Behavioral Theory of Systems (EE 714)

Problem Set 2

1. Determine the behavior $\mathfrak{B}$ associated with the differential equation

$$-32w + 22\frac{d^2}{dt^2}w + 9\frac{d^3}{dt^3}w + \frac{d^4}{dt^4}w = 0.$$

2. Let $P_i(\xi) \in \mathbb{R}[\xi], (i = 1, 2)$. Denote the corresponding behaviors by $\mathfrak{B}_i$. Assume that $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$. Prove that the polynomial $P_1(\xi)$ divides $P_2(\xi)$.

3. Many differential equations occurring in physical applications, e.g., in mechanics, contain even derivatives only. Consider the behavioral equation

$$P\left(\frac{d^2}{dt^2}\right)w = 0,$$

with $P(\xi) \in \mathbb{R}^{q\times q}[\xi], \det P(\xi) \neq 0$. Assume that the roots of $\det P(\xi)$ are real and simple (multiplicity one). Describe the real behavior of this system in terms of the roots λ_k of $\det P(\xi)$ and the kernel of $P(\lambda_k)$.

4. Consider the set of differential equations

$$\begin{aligned} w_1 + \frac{d^2}{dt^2}w_1 - 3w_2 - \frac{d}{dt}w_2 + \frac{d^2}{dt^2}w_2 + \frac{d^3}{dt^3}w_2 &= 0. \\ w_1 - \frac{d}{dt}w_1 - w_2 + \frac{d}{dt}w_2 &= 0. \end{aligned} \tag{1}$$

   (a) Determine the matrix $P(\xi) \in \mathbb{R}^{2\times 2}[\xi]$ such that equations (1) is equivalent to $P(\frac{d}{dt})w = 0$.

   (b) Determine the roots of $\det P(\xi)$.

   (c) Prove that every (strong) solution of (1) can be written as

$$w(t) = \begin{bmatrix} \alpha_1 - 3\alpha_2 \\ \alpha_1 \end{bmatrix} e^t + \begin{bmatrix} \alpha_2 \\ \alpha_2 \end{bmatrix} te^t + \begin{bmatrix} \beta \\ \beta \end{bmatrix} e^{-2t} + \begin{bmatrix} \gamma \\ \gamma \end{bmatrix} e^{-t}.$$

5. (a) Show that the polynomial matrix $U(\xi) \in \mathbb{R}^{2\times 2}[\xi]$ given by

$$U(\xi) := \begin{bmatrix} 1 + 3\xi + \xi^2 & -2\xi - \xi^2 \\ -2 - \xi & 1 + \xi \end{bmatrix}$$

   is unimodular, and determine $(U(\xi))^{-1}$.

   (b) Write $U(\xi)$ as a product of elementary unimodular matrices.

(c) Determine the behavior of $U(\frac{d}{dt})w = 0$. What general principle lies behind your answer?

6. Determine the behavior $\mathfrak{B}$ associated with $P(\frac{d}{dt})w = 0$, where

$$P(\xi) = \begin{bmatrix} 2+\xi^2 & 1 \\ 2-2\xi-4\xi^2 & 1+\xi \end{bmatrix}.$$

7. Different polynomial matrices may have the same determinant. Let $P(\xi) \in \mathbb{R}^{2\times 2}[\xi]$ be a diagonal matrix. Given $\det P(\xi) = -2-\xi+2\xi^2+\xi^3$, how many different behaviors correspond to this determinant?

8. Let $P(\xi)$ be given by

$$P(\xi) := \begin{bmatrix} P_{11}(\xi) & 0 \\ P_{21}(\xi) & P_{22}(\xi) \end{bmatrix}$$

Consider the behavior associated with $P(\frac{d}{dt})w = 0$.

(a) Take $P_{11}(\xi) = 1-2\xi+\xi^2, P_{21}(\xi) = -3+\xi$, and $P_{22}(\xi) = 1+\xi$. Determine a basis of the corresponding behavior $\mathfrak{B}_a$ and conclude that $\mathfrak{B}_a$ is a linear subspace of dimension three.

(b) Take $P_{11}(\xi)$ and $P_{22}(\xi)$ as in the previous part and $P_{21}(\xi) = -3+2\xi-2\xi^2+\xi^3$. Prove that the corresponding behavior, $\mathfrak{B}_b$, equals $\mathfrak{B}_a$.

(c) Now let $P_{11}(\xi) \neq 0, P_{22}(\xi) \neq 0$, and $P_{22}(\xi)$ arbitrary. Prove that the corresponding behavior is a linear subspace of dimension equal to the degree of $P_{11}(\xi)P_{22}(\xi)$.

(d) Consider the more general case

$$P(\xi) := \begin{bmatrix} P_{11}(\xi) & P_{12}(\xi) \\ P_{21}(\xi) & P_{22}(\xi) \end{bmatrix}.$$

Prove that P can be brought into lower triangular form by elementary row operations. Use this to prove that the dimension of the corresponding behavior is equal to the degree of the determinant of $P(\xi)$.

(e) Use induction on q to prove the following theorem.

Theorem 0.1. *Let $P(\xi) \in \mathbb{R}^{q\times q}[\xi]$ and let $\lambda_i \in \mathbb{C}, i = 1,\ldots,N$, be the distinct roots of $\det P(\xi)$ of multiplicity n_i : $\det P(\xi) = c\prod(\xi-\lambda_k)^{n_k}$ for some nonzero constant c. The corresponding behavior $\mathfrak{B}$ is autonomous and is a finite-dimensional subspace of $\mathcal{C}^\infty(\mathbb{C},\mathbb{C}^q)$ of dimension $n = \deg\det P(\xi)$. Moreover, $w \in \mathfrak{B}$ if and only if it is of the form*

$$w(t) = \sum_{i=1}^{N}\sum_{j=0}^{n_i-1} B_{ij}t^j e^{\lambda_i t} \tag{2}$$

where the vectors $B_{ij} \in \mathbb{C}^q$ satisfy the relations

$$\sum_{j=l}^{n_i-1} \binom{j}{l}\ P^{(j-l)}(\lambda_j)B_{ij} = 0, i = 1,\ldots,N; l = 0,\ldots,n_i-1$$

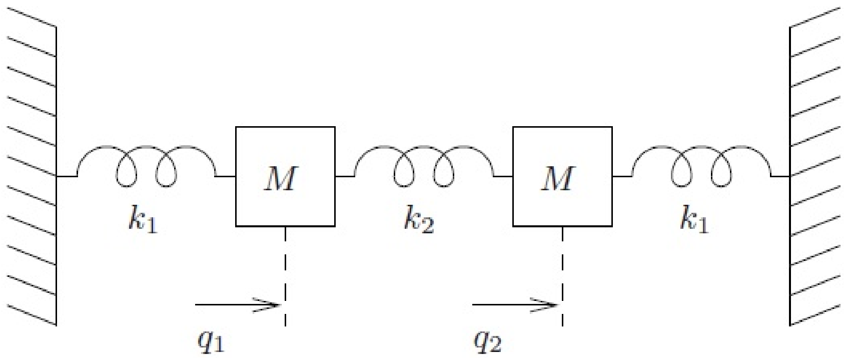

Figure 1: Mass-spring system

9. Consider the mechanical system shown in Figure 1. Assume that $q_1 = 0$ corresponds to the equilibrium position of the mass on the left-hand side and that $q_2 = 0$ corresponds to that of the other mass.

   (a) Determine for each of the cases below the differential equations describing
      i. $\mathrm{col}(q_1, q_2)$,
      ii. q_1,
      iii. q_2.

   (b) Use Theorem 0.1 to determine the behavior for the three cases above.

   (c) Consider the behavior $\mathfrak{B}$ of $\mathrm{col}(q_1, q_2)$. It is of interest to see how the time behavior of q_1 relates to that of q_2. Show that the behavior $\mathfrak{B}$ may be written as $\mathfrak{B} = \mathfrak{B}_s + \mathfrak{B}_a$ (subscript s for symmetric, a for antisymmetric), with $\mathfrak{B}_s$ consisting of elements of $\mathfrak{B}$ of the form $(q_1, q_2) = (q, q)$ and $\mathfrak{B}_a$ consisting of elements of the form $(q, -q)$. Derive differential equations describing $\mathfrak{B}_s$ and $\mathfrak{B}_a$.

   (d) Prove that also $\mathfrak{B}_s$ and $\mathfrak{B}_a$ consist of pure sinusoids. Denote the respective frequencies by ω_s and ω_a. Discuss these frequencies for the cases
      i. $\frac{k_1}{k_2} \ll 1$.
      ii. $\frac{k_1}{k_2} \gg 1$.

10. Consider the one-dimensional horizontal motion of the mechanical system depicted in Figure 2. Let q_1 denote the displacement of M_1 from some reference point, and q_2 the displacement of M_2 from its equilibrium when M_1 is in the position corresponding to $q_1 = 0$. Assume that external forces F_1, F_2 act on the masses M_1 and M_2 respectively.

   (a) Derive differential equations relating q_1, q_2, F_1, F_2.

   (b) Derive all possible input/output partitions of q_1, q_2, F_1, F_2.

   (c) Derive an integral expression relating the input $\mathrm{col}(F_1, F_2)$ to $\mathrm{col}(q_1, q_2)$.

11. (a) Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. The Vandermonde matrix M is given by

$$M = \begin{bmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_n \\ \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix}$$

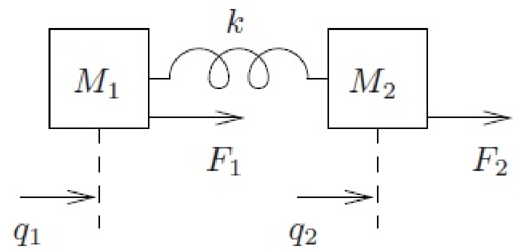

Figure 2: Mass-spring system

Prove that M is nonsingular if and only if the λ_is are mutually distinct.

(b) Let $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$. Let $n_1, \ldots, n_N \in \mathbb{N}$ and define $n := \sum_{i=1}^{N} n_i$. Suppose we have a polynomial $P(\xi) \in \mathbb{C}[\xi]$ given by

$$P(\xi) = \prod_{i=1}^{N} (\xi - \lambda_i)^{n_i}.$$

Recall that we have proved in class that the trajectories in the behavior $\mathfrak{B} = \ker P\left(\frac{d}{dt}\right)$ are given by

$$w(t) = \sum_{i=1}^{N} \sum_{j=0}^{n_i - 1} k_{i,j} t^j e^{\lambda_i t}, \quad k_{i,j} \in \mathbb{R}.$$

Find out the matrix M that relates the vector $\mathrm{col}(k_{1,0}, k_{1,1}, \ldots, k_{1,n_1-1}, k_{2,0}, k_{2,1}, \ldots, k_{N,n_N})$ with the vector $\mathrm{col}(w(0), \frac{dw}{dt}(0), \ldots, \frac{d^{n-1}w}{dt^{n-1}}(0))$ as

$$M \begin{bmatrix} k_{1,0} \\ k_{1,1} \\ \vdots \\ k_{N,n_N} \end{bmatrix} = \begin{bmatrix} w(0) \\ \frac{dw}{dt}(0) \\ \vdots \\ \frac{d^{n-1}w}{dt^{n-1}}(0)) \end{bmatrix}.$$

Prove that the matrix M is invertible if and only if the λ_is are mutually distinct.

12. Determine the partial fraction expansion of $\frac{1-6\xi+\xi^2}{-36+5\xi^2+\xi^4}$.

13. Consider the i/o system defined by

$$p(\frac{d}{dt})y = q(\frac{d}{dt})u \tag{3}$$

with $p(\xi) = \xi - 2\xi^2 + \xi^3$ and $q(\xi) = -1 + \xi^2$.

(a) Determine the partial fraction expansion of $\frac{q(\xi)}{p(\xi)}$.

(b) Give an explicit characterization of the behavior $\mathfrak{B}$ of (3).

(c) Consider now

$$\widetilde{p}(\frac{d}{dt})y = \widetilde{q}(\frac{d}{dt})u \tag{4}$$

with $\widetilde{p}(\xi) = -\xi + \xi^2$ and $\widetilde{q}(\xi) = 1 + \xi$. Determine the partial fraction expansion of $\frac{\widetilde{q}(\xi)}{\widetilde{p}(\xi)}$. What strikes you?

(d) Give an explicit characterization of the behavior $\widetilde{\mathfrak{B}}$ of (4).

(e) In what sense are $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$ different?

(f) Give convolution representations of $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$.

14. Let the polynomial matrix $R(\xi)$ be given by

$$R(\xi) := \begin{bmatrix} -5\xi + \xi^2 & -5 + \xi \\ -\xi + \xi^2 & -1 + \xi \end{bmatrix}$$

Show that $R(\frac{d}{dt})w = 0$ does not define an autonomous system. Write this system in input/output form. Indicate clearly which component of w is considered input and which is the output.

15. Recall the definition of autonomous systems: $\mathfrak{B}$ *is said to be* autonomous *if and only if for all $w_1, w_2 \in \mathfrak{B}$, the condition $w_1(t) = w_2(t)$ for $t \leqslant 0$ implies $w_1(t) = w_2(t)$ for all $t \in \mathbb{R}$.* Now prove that if $\mathfrak{B}$ is linear then $\mathfrak{B}$ is autonomous if and only if for all $w \in \mathfrak{B}$, the condition $w(t) = 0$ for all $t \leqslant 0$ implies $w(t) = 0$ for all $t \in \mathbb{R}$.

16. Recall the definition of autonomous systems: $\mathfrak{B}$ *is said to be* autonomous *if and only if for all $w_1, w_2 \in \mathfrak{B}$, the condition $w_1(t) = w_2(t)$ for $t \leqslant 0$ implies $w_1(t) = w_2(t)$ for all $t \in \mathbb{R}$.* Now prove that if $\mathfrak{B}$ is time-invariant and $\tau \in \mathbb{R}$ is arbitrary then $\mathfrak{B}$ is autonomous if and only if for all $w_1, w_2 \in \mathfrak{B}$, the condition $w_1(t) = w_2(t)$ for all $t \leqslant \tau$ implies $w_1(t) = w_2(t)$ for all $t \in \mathbb{R}$.

17. Recall the result proved in class: *Suppose $\mathfrak{B}_1 = \ker R_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \ker R_2(\frac{d}{dt})$, where $R_1(\xi), R_2(\xi) \in \mathbb{R}^{g \times q}[\xi]$. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the two equations modules corresponding to $R_1(\xi)$ and $R_2(\xi)$, respectively. Then $\mathfrak{B}_1 \cap \mathfrak{C}^\infty = \mathfrak{B}_2 \cap \mathfrak{C}^\infty$ if and only if $\mathcal{R}_1 = \mathcal{R}_2$.*

Now prove that $\mathfrak{B}_1 \cap \mathfrak{L}_1^{\text{loc}} = \mathfrak{B}_2 \cap \mathfrak{L}_1^{\text{loc}}$ if and only if $\mathcal{R}_1 = \mathcal{R}_2$.