# Behavioral Theory of Systems (EE 714)

Problem Set 4

1. Consider the following behaviors in *kernel representation* $\mathfrak{B} = \left\{ w : R(\frac{d}{dt})w = 0 \right\}$ where $R(\xi)$ are given by

   (a) $\begin{bmatrix} \xi^2 + 5\xi + 6 & -\xi - 1 \end{bmatrix}$.

   (b) $\begin{bmatrix} \xi^2 + \xi + 1 & \xi^2 + \xi + 1 & \xi^2 + 3\xi + 3 \\ 2\xi + 1 & \xi^2 + 2\xi & \xi^2 + 4\xi + 2 \end{bmatrix}$.

   Are these systems behavior controllable?

2. Consider the following behaviors in *image representation*

   (a) $\mathfrak{B}_1 = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & \xi \\ \xi + 1 & 1 \\ 2 & \xi + 1 \end{bmatrix} \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \right\}$, and

   (b) $\mathfrak{B}_2 = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \xi + 1 & \xi^2 + \xi + 1 \\ \xi + 2 & 2\xi + 2 \\ \xi + 3 & \xi^2 + 2\xi + 3 \end{bmatrix} \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \right\}$

   In these behaviors, is $\ell$ observable from $w$? In the manifest behaviors, is $w_3$ observable from $(w_1, w_2)$?

   In each case find out kernel representations of the manifest behaviors by eliminating $\ell$, and construct uncontrollable behaviors for each case whose controllable part is the manifest behavior you have obtained.

3. (a) Let $r(\xi) \in \mathbb{R}[\xi]$, $w = \mathrm{col}(w_1, w_2)$, where $w_1$ is $q_1$-dimensional and $w_2$ is $q_2$-dimensional, $A \in \mathbb{R}^{q_1 \times q_1}$ and $B \in \mathbb{R}^{q_1 \times q_2}$. Assume that $r(\xi)$ is a polynomial of degree at least one. Prove that $r(\frac{d}{dt})w_1 + Aw_1 = Bw_2$ is controllable if and only if rank $\begin{bmatrix} B & AB & \dots & A^{q_1-1}B \end{bmatrix} = q_1$.

   (b) Mechanical systems are often described by second-order differential equations. In the absence of damping, they lead to models of the form

   $$M\frac{d^2q}{dt^2} + Kq = BF$$

   with $q$ the vector of (generalized) positions, assumed $n$-dimensional; $F$ the external forces; and $M, K$, and $B$ matrices of suitable dimension; $M$ is the mass matrix and $K$ the matrix of spring constants. Assume that $M$ is square and nonsingular. Prove that with $w = \mathrm{col}(q, F)$, this system is controllable if and only if rank $\begin{bmatrix} B & KM^{-1}B & \dots & (KM^{-1})^{n-1}B \end{bmatrix} = n$.

4. Consider the i/o behavior $\mathfrak{B}$ defined by

$$-y + \frac{d^2}{dt^2}y = -u + \frac{d}{dt}u.$$

(a) Is this system controllable?

(b) Show trajectories in $\mathfrak{B}$ that are not patchable with each other.

(c) Write $\mathfrak{B}$ as the direct sum of an autonomous part and a controllable part.

(d) Define $\mathfrak{B}_{\text{aut}} := \{(u,y) | -y + \frac{d}{dt}y = 0, u = 0\}$ and $\mathfrak{B}_{\text{contr}} := \{(u,y) | y + \frac{d}{dt}y = u\}$. Prove that $\mathfrak{B} = \mathfrak{B}_{\text{aut}} \oplus \mathfrak{B}_{\text{contr}}$.

5. (a) Consider the behavior $\mathfrak{B}$ of $R(\frac{d}{dt})w = 0$ with $R(\xi) = \begin{bmatrix} \xi^2 - 1 & \xi + 1 \end{bmatrix}$. Provide two different decompositions of $\mathfrak{B}$ as a direct sum of a controllable and an autonomous part.

(b) Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ be of full row rank and let $\mathfrak{B}$ be the behavior of $R(\frac{d}{dt})w = 0$. Let $U(\xi) \in \mathbb{R}^{g \times g}[\xi]$ and $V(\xi) \in \mathbb{R}^{q \times q}[\xi]$ be unimodular matrices that transform $R(\xi)$ into Smith form:

$$U(\xi)R(\xi)V(\xi) = \begin{bmatrix} D(\xi) & 0 \end{bmatrix}$$

We know that, a decomposition of the behavior $\mathfrak{B}$ into a controllable and an autonomous part is obtained by defining

$$R_{\text{contr}}(\xi) = \begin{bmatrix} I & 0 \end{bmatrix} V^{-1}(\xi), \; R_{\text{aut}}(\xi) = \begin{bmatrix} D(\xi) & 0 \\ 0 & I \end{bmatrix} V^{-1}(\xi).$$

Let $W(\xi) \in R^{q \times q}[\xi]$ be a unimodular matrix with the property that

$$\begin{bmatrix} D(\xi) & 0 \end{bmatrix} W(\xi) = \begin{bmatrix} D(\xi) & 0 \end{bmatrix},$$

and define $R'_{\text{aut}}(\xi) = \begin{bmatrix} D(\xi) & 0 \\ 0 & I \end{bmatrix} W^{-1}(\xi)V^{-1}(\xi)$. Prove that $R_{\text{contr}}(\xi), R'_{\text{aut}}(\xi)$ also provides a decomposition of $\mathfrak{B}$ into a direct sum of a controllable and an autonomous part.

(c) In order to classify all possible decompositions of $\mathfrak{B}$ into a direct sum of a controllable and an autonomous part, we first classify all such decompositions of $\widetilde{\mathfrak{B}}$, the behavior of $\begin{bmatrix} D(\xi) & 0 \end{bmatrix}$. Let $\widetilde{R}_{\text{contr}}(\xi), \widetilde{R}_{\text{aut}}(\xi)$ define such a decomposition. Assume that both $\widetilde{R}_{\text{contr}}(\xi)$ and $\widetilde{R}_{\text{aut}}(\xi)$ are of full row rank. Prove that there exist unimodular matrices $U(\xi) \in \mathbb{R}^{g \times g}[\xi]$, and $W(\xi) \in \mathbb{R}^{q \times q}[\xi]$ such that

$$\widetilde{R}_{\text{contr}}(\xi) = U(\xi) \begin{bmatrix} I & 0 \end{bmatrix}, \quad \begin{bmatrix} D(\xi) & 0 \end{bmatrix} W(\xi) = \begin{bmatrix} D(\xi) & 0 \end{bmatrix},$$
$$\widetilde{R}_{\text{aut}}(\xi) = U(\xi) \begin{bmatrix} D(\xi) & 0 \\ 0 & I \end{bmatrix} W(\xi).$$

(d) Let $\mathfrak{B} = \mathfrak{B}_{\text{contr}} \oplus \mathfrak{B}_{\text{aut}}$ be a decomposition into a controllable part and an autonomous part defined by polynomial matrices $R'_{\text{contr}}(\xi)$ and $R'_{\text{aut}}(\xi)$. Assume that both $\widetilde{R}_{\text{contr}}(\xi)$ and $\widetilde{R}_{\text{aut}}(\xi)$ are of full row rank. Prove that there exist unimodular matrices $U(\xi) \in R^{g \times g}[\xi]$, and $W(\xi) \in \mathbb{R}^{q \times q}[\xi]$ such that

$$R'_{\text{contr}}(\xi) = U(\xi)R_{\text{contr}}(\xi), \quad \begin{bmatrix} D(\xi) & 0 \end{bmatrix} W() = \begin{bmatrix} D(\xi) & 0 \end{bmatrix},$$
$$R'_{\text{aut}}(\xi) = U(\xi)R_{\text{aut}}(\xi)W^{-1}(\xi)$$

(e) Characterize all unimodular matrices $W(\xi) \in \mathbb{R}^{q \times q}[\xi]$ with the property that

$$\begin{bmatrix} D(\xi) & 0 \end{bmatrix} W(\xi) = \begin{bmatrix} D(\xi) & 0 \end{bmatrix}.$$

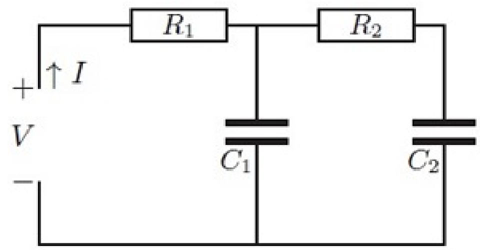


Figure 1: Electrical Circuit

6. Consider the circuit in Figure 1. Let $\begin{bmatrix} V \\ I \end{bmatrix}$ form trajectories in a behavior.

   (a) Using the voltages across the two capacitors as latent variables write down a latent variable representation of the behavior.

   (b) Find a *syzygy* matrix for the operator acting on the latent variables.

   (c) Eliminate the latent variables to obtain a *kernel representation* of this behavior.

   (d) For what values of $R_1, R_2, C_1, C_2$ is the system behavior controllable.

   (e) For what values of $R_1, R_2, C_1, C_2$ is the system observable.

   (f) Find a *controllable-autonomous* decomposition for this behavior.