + a_2\xi^2 + \dots + a_r\xi^r$. Choosing the constant terms from each polynomial and writing them in their respective (i, j) positions we get the constant matrix R_0., Similarly writing the coefficients of ξ in their respective positions we get the matrix R_1. Continuing this in a similar fashion for the highest degree in the polynomial we obtain the matrix $R_n \neq 0$ corresponding to the n-th degree of ξ. So, $R(\xi)$ can be written as

$$R(\xi) = R_0 + R_1\xi + R_2\xi^2 + \dots + R_n\xi^n$$

Thus a polynomial matrix (matrix with polynomial entries) can be written as a polynomial with matrix coefficients.

Considering $w : \mathbb{R} \to \mathbb{R}^q$, then $R(\xi)w = 0$ can be written as

$$R_0 w + R_1 \xi w + R_2 \xi^2 w + \cdots + R_n \xi^n w = 0 \tag{3}$$
$$R_0 w + R_1 \frac{dw}{dt} + R_2 \frac{d^2 w}{dt^2} + \cdots + R_n \frac{d^n w}{dt^n} = 0$$

Integrating (3) once we get,

$$R_0 \int_0^t w(\tau) d\tau + R_1 w(t) + R_2 \frac{dw}{dt} + \cdots + R_n \frac{d^{n-1} w}{dt^{n-1}} = C_0$$

Integrating it again

$$R_0 \int_0^t \int_0^{\tau_1} w(\tau) d\tau ds + R_1 \int_0^t w(s) ds + R_2 w(t) + \cdots + R_n \frac{d^{n-2} w}{dt^{n-2}} = C_0 t + C_1$$

Integrating it n times

$$(R_0 (\int)^n + R_1 (\int)^{n-1} + R_2 (\int)^{n-2} + \cdots + R_n) w = C_0 t^{n-1} + C_1 t^{n-2} + \cdots + C_{n-1}$$
$$(R^*(\int)) w(t) = C_0 t^{n-1} + C_1 t^{n-2} + \cdots + C_{n-1}$$

where R^* refers to the **reciprocal polynomial matrix**. The reciprocal polynomial matrix of a polynomial matrix $R(\xi)$ is defined as follows: Given a polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ in the indeterminate x, then the reciprocal polynomial is defined as $p^*(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$. Thus $p^*(x) := x^n p(1/x)$. Similarly, for a polynomial with matrix $R(\xi)$ written as a polynomial with real constant matrix coefficients

$$R(\xi) = R_0 + R_1 \xi + R_2 \xi^2 + \cdots + R_n \xi^n$$
$$R^*(\xi) := \xi^n R(1/\xi) = R_n + R_{n-1} \xi + R_{n-2} \xi^2 + \cdots + R_0 \xi^n$$

Lecture 3

Definition 4. $\mathfrak{C}^\infty$ ***functions:*** *A function $w : \mathbb{R} \to \mathbb{R}^q$ is called* infinitely differentiable *if w is k times differentiable for all $k \in \mathbb{N}$. This space of infinitely differentiable functions is denoted by $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$.*

$\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ functions are good in the sense that they can be directly tested for satisfaction of differential equations, without us having to worry whether we pick up impulses (which are *not* functions!) by blindly differentiating non-smooth functions. Thus, $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ solutions of *ker* $R(\frac{d}{dt})$ are strong solutions. However, there could be strong solutions in *ker* $R(\frac{d}{dt})$ which are *not* $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$; these solutions may be sufficiently differentiable allowing us to directly verify satisfaction of differential equations, but they may not be *infinite* times differentiable.

Theorem 5. *Consider the behavior defined by*

$$R(\frac{d}{dt}) w = 0 \tag{4}$$

1. *Every strong solution of* (4) *is also a weak solution.*
2. *Every weak solution that is sufficiently smooth (belongs to C^∞)is also a strong solution.*

Topological Properties of Behavior

Given a set $\mathcal{S}$ then the elements $w_1, w_2, \ldots, w_k$ form a sequence represented by $\{w_k\}$ where $k \in \mathbb{N}$. The distance between two elements of the set is $|w_i - w_j|$. If $\exists w \in \mathbb{R}$ such that

$$\lim_{k\to\infty} |w_k - w| = 0$$

then $\{w_k\}$ is called a convergent sequence.

Convergence of functions in the sense of $\mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q)$:

A sequence $\{w_k\}$ in $\mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q)$ converges to $w \in \mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q)$ in the sense of $\mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q)$ if for all $a, b \in \mathbb{R}$,

$$\lim_{k\to\infty} \int_a^b \|w(t) - w_k(t)\|_2 = 0$$

Example: Consider the sequence of functions $\{w_k\}$ with $w_k(t)$ defined by

$$w_k(t) = \begin{cases} 0 \text{ for } |t| > 1/k \\ 1 \text{ for } |t| \leq 1/k \end{cases}$$

The sequence converges to zero in the sense of $\mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q)$ because for $k = 1$ the function is a positive pulse of height 1 and width 2. As k increases the width of the function decreases and at $k \to \infty$ it can be thought of as a spike of height 1. Such a function when square integrated over any finite interval results in the zero function. If the function is changed to

$$w_k(t) = \begin{cases} 0 \text{ for } |t| \geq 1/k \\ k \text{ for } |t| < 1/k \end{cases}$$

the sequence doesnot converge in the sense of $\mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q)$ because the magnitude of the sequences goes on increasing as k increases.

Lemma 6. *Suppose $\mathfrak{B} = kerR(\frac{d}{dt})$. If $\{w_k\} \subseteq \mathfrak{B}$ such that w_k converges to w in the sense of $\mathfrak{L}_1^{loc}$ then $w(t) \in \mathfrak{B}$*

Bump Function :

A function ϕ defined as

$$\phi(t) = \begin{cases} 0 \text{ for } |t| \geq 1 \\ e^{-\frac{1}{1-t^2}} \text{ for } |t| < 1 \end{cases} \tag{5}$$

is an infinitely differentiable function.

Lemma 7. *Let $w \in \mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q)$ and let ϕ be given by (5). Define the function v by*

$$v(t) := \int_{-\infty}^{\infty} \phi(\tau) w(t - \tau) d\tau \tag{6}$$

Then v is infinitely differentiable.

Lemma 8. *For every $w \in \mathfrak{B}$ we have $v = \phi * w \in \mathfrak{B}$.*

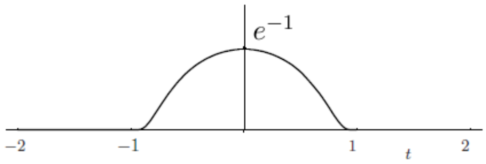

Figure 2: Graph of the bump function defined in (5)

Theorem 9. *Let $w \in \mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q)$. There exists a sequence $\{w_k\}$ in $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ that converges to w in the sense of $\mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q)$*

This theorem shows that for every weak solution there is a sequence of strong solutions or in other words, the $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ is dense in $\mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q)$. The strong solutions are dense means every weak solution can be approximated by a strong solution.

Theorem 10. *Let $R_1(\xi) \in \mathbb{R}^{g_1 \times q}[\xi]$ and $R_2(\xi) \in \mathbb{R}^{g_2 \times q}[\xi]$. The corresponding behaviors are denoted by $\mathfrak{B}_1$ and $\mathfrak{B}_2$. If $\mathfrak{B}_1 \cap \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) = \mathfrak{B}_2 \cap \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$, then $\mathfrak{B}_1 = \mathfrak{B}_2$*