Appendix to the paper titled "Dissipativity analysis of SISO systems using Nyquist-Plot-Compatible (NPC) supply rates"

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Lemma 3.3: Let $T = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$ be a non-singular matrix and $\sum_{br} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ be the small-gain supply rate. Then the new supply rate $\Phi = T^T \sum_{br} T$ is an NPC supply rate, and its corresponding $\mathcal{A}_\Phi^+$ is one of the following:

1) If $b = d$, then the the boundary, $\mathcal{A}_\Phi^0$, is a line parallel to the imaginary axis. Further, if $ab - cd > 0$ (or, if $ab - cd < 0$) then $\mathcal{A}_\Phi^+$ is the RHS (LHS) of the line $\mathcal{A}_\Phi^0$.
2) If $b \neq d$ then the boundary, $\mathcal{A}_\Phi^0$, is a circle with center on the real axis. Further, the corresponding $\mathcal{A}_\Phi^+$ is the interior (or the exterior) of the circle if $b^2 - d^2 < 0$ ($b^2 - d^2 > 0$).

Proof. First, note that Φ can be written explicitly as

$$\Phi = \begin{bmatrix} a^2 - c^2 & ab - cd \\ ab - cd & b^2 - d^2 \end{bmatrix}.$$

Now, for a transfer function $G(s) = N(s)/D(s)$, let us denote $x := \text{Re}\, G(j\omega)$ and $y := \text{Imag}\, G(j\omega)$. It then follows from the definition of NPC supply rates (Definition 3.1) that the Nyquist plot of G being contained in $\mathcal{A}_\Phi^0$ for some non-negative frequency is equivalent to

$$(x^2 + y^2)(b^2 - d^2) + 2x(ab - cd) + (a^2 - c^2) = 0.$$

Thus, $\mathcal{A}_\Phi^0 =$

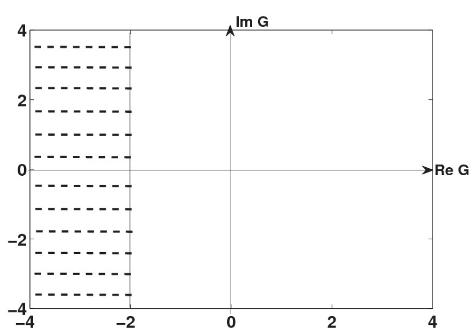

Fig. 1. Associated region of shifted passivity supply rate ($b = d$ in T).

$$\{x + iy \mid (x^2 + y^2)(b^2 - d^2) + 2x(ab - cd) + (a^2 - c^2) = 0\}.$$

Likewise, $\mathcal{A}_\Phi^+ =$

$$\{x + iy \mid (x^2 + y^2)(b^2 - d^2) + 2x(ab - cd) + (a^2 - c^2) > 0\}. \quad (1)$$

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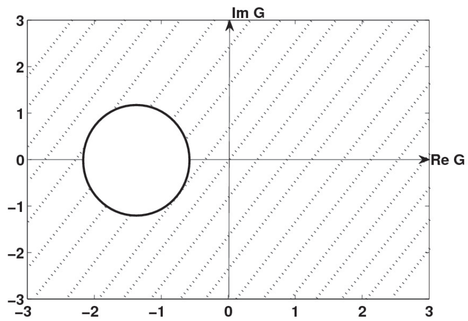

Fig. 2. Associated region of shifted small-gain supply rate ($b \neq d$ in T).

Using this we now prove statements 1) and 2).

If $b = d$, then (1) can be simplified as:

$$\mathcal{A}_\Phi^+ = \{x + iy \mid (a - c)(2bx + (a + c)) > 0\}.$$

It then follows that $\mathcal{A}_\Phi^0$ is given by the vertical line $x = -\frac{a+c}{2b}$. Further, when $ab - cd > 0$, $\mathcal{A}_\Phi^+$ turns out to be the RHS of this line, and, when $ab - cd < 0$, $\mathcal{A}_\Phi^+$ is the LHS of this line. Note that here $ab - cd = ad - bc = \det T \neq 0$ because T must be non-singular. This proves statement 1). An example of such an NPC-region is shown in Figure 4.

For statement 2), that is, when $b \neq d$, observe that the equation for the NPC-boundary $\mathcal{A}_\Phi^0$ matches with the generic equation of a circle with finite radius and center on the x-axis (because there is no y, or xy terms in the equation for $\mathcal{A}_\Phi^0$). Note also that the signs of the quadratic terms x^2 and y^2 are given by the sign of $b^2 - d^2$. Therefore, the corresponding $\mathcal{A}_\Phi^+$ is the interior or the exterior of the circle depending on whether $b^2 - d^2 < 0$ or $b^2 - d^2 > 0$, respectively. Figure 5 shows such an NPC-region. □

Theorem 4.1: Consider a SISO LTI system given by the transfer function $G(s)$ and let $\mathfrak{B}_G = \text{im} M(\frac{d}{dt})$ be its image representation. Let Φ_1 and Φ_2 be NPC supply rates. Then the following two statements are equivalent:

1) G has Nyquist plot contained in $\mathcal{A}_{\Phi_1}^+ \cup \mathcal{A}_{\Phi_2}^+$ for almost all $\omega \geqslant 0$.
2) There exist $p, q \in \mathbb{R}[\xi]$ such that $\mathfrak{B}_G$ is strictly dissipative with respect to

$$\Phi(\zeta, \eta) := p(\zeta)\Phi_1(\zeta, \eta)p(\eta) + q(\zeta)\Phi_2(\zeta, \eta)q(\eta) \quad (2)$$

Proof. 1) $\Rightarrow$ 2): We assume that G has Nyquist plot contained in $\mathcal{A}_{\Phi_1}^+ \cup \mathcal{A}_{\Phi_2}^+$ for almost all $\omega \geqslant 0$, we have to prove the existence of polynomials $p, q \in \mathbb{R}[\xi]$ such that $\mathfrak{B}_G$ is strictly dissipative with respect to $\Phi(\zeta,\eta) := p(\zeta)\Phi_1(\zeta,\eta)p(\eta) + q(\zeta)\Phi_2(\zeta,\eta)q(\eta)$. This, by Proposition 2.1, is equivalent to

$$M^T(-j\omega)\partial\Phi(j\omega)M(j\omega) > 0 \quad \text{for almost all } \omega \in \mathbb{R}. \quad (3)$$

$$\iff M^T(-j\omega)[p(-j\omega)\partial\Phi_1(j\omega)p(j\omega)+ q(-j\omega)\partial\Phi_2(j\omega)q(j\omega)]M(j\omega) > 0 \quad \text{for almost all } \omega \in \mathbb{R}. \quad (4)$$

We define the following two functions of ω:

$$\left.\begin{array}{lcl} \Gamma(\omega) & := & M^T(-j\omega)\partial\Phi_1(j\omega)M(j\omega) \\ \Pi(\omega) & := & M^T(-j\omega)\partial\Phi_2(j\omega)M(j\omega) \end{array}\right\} \quad (5)$$

Using equation (5) equation (4) can be rewritten as

$$p(-j\omega)\Gamma(\omega)p(j\omega) + q(-j\omega)\Pi(\omega)q(j\omega) > 0$$

for almost all $\omega \in \mathbb{R}$. This can be written in a matrix-vector form as

$$\begin{bmatrix} p(-j\omega) \\ q(-j\omega) \end{bmatrix}^T \begin{bmatrix} \Gamma(\omega) & 0 \\ 0 & \Pi(\omega) \end{bmatrix} \begin{bmatrix} p(j\omega) \\ q(j\omega) \end{bmatrix} > 0 \quad (6)$$

for almost all $\omega \in \mathbb{R}$. At this point we claim that for any $\omega \geqslant 0$, the two functions $\Gamma(\omega)$ and $\Pi(\omega)$ cannot both be negative simultaneously. Indeed, if for some $\omega \geqslant 0$, $\Gamma(\omega) < 0$ then, since Φ_1 is an NPC supply rate, by Definition 3.1, it means that the Nyquist plot of G at that ω is contained in $\mathcal{A}^-_{\Phi_1}$. If $\Pi(\omega)$, too, is less than zero at that frequency ω, then the Nyquist plot of G at ω is contained in $\mathcal{A}^-_{\Phi_2}$ because Φ_2 also is an NPC supply rate. These two facts together means that at the frequency ω the Nyquist plot of G is contained in $\mathcal{A}^-_{\Phi_1} \cap \mathcal{A}^-_{\Phi_2}$. Since, $\Gamma(\omega)$ and $\Pi(\omega)$ are continuous functions of ω, if they are negative at some frequency ω then there exists an open interval containing ω over which they continue to be simultaneously negative. This means that over a continuous band of frequencies the Nyquist plot is contained in $\mathcal{A}^-_{\Phi_1} \cap \mathcal{A}^-_{\Phi_2}$. Since $\mathcal{A}^-_{\Phi_1} \cap \mathcal{A}^-_{\Phi_2} \subseteq \mathbb{C} \setminus \left(\mathcal{A}^+_{\Phi_1} \cup \mathcal{A}^+_{\Phi_2}\right)$, the last statement clearly contradicts the assumption of statement 1).

Now consider the matrix

$$S(\xi) = \begin{bmatrix} \Gamma(-j\xi) & 0 \\ 0 & \Pi(-j\xi) \end{bmatrix}. \quad (7)$$

Note that from equation (5) it follows that both $\Gamma(-j\xi)$ an $\Pi(-j\xi)$ are polynomials in ξ with real constant coefficients. Therefore, $S(\xi) \in \mathbb{R}^{2\times 2}[\xi]$. Moreover, it can be easily checked that $S(\xi)$ is para-Hermitian. Recalling the definition of worst inertia of a para-Hermitian polynomial matrix (Definition 2.2) it follows that $S(\xi)$ has worst inertia either $(0,2)$, or $(1,1)$. This is because, in order for the worst inertia to be anything other than $(0,2)$, or $(1,1)$, $S(j\omega)$ must be negative definite at some frequency $\omega \geqslant 0$. This is equivalent to $\Gamma(\omega) < 0$ and $\Pi(\omega) < 0$ at the frequency ω, which is not possible as argued in the last paragraph.

For the case when $S(\xi)$ has worst inertia $(0,2)$ it follows that $S(j\omega)$ then is positive definite for almost all $\omega \in \mathbb{R}$. This means for *any* coprime pair of polynomials (p,q) inequality (6) holds.

For the latter case, i.e., when $S(\xi)$ is having worst inertia $(1,1)$, it follows from Proposition 2.3 that $S(j\omega)$ can be written as

$$S(j\omega) = K^T(-j\omega)J_{worst}K(j\omega) + L^T(-j\omega)L(j\omega) \quad (8)$$

where $J_{worst} = \mathrm{diag}(1,-1)$ and matrices $K(\xi) \in \mathbb{R}^{2\times 2}[\xi]$, $L(\xi) \in \mathbb{R}^{\bullet\times 2}[\xi]$, with $\det\, K(\xi) \not\equiv 0$.

At this point we follow a construction similar to proof of Theorem 6.6 in [1]. There, the above equation is used to find the polynomials p,q that meet the requirements of inequality (6) as follows:

1) Choose $p_1(\xi), q_1(\xi) \in \mathbb{R}[\xi]$ coprime such that

$$p_1(-j\omega)p_1(j\omega) - q_1(-j\omega)q_1(j\omega) > 0 \;\forall \omega \in \mathbb{R}. \quad (9)$$

2) Next, construct $\mathrm{adj}\, K(\xi) \in \mathbb{R}^{2\times 2}[\xi]$ the adjugate of $K(\xi)$. The required p,q are given by

$$\begin{bmatrix} p(\xi) \\ q(\xi) \end{bmatrix} := \mathrm{adj}\; K(\xi) \begin{bmatrix} p_1(\xi) \\ q_1(\xi) \end{bmatrix}.$$

Indeed, putting this p,q in the LHS of inequality (6) and utilizing the factorization of $S(j\omega)$ given by equation (8) we get

$$\begin{bmatrix} p(-j\omega) \\ q(-j\omega) \end{bmatrix}^T S(j\omega) \begin{bmatrix} p(j\omega) \\ q(j\omega) \end{bmatrix} \geqslant$$
$$\begin{bmatrix} p(-j\omega) \\ q(-j\omega) \end{bmatrix}^T \left(K^T(-j\omega)J_{worst}K(j\omega)\right) \begin{bmatrix} p(j\omega) \\ q(j\omega) \end{bmatrix} =$$
$$\delta(-j\omega)\delta(j\omega) \begin{bmatrix} p_1(-j\omega) \\ q_1(-j\omega) \end{bmatrix}^T J_{worst} \begin{bmatrix} p_1(j\omega) \\ q_1(j\omega) \end{bmatrix} > 0$$

for almost all $\omega \in \mathbb{R}$, where $\delta(\xi) = \det\, K(\xi)$. The last inequality follows from equation (9). Thus, the p,q chosen above satisfy inequality (6). This completes the proof of 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 1): Statement 2) says that there exist $p,q \in \mathbb{R}[\xi]$ such that $\mathfrak{B}_G$ is strictly dissipative with respect to $\Phi(\zeta,\eta) = p(\zeta)\Phi_1(\zeta,\eta)p(\eta) + q(\zeta)\Phi_2(\zeta,\eta)q(\eta)$. From this we have to prove that G has Nyquist plot contained in $\mathcal{A}^+_{\Phi_1} \cup \mathcal{A}^+_{\Phi_2}$ for all $\omega \geqslant 0$. From Proposition 2.1, statement 2) is equivalent to $M^T(-j\omega)\partial\Phi(j\omega)M(j\omega) > 0$ for almost all $\omega \in \mathbb{R}$. Now note that using equation (5), and the definition of Φ, the last inequality reduces to

$$p(-j\omega)\Gamma(\omega)p(j\omega) + q(-j\omega)\Pi(\omega)q(j\omega) > 0 \quad (10)$$

for almost all $\omega \in \mathbb{R}$. This means there cannot exist any $\omega \geqslant 0$ for which $\Gamma(\omega) < 0$ and $\Pi(\omega) < 0$ simultaneously. For if there is some ω, then by continuity of Γ and Π, there is an interval around ω over which Γ and Π would be negative. Because $p(-j\omega)p(-j\omega)$ and $q(-j\omega)q(j\omega)$ are non-negative for each $\omega \in \mathbb{R}$, the last statement implies inequality (10) is violated over a continuous band of frequencies. Hence we infer that for all $\omega \geqslant 0$, either $\Gamma(\omega) \geqslant 0$ or $\Pi(\omega) \geqslant 0$ (or both). Since, Γ and Π both cannot be identically zero polynomials, we conclude that for almost all $\omega \geqslant 0$ either $\Gamma(\omega) > 0$ or $\Pi(\omega) > 0$ (or both). By using Definition 3.1 and the fact that Φ_1 and Φ_2 are NPC supply rates, it follows that for almost all $\omega \geqslant 0$, the Nyquist plot of G is contained in $\mathcal{A}_{\Phi_1}^{+} \cup \mathcal{A}_{\Phi_2}^{+}$. This completes the proof of $2) \Rightarrow 1)$. $\square$

References

[1] I. Pendharkar, H. K. Pillai, *A parametrization for dissipative behaviors,* Systems and Control Letters, Vol. 51 , pp.123-132, 2004.