Behavioral Theory of Systems (EE 714)

Problem set 1

Submit solutions to problems 1, 3, 4, 5, 6, 8, 9 by Tuesday the 27th January, 2015. Write on A4 sheets (both sides) staple them and put the stapled sheets in my mailbox kept at the EE Office by 5.00 PM. Solutions typed in LaTeX 2_ε and printed, would be preferable, but it is not mandatory; hand-written solutions are also equally welcome.

1. Consider the differential equation

$$\frac{d}{dt}w_2 = w_1 + w_2. \tag{1}$$

The function (w_1, w_2) is given by

$$(w_1(t), w_2(t)) = \begin{cases} (0,0) & t < 0, \\ (1, e^t - 1) & t \geq 0. \end{cases} \tag{2}$$

Prove that the function (w_1, w_2) in (2) is a weak solution of (1).

2. Consider the RC circuit of Figure 1.

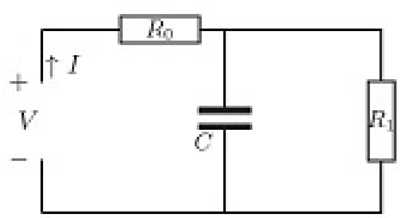

Figure 1: Electrical Circuit

Assume for ease of calculation that $R_0 = R_1 = 1, C = 1$. Prove that with the port voltage

$$V(t) = \begin{cases} 0 & t < 0, \\ 1 & t \geq 0 \end{cases}$$

the current

$$I(t) = \begin{cases} 0 & t < 0, \\ \frac{1}{2} + \frac{1}{2}e^{-2t} & t \geq 0 \end{cases}$$

yields a weak solution of

$$V + CR_1\frac{d}{dt}V = (R_0 + R_1)I + CR_0R_1\frac{d}{dt}I. \tag{3}$$

Assume instead that a current source

$$I(t) = \begin{cases} 0 & t < 0, \\ 1 & t \geq 0. \end{cases}$$

is switched on at $t = 0$. Compute, analogously to the previous situation, a corresponding weak solution of (3).

3. Let the polynomial matrix $U(\xi)$ be given by

$$\begin{bmatrix} -1 + \xi^2 + \xi^3 & \xi \\ \xi + \xi^2 & 1 \end{bmatrix}$$

Is $U(\xi)$ unimodular? Write it as a product of elementary matrices.

4. Let $r_1(\xi) = 2 - 3\xi + \xi^2, r_2(\xi) = 6 - 5\xi + \xi^2$, and $r_3(\xi) = 12 - 7\xi + \xi^2$.

   (a) Find a unimodular matrix $U(\xi) \in \mathbb{R}^{3\times 3}[\xi]$ such that the last row of $U(\xi)$ equals $[r_1(\xi), r_2(\xi), r_3(\xi)]$.

   (b) Find polynomials $a_1(\xi), a_2(\xi), a_3(\xi)$ such that

$$r_1(\xi)a_1(\xi) + r_2(\xi)a_2(\xi) + r_3(\xi)a_3(\xi) = 1$$

5. A polynomial matrix $U(\xi) \in \mathbb{R}^{g\times g}[\xi]$ has a polynomial inverse $V(\xi) \in \mathbb{R}^{g\times g}[\xi]$ if and only if $\det U(\xi)$ equals a nonzero constant. Prove this statement.

6. Consider the differential system

$$\begin{aligned} -w_1 + \frac{d^2}{dt^2}w_1 + w_2 + \frac{d}{dt}w_2 &= 0, \\ -\frac{d}{dt}w_1 + \frac{d^2}{dt^2}w_1 + \frac{d}{dt}w_2 &= 0. \end{aligned}$$

Is it a full row rank representation? If not, construct an equivalent full row rank representation.

7. Consider the Electrical circuit of Figure 2. Write the following equations

   (a) *Constitutive equations* (also known as) *device characteristic equations*

   (b) *Kirchhoff's current laws*

   (c) *Kirchhoff's voltage laws*

in polynomial matrix form $R(\frac{d}{dt})w = 0$ with $w = col(V, I, V_{R_C}, I_{R_C}, V_{R_L}, I_{R_L}, V_C, I_C, V_L, I_L)$

8. Define the functions $w_n \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R})$ as

$$w_n(t) = \begin{cases} 0 & |t| < n, \\ n & |t| \geq n. \end{cases}$$

Prove that w_k converges to the zero function in the sense of $\mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R})$.

9. (a) Does *pointwise* convergence ($\lim_{k\to\infty} w_k(t) = w(t)$ for all t) imply convergence in the sense $\mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$? Provide a counterexample.

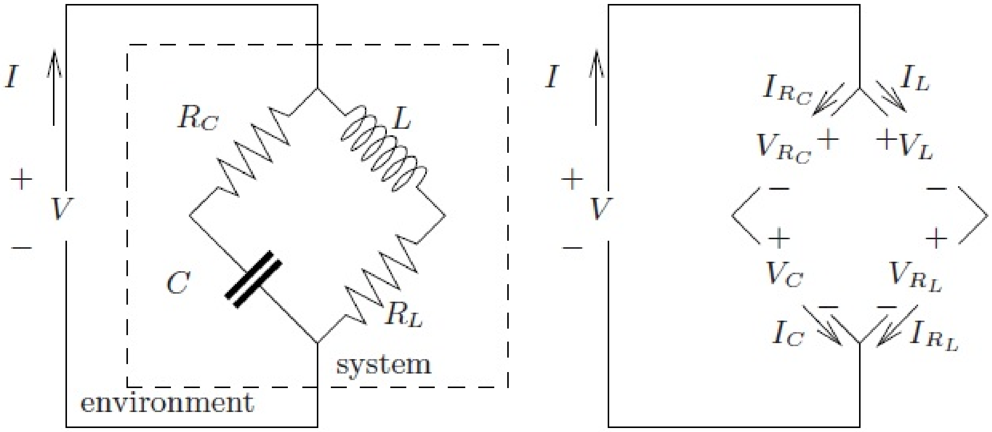

Figure 2: Electrical Circuit

(b) The sequence of functions $\{w_k\}$ is said to converge to w uniformly in t if for all $\epsilon > 0$ there exists an N such that for all t and for all $k \geq N$, $||w_k(t) - w(t)|| < \epsilon$. The difference with pointwise convergence is that N is not allowed to depend on t. Equivalently, $\lim_{k\to\infty}(\sup_t ||w_k(t) - w(t)||) = 0$. Does uniform convergence imply convergence in the sense of $\mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$?

10. Let $w \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ be a weak solution of $R(\frac{d}{dt})w = 0$, where $R(\xi) \in \mathbb{R}^{g\times q}[\xi]$.

    (a) Let $\psi_i \in \mathbb{R}$ and $\tau_i \in \mathbb{R}$, $i = 1, \dots, N$. Prove that $\sum_{i=1}^{N} \psi_i w(t-\tau_i)$ is a weak solution of $R(\frac{d}{dt})w = 0$.

    (b) Assume that $\psi \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R})$ is such that

    $$(\psi * w)(t) := \int_{-\infty}^{\infty} \psi(\tau) w(t-\tau) d\tau$$

    is a well-defined integral with $\psi * w \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$. Prove that $\psi * w$ is also a weak solution of $R(\frac{d}{dt})w = 0$.

11. Let $\psi \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})$ and $w \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R})$. Define v by

    $$v(t) := \int_0^t w(s) ds.$$

    Show that

    $$\int_0^t (\psi * w)(\tau) d\tau = (\psi * v)(t).$$

12. Prove that integration is a continuous operation on $\mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$; i.e., show that if $\lim_{k\to\infty} w_k = w$ in the sense of $\mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$, then

    $$\lim_{k\to\infty} \int_a^b w_k(t)(dt) = \int_a^b w(t) dt.$$

13. Let $w_k(t) = c_{0,k} + \cdots + c_{n,k}t^n, c_{i,k} \in \mathbb{R}^q$. Prove that if the sequence $\{w_k\}$ converges in the sense of $\mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q)$, then the sequences $c_{i,k}$ converge in $\mathbb{R}^q$ for all i. For simplicity you may confine yourself to the case $q = 1, n = 1$.

14. Let $a(\xi), b(\xi) \in \mathbb{R}[\xi]$ be polynomials. Prove that $a(\xi)$ and $b(\xi)$ are coprime if and only if there exist polynomials $p(\xi)$ and $q(\xi)$ such that $a(\xi)p(\xi) + b(\xi)q(\xi) = 1$.

15. Consider $\mathbb{R}^{g\times g}[\xi]$. Obviously, addition, $+$, and multiplication, $\bullet$, each define binary operations on $\mathbb{R}^{g\times g}[\xi]$. Prove that $(\mathbb{R}[\xi], \bullet, +)$ defines a ring. Let

$$\mathcal{U} := \{U(\xi) \in \mathbb{R}^{g\times g}[\xi] | U(\xi) \text{is unimodular}\}.$$

Prove that $(\mathcal{U}, \bullet)$ forms a group.

16. If the real vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ are linearly dependent over $\mathbb{R}$, then at least one of these vectors can be written as a linear combination (over $\mathbb{R}$) of the others. Show by means of an example that this is not true for polynomial vectors. (Recall that (in)dependence of polynomial vectors assumes the coefficients are coming from the ring $\mathbb{R}[\xi]$.)