Behavioral Theory of Systems (EE 714)

Problem Set 3

1. Consider the electrical circuit consisting of a resistor, a capacitor, an inductor, and an external port shown in Figure 1 . Determine the relation between V and I by applying the general elimination procedure.

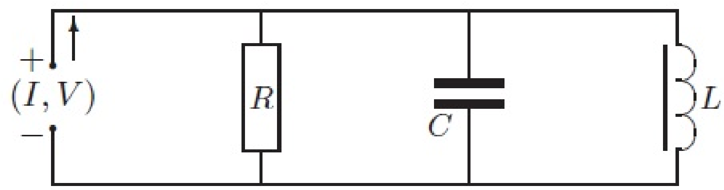

Figure 1: Electrical circuit

2. Let $R(\xi), M(\xi) \in \mathbb{R}^{2\times 1}[\xi]$ and consider

$$R(\frac{d}{dt})w = M(\frac{d}{dt})\ell, \tag{1}$$

with $R(\xi) = \begin{bmatrix} R_1(\xi) & R_2(\xi) \end{bmatrix}^T$ and $M(\xi) = \begin{bmatrix} M_1(\xi) & M_2(\xi) \end{bmatrix}^T$. We want to eliminate ℓ.

   (a) Assume that $M_1(\xi)$ and $M_2(\xi)$ have no common factors. Prove that the manifest behavior $\mathfrak{B}$ defined as

   $$\mathfrak{B} := \{w \in \mathfrak{C}^\infty \mid \exists \ell \in \mathfrak{C}^\infty \text{ such that equation (1) is satisfied}\}$$

   is described by

   $$\left(M_2(\frac{d}{dt})R_1(\frac{d}{dt}) - M_1(\frac{d}{dt})R_2(\frac{d}{dt}) \right) w = 0.$$

   (b) Determine the differential equation for the manifest behavior when $M_1(\xi)$ and $M_2(\xi)$ may have a common factor.

3. Consider the SISO systems

$$\Sigma_1 : p_1(\frac{d}{dt})y_1 = q_1(\frac{d}{dt})u_1, \Sigma_2 : p_2(\frac{d}{dt})y_2 = q_2(\frac{d}{dt})u_2. \tag{2}$$

Define the *feedback interconnection* of Σ_1 and Σ_2 by (2) and the interconnection equations $u_2 = y_1, u_1 = u + y_2$, and $y = y_1$. Here u is the external input and y is the external output; see Figure 2

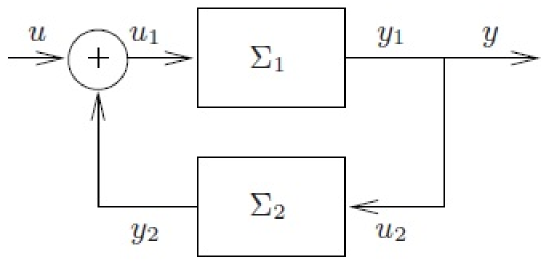

Figure 2: Feedback interconnection of Σ_1 and Σ_2

We are interested in the relation between u and y. To that end we have to eliminate u_1, u_2, y_1, y_2. Elimination of u_2 and y_1 is straightforward, since $u_2 = y_1 = y$. In order to eliminate u_1 and y_2, define ℓ and w as

$$\ell := \begin{bmatrix} u_1 \\ y_2 \end{bmatrix}, w := \begin{bmatrix} u \\ y \end{bmatrix}.$$

(a) Determine matrices $R(\xi), M(\xi)$ of appropriate dimensions such that the behavior with these latent variables is described by $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$.

(b) Eliminate ℓ from $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$. Conclude that the relation between u and y is given by

$$(p_1(\frac{d}{dt})\overline{p}_2(\frac{d}{dt}) - \overline{q}_1(\frac{d}{dt})q_2(\frac{d}{dt}))y = \overline{p}_2(\frac{d}{dt})q_1(\frac{d}{dt})u,$$

with $p_2(\xi) = c(\xi)\overline{p}_2(\xi)$ and $q_1(\xi) = c(\xi)\overline{q}_1(\xi)$, such that $\overline{p}_2(\xi)$ and $\overline{q}_1(\xi)$ have no common factors.

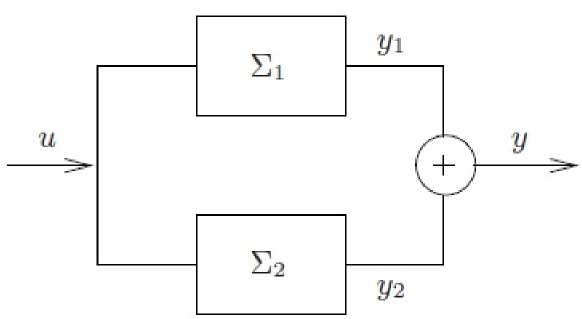

Figure 3: Parallel interconnection of Σ_1 and Σ_2

4. Repeat 3 for the *parallel interconnection* $p_1(\frac{d}{dt})y_1 = q_1(\frac{d}{dt})u, p_2(\frac{d}{dt})y_2 = q_1(\frac{d}{dt})u, y = y_1 + y_2$. See Figure 3 . The answer in this case is

$$(\overline{p}_2(\frac{d}{dt})q_1(\frac{d}{dt}) + \overline{p}_1(\frac{d}{dt})p_2(\frac{d}{dt}))y = \overline{p}_1(\frac{d}{dt})q_2(\frac{d}{dt})u,$$

where $p_1(\xi) = c(\xi)\overline{p}_1(\xi)$ and $p_2(\xi) = c(\xi)\overline{p}_2(\xi)$, such that $\overline{p}_1(\xi)$ and $\overline{p}_2(\xi)$ have no common factors.

5. Suppose we have

$$\begin{bmatrix} R_1(\frac{d}{dt}) & R_2(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0,$$

where $R_1(\xi) \in \mathbb{R}^{g\times q_1}[\xi]$ and $R_2(\xi) \in \mathbb{R}^{g\times q_2}[\xi]$. Further, assume that $[R_1(\xi)\ R_2(\xi)]$ is full row rank. Prove that w_2 is an input iff $R_1(\xi)$ is full row rank. Is the statement necessarily true if we remove the assumption that $[R_1(\xi)\ \ R_2(\xi)]$ is full row rank?

6. Suppose we have a kernel representation for $\mathfrak{B}$ as

$$\begin{bmatrix} R_1(\frac{d}{dt}) & R_2(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0,$$

where $R_1(\xi) \in \mathbb{R}^{g\times q_1}[\xi]$ and $R_2(\xi) \in \mathbb{R}^{g\times q_2}[\xi]$. In class we have looked at projections of $\mathfrak{B}$ onto one or more variables. The dual of this action is *nullification*. Define the set

$$\mathfrak{B}_{w_1,0} := \{w_1 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q_1}) | (w_1, 0) \in \mathfrak{B}\}.$$

   (a) Show that $\mathfrak{B}_{w_1,0} \subseteq \mathfrak{B}$.

   (b) Show that $\mathfrak{B}_{w_1,0}$ itself is a behavior by obtaining a kernel representation of it.

7. Consider the latent variable representation

$$R(\frac{d}{dt})w + M(\frac{d}{dt})\ell = 0 \tag{3}$$

Suppose by unimodular row operation $[R(\xi)\ M(\xi)]$ is brought to the form $\begin{bmatrix} R_1(\xi) & \widetilde{M}(\xi) \\ R_2(\xi) & 0 \end{bmatrix}$ where $\widetilde{M}(\xi)$ is full row rank. Prove, WITHOUT USING the fundamental principle, that the manifest behavior

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid \exists \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q'}) \text{ such that equation (3) is satisfied}\}$$

has kernel representation given by $\mathfrak{B} = \ker R_2(\frac{d}{dt})$.

8. Consider the polynomial matrix

$$M(\xi) = \begin{bmatrix} C\xi & 1 \\ 0 & L\xi + R_L \\ R_c C\xi + 1 & 0 \end{bmatrix}$$

which made an appearance in the Wheatstone bridge circuit problem. Recall that an MLA of $M(\xi)$ was calculated as

$$\widetilde{R}(\xi) = \begin{bmatrix} (L\xi + R_L)(R_c C\xi + 1) & -(R_c C\xi + 1) & -(L\xi + R_L)C\xi \end{bmatrix}.$$

Now suppose $\frac{R_L}{L} = \frac{1}{R_c C}$. Prove that in this case $\widetilde{R}(\xi)$ is NOT an MLA of $M(\xi)$. Find out an MLA for this case.

9. Suppose $M(\xi) \in \mathbb{R}^{g \times q'}[\xi]$. Further let $U(\xi) \in \mathbb{R}^{g \times g}[\xi]$ be a unimodular matrix that brings $M(\xi)$ to the form

$$U(\xi)M(\xi) = \begin{bmatrix} \widetilde{M}(\xi) \\ 0 \end{bmatrix}$$

where $\widetilde{M}(\xi)$ is full row rank, and number of rows in it is g'. Prove that

$$\widetilde{R}(\xi) := \begin{bmatrix} \mathbf{0}_{(g-g') \times g'} & I_{g-g'} \end{bmatrix} U(\xi)$$

is an MLA of $M(\xi)$. Further, prove that if $\widetilde{R}_1(\xi)$ is any other MLA of $M(\xi)$ then there exists $U_1(\xi) \in \mathbb{R}^{(g-g') \times (g-g')}[\xi]$ such that $\widetilde{R}_1(\xi) = U_1(\xi)\widetilde{R}(\xi)$. Note that this completely characterizes all MLAs of $M(\xi)$. From this deduce that every MLA of $M(\xi)$ has rank $g - g'$ where g' is the rank of $M(\xi)$.