EE752: Assignment

April 10, 2015

1. The power associated with a mechanical system is given by $P = F\dot{x}$, where F is the force and x the position associated with the system.
   (a) What is the two variable symmetric polynomial matrix $\Phi(\zeta, \eta)$ associated with the supply rate P.
   (b) Show that $\Phi(\zeta, \eta)$ is unique for the given P.

2. (a) Consider the 2×2 two variable polynomial matrix

$$\Phi(\zeta, \eta) = \begin{bmatrix} \zeta + \eta & 3\eta \\ 3\zeta & 5\zeta^2 + 3\eta \end{bmatrix}$$

      i. Find the QDF associated with $\Phi(\zeta, \eta)$.
      ii. Is the QDF unique? If yes prove uniqueness otherwise provide a counter example.

   (b) Repeat question 2(a) for the 3×3 two variable polynomial matrix given below

$$\Phi(\zeta, \eta) = \begin{bmatrix} \zeta^2 + \eta^2 & 3\eta^2 & 0 \\ 3\zeta^2 & \zeta\eta & 4\zeta \\ 0 & 4\eta & \zeta + \eta \end{bmatrix}$$

3. Consider the supply rate induced by the matrix $\Phi = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. Then

   (a) For a behavior $\mathfrak{B} = \text{ img } \begin{bmatrix} \xi + 1 \\ 1 \end{bmatrix}$, show that $\Phi \overset{\mathfrak{B}}{>} 0$ i.e. the QDF is positive on $\mathfrak{B}$.
   (b) Consider the behavior $\mathfrak{B}_s = \text{ img } \begin{bmatrix} \xi^2 + \xi \\ \xi \end{bmatrix}$. Verify that the behavior $\mathfrak{B}_s$ and $\mathfrak{B}$ is the same.
   (c) For the behavior $\mathfrak{B}_s$ show that $\Phi \overset{\mathfrak{B}_s}{\geqslant} 0$ i.e. the QDF is non-negative on $\mathfrak{B}_s$.
   (d) From part (b) of the question it is clear that both $\mathfrak{B}$ and $\mathfrak{B}_s$ are the same behavior. Average positivity or non-negativity is a system property then how can the same behavior be average positive (as seen in part (a)) and at the same time be average non-negative (as seen in part (c)).
   What is the actual property of the system?
      i. Average positivity
      ii. Average non-negativity
      iii. None of the above

   Give reasons for your answer. Write down the reason why we had two different conclusions from (a) and (c) for the same behavior $\mathfrak{B}$ and $\mathfrak{B}_s$.

4. Consider the following systems and let the supply rate Q_Σ be induced by the matrix Σ.

   (a) $\mathfrak{B} = \text{im} \begin{bmatrix} \xi^2 - \xi + 1 \\ \xi^2 + \xi + 1 \end{bmatrix}$ with supply rate $\Sigma = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$
   (b) $\mathfrak{B} = \text{im} \begin{bmatrix} \xi^2 - \xi + 1 \\ \xi^2 + \xi + 1 \end{bmatrix}$ with supply rate $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$
   (c) $\mathfrak{B} = \text{im} \begin{bmatrix} \xi^4 + 5\xi^2 + 4 \\ 5\xi + 2\xi^3 \end{bmatrix}$ with supply rate $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$
   (d) $\mathfrak{B} = \text{im} \begin{bmatrix} \xi^4 + 5\xi^2 + 4 \\ 5\xi + 2\xi^3 \end{bmatrix}$ with supply rate $\Sigma = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

   (i) Check if the systems are conservative.
   (ii) Find the storage function for the behaviors/system that are conservative.

(iii) For the conservative systems show that every storage function is a state function i.e. every storage function Q_Ψ can be written as $x^T K x$ where K is a constant matrix and x is related to the states of the system.

5. Consider the mechanical system in figure 1 with spring constant $K = 2\ kg/sec^2$ and mass $M = 1\ kg$. The supply rate for the system is $Q_\Sigma = F\dot{x}$.

   (a) Show that the system is conservative.

   (b) From the system it is clear that the McMillan degree of the system is 2. Find the storage function with respect to the system states and verify that the storage function with respect to the system states are not static relations but have derivative terms in it.

   (c) Rewrite the obtained storage function as $p^T K p$ where K is a constant matrix and verify that p turns out to be a state vector with more states compared to the states of the system. This will show that to express the storage function as a state function the number of states required is more than the McMillan degree of the system. This is due to the fact that the supply rate has derivative term in it.

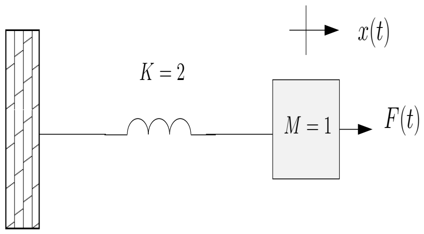

Figure 1: Mechanical system Q5

6. Show that conservative systems admit unique storage function.

7. (a) For each of the behavior $\mathfrak{B}$, test whether the systems are dissipative with respect to the supply rate given i.e. $\int Q_\Sigma \overset{\mathfrak{B}}{\geqslant} 0$.

   i. $\mathfrak{B} = \operatorname{img} \begin{bmatrix} \xi+2 \\ \xi+1 \end{bmatrix}$ with respect to positve real supply rate $u^T y$.

   ii. $\mathfrak{B} = \operatorname{img} \begin{bmatrix} \xi^2+1 \\ 1 \end{bmatrix}$ with respect to bounded real suply rate $u^T u - y^T y$.

   iii. $\mathfrak{B} = \operatorname{img} \begin{bmatrix} \xi \\ 1 \end{bmatrix}$ with respect to LQ supply rate $x^T x + u^T u$.

   iv. $\mathfrak{B} = \operatorname{img} \begin{bmatrix} \xi+1 & 1 \\ 1 & \xi^2 \\ \xi & \xi^2+1 \end{bmatrix}$ with respect to $u^T u - y^T y$.

   v. $\mathfrak{B} = \operatorname{img} \begin{bmatrix} 2\xi & 1 \\ \xi^2 & \xi \\ 1 & 0 \end{bmatrix}$ with respect to supply rate induced by $\Sigma = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$.

   (b) Is the behavior in question $7a$(v) dissipative with respect to $u^T y$?

   (c) Which of the dissipative behaviors in question $7a$ satisfies $\int Q_\Sigma \overset{\mathfrak{B}}{>} 0$.

8. (a) Consider the behavior $\mathfrak{B} = \operatorname{img} \begin{bmatrix} \xi^2+2\xi+1 \\ 0.5 \end{bmatrix}$ is dissipative with respect to the supply rate induced by the matrix $\Sigma = \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix}$ where $\gamma \in \mathbb{R}$ and $\gamma > 0$. The manifest variable of the behavior is $\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right)$ where u, y is the input and output of the behavior respectively. What is the minimum value of γ for the behavior to be Σ-dissipative?

   (b) Find the maximum value of the magnitude Bode plot of the behavior $\mathfrak{B}$ in question (a). Verify that the value of γ is the same as the maximum value of the magnitude Bode plot.

   Note: For a system with no poles on the closed right half plane (RHP + imaginary axis) of the $\mathbb{C}$ plane, the maximum value of Bode magnitude plot is called the H_∞-norm of a SISO system. The exercise shows that the value of γ and H_∞-norm is the same. In general given a proper real transfer matrix $G(s)$ with no poles on the closed right half plane, H_∞ norm is be defined as

   $$||G(s)||_{H_\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{max}\{G(j\omega)\}$$

   For SISO case the $\sigma_{max}\{G(j\omega)\}$ = maximum magnitude of the Bode plot.

9. Compute the extremum storage functions Ψ_- and Ψ_+ for the following systems

   (a) i. Consider the supply rate $v \times i$, where v and i are the current and voltage of the system respectively in figure 2.

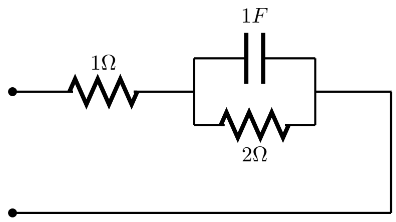

Figure 2: Circuit for problem 7(a)i

   ii. $\mathfrak{B} = \text{ img } \begin{bmatrix} 2\xi + 11 \\ 1 \end{bmatrix}$ with supply rate induced by $\Sigma = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

   iii. $\mathfrak{B} = \text{ img } \begin{bmatrix} \xi^2 - 2\xi + 1 \\ \xi \end{bmatrix}$ with supply rate induced by $\Sigma = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

   iv. $\mathfrak{B} = \text{ img } \begin{bmatrix} \xi^2 - 2\xi + 1 \\ \xi \end{bmatrix}$ with supply rate induced by $\Sigma = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$

   (b) Verify for the behaviors in question 9(a) that $\frac{1}{2}(\Psi_- + \Psi_+)$ is also storage function.

10. Given a dissipative system there are four types of behavior possible

   (a) $\int Q_\Phi \geqslant 0 \Leftrightarrow \partial\Phi(i\omega) \geqslant 0 \quad \forall \quad \omega \in \mathbb{R}$
   (b) $\int Q_\Phi > 0 \Leftrightarrow \partial\Phi(i\omega) \geqslant 0 \quad \forall \quad \omega \in \mathbb{R}$ with $\det \partial\Phi \neq 0$
   (c) $\int Q_\Phi \overset{\text{per}}{>} 0 \Leftrightarrow \partial\Phi(i\omega) > 0 \quad \forall \quad \omega \in \mathbb{R}$
   (d) $\int Q_\Phi \gg 0 \Leftrightarrow \partial\Phi(i\omega) > 0 \quad \forall \quad \omega \in \mathbb{R}$ with $\det \partial\Phi = 2\text{n}$ where n is the *McMillan degree* of M where $\Phi(\zeta, \eta) = M(\zeta)^T \Sigma M(\eta)$

   (i) Verify the following for $\Sigma = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

   - System with transfer matrix $G(s) = \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$ satisfies statement (a) only. Give example of a MIMO system satisfying statement (a) only.
   - System with transfer function $G(s) = \frac{1}{s+1}$ satisfies statement (b) but not (c) and (d). Give example of a MIMO system satisfying statement (b) but not (c) and (d).
   - System with transfer function $G(s) = \frac{s-0.8}{s+1}$ satisfies statement (c) but not (d). Give example of a MIMO system satisfying statement (c) but not (d).
   - System with transfer function $G(s) = \frac{0.2}{s+1}$ satisfies statement (d). Give example of a MIMO system satisfying (d).

   (ii) Construct an example of a SISO system that satisfies condition (a) but donot satisfy condition (b).

11. (a) Consider the following controllable behaviors

$$\mathfrak{B}_{sg} = \{ \begin{pmatrix} u \\ y \end{pmatrix} \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}}) \text{ such that } \begin{pmatrix} u \\ y \end{pmatrix} = M_1(\frac{d}{dt})\ell \}$$

$$\mathfrak{B}_{pr} = \{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}}) \text{ such that } \begin{pmatrix} p \\ q \end{pmatrix} = M_2(\frac{d}{dt})\ell \}$$

   u and y are the input and output of the behavior $\mathfrak{B}_{sg}$ respectively. Similarly p and q is the input and output of $\mathfrak{B}_{pr}$ respectively. The input and output cardinality of each behavior is the same.
   Consider matrices $\Sigma_{sg} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ and $\Sigma_{pr} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.
   Considering that $p = u - y$ and $q = u + y$, show that

$$\mathfrak{B}_{sg} \text{ is } \Sigma_{sg} - \text{dissipative if and only if } \mathfrak{B}_{pr} \text{ is } \Sigma_{pr} - \text{dissipative.}$$

   [Hint: use the transformation $u^T u - y^T y = (u - y)^T (u + y)$].

   (b) Consider a system with transfer function $G(s)$ is Σ_{pr}-dissipative. Verify that system with transfer function $\frac{1-G(s)}{1+G(s)}$ is Σ_{sg}-dissipative.